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Stochastic homogenization of convolution type operators

A. Piatnitski [a,b,*], E. Zhizhina [b]

[a] *The Arctic University of Norway, Campus Narvik, P.O. Box 385, 8505 Narvik, Norway*
[b] *Institute for Information Transmission Problems RAS, Bolshoy Karetny 19, Moscow, Russia*

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ABSTRACT

This paper deals with the homogenization problem for convolution type non-local operators in random statistically homogeneous ergodic media. Assuming that the convolution kernel has a finite second moment and satisfies certain symmetry and uniform ellipticity conditions, we prove the almost sure homogenization result and show that the limit operator is a second order elliptic differential operator with constant deterministic coefficients.

RÉSUMÉ

Ce papier traite du problème d'homogénéisation pour les opérateurs non locaux de type à convolution dans des milieux ergodiques statistiquement homogènes et aléatoires. En supposant que le noyau de convolution ait un deuxième moment fini et satisfait à certaines conditions de symétrie et d'ellipticité uniformes, nous prouvons le résultat d'homogénéisation presque sûr et nous montrons que l'opérateur limite est un opérateur elliptique différentiel du second ordre à coefficients déterministes constants.

1. Introduction

The paper deals with homogenization problem for integral operators of convolution type in $\mathbb{R}^d$ with dispersal kernels that have random statistically homogeneous ergodic coefficients. For such operators, under natural integrability, moment and uniform ellipticity conditions as well as the symmetry condition we prove the homogenization result and study the properties of the limit operator.

The integral operators with a kernel of convolution type are of great interest both from the mathematical point of view and due to various important applications in other fields. Among such applications are models of population dynamics and ecological models, see [16], [6] and references therein, non-local diffusion prob-

* Corresponding author at: The Arctic University of Norway, Campus Narvik, P.O. Box 385, 8505 Narvik, Norway.
E-mail addresses: apiatnitski@gmail.com (A. Piatnitski), ejj@iitp.ru (E. Zhizhina).

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lems, see [1,4], continuous particle systems, see [7,11], image processing algorithms, see [9]. In the cited works only the case of homogeneous environments has been considered. In this case the corresponding dispersal kernel depends only on the displacement $y - x$. However, many applications deal with non-homogeneous environments. Such environments are described in terms of integral operator whose dispersal kernels depend not only on the displacement $x - y$ but also on the starting and the ending positions x, y.

When studying the large-time behavior of evolution processes in these environments it is natural to make the diffusive scaling in the corresponding integral operators and to consider the homogenization problem for the obtained family of operators with a small positive parameter. In what follows we call this parameter ε.

The case of environments with periodic characteristics has been studied in the recent work [18]. It has been shown that under natural moment and symmetry conditions on the kernel the family of rescaled operators admits homogenization, and that for the corresponding jump Markov process the Central Limit Theorem and the Invariance Principle hold. Interesting homogenization problems for periodic operators containing both second order elliptic operator and nonlocal Levy type operator have been considered in [2] and [20]. In [8] the limit theorem for periodic jump-diffusion driven by Levy-type noise with drift was proved.

In the present paper we consider the more realistic case of environments with random statistically homogeneous characteristics. More precisely, we assume that the dispersal kernel of the studied operators has the form $\Lambda(x, y)a(x-y)$, x, $y \in \mathbb{R}^d$, where $a(z)$ is a deterministic even function that belongs to $L^1(\mathbb{R}^d) \cap L^2_{\rm loc}(\mathbb{R}^d)$ and has finite second moments, while $\Lambda(x, y) = \Lambda(x, y, \omega)$ is a statistically homogeneous symmetric ergodic random field that satisfies the uniform ellipticity conditions $0 < \Lambda^- \le \Lambda(x, y) \le \Lambda^+$.

Making a diffusive scaling we obtain the family of operators

$$(L^\varepsilon u)(x) \; = \; \varepsilon^{-d-2} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\Lambda\Big(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\Big)(u(y) - u(x))dy, \tag{1}$$

where a positive scaling factor ε is a parameter.

For the presentation simplicity we assume in this paper that $\Lambda(x, y) = \mu(x)\mu(y)$ with a statistically homogeneous ergodic field μ. However, all our results remain valid for the generic statistically homogeneous symmetric random fields $\Lambda(x, y)$ that satisfy the above ellipticity conditions, as well as for symmetrizable operators with $\Lambda(x, y) = \lambda(x)\mu(y)$. The latter case is considered in Section 7.1.

The main goal of this work is to investigate the limit behavior of L^ε as $\varepsilon \to 0$. We are going to show that the family L^ε converges almost surely to a second order elliptic operator with constant deterministic coefficient in the so-called G-topology, that is for any $m > 0$ the family of operators $(-L^\varepsilon + m)^{-1}$ almost surely converges strongly in $L^2(\mathbb{R}^d)$ to the operator $(-L^0 + m)^{-1}$ where $L^0 = \Theta^{ij} \frac{\partial^2}{\partial x^i \partial x^j}$, and Θ is a positive definite constant matrix.

There is a vast existing literature devoted to homogenization theory of differential operators, at present it is a well-developed area, see for instance monographs [3], [5] and [10]. The first homogenization results for divergence form differential operators with random coefficients were obtained in pioneer works [12] and [17]. In these works it was shown that the generic divergence form second order elliptic operator with random statistically homogeneous coefficients admits homogenization. Moreover, the limit operator has constant coefficients, in the ergodic case these coefficients are deterministic.

Later on a number of important homogenization results have been obtained for various elliptic and parabolic differential equations and system of equations in random stationary media. The reader can find many references in the book [10].

Homogenization of elliptic difference schemes and discrete operators in statistically homogeneous media has been performed in [13], [14]. Also, in [14] several limit theorems have been proved for random walks in stationary discrete random media that possess different types of symmetry.

To our best knowledge in the existing literature there are no results on stochastic homogenization of convolution type integral operators with a dispersal kernel that has stationary rapidly oscillating coefficients.

In the one-dimensional case a homogenization problem for the operators that have both local and non-local parts has been considered in the work [19]. This work deals with scaling limits of the solutions to stochastic differential equations in dimension one with stationary coefficients driven by Poisson random measures and Brownian motions. The annealed convergence theorem is proved, in which the limit exhibits a diffusive or superdiffusive behavior, depending on whether the Poisson random measure has a finite second moment or not. It is important in this paper that the diffusion coefficient does not degenerate.

Our approach relies on asymptotic expansion techniques and using the so-called corrector. As often happens in the case of random environments we cannot claim the existence of a stationary corrector. Instead, we construct a corrector which is a random field in $\mathbb{R}^d$ with stationary increments and almost surely has a sublinear growth in $L^2(\mathbb{R}^d)$.

When substituting two leading terms of the expansion for the solution of the original equation, we obtain the discrepancies being oscillating functions with zero average. Some of these functions are not stationary. In order to show that the contributions of these discrepancies are asymptotically negligible we add to the expansion two extra terms. The necessity of constructing these terms is essentially related to the fact that, in contrast with the case of elliptic differential equations, the resolvent of the studied operator is not locally compact in $L^2(\mathbb{R}^d)$.

The paper is organized as follows:

In Section 2 we provide the detailed setting of the problem and formulate the main result of this work.

The leading terms of the ansatz for a solution of equation $(L^\varepsilon - m)u^\varepsilon = f$ with $f \in C_0^\infty(\mathbb{R}^d)$ are introduced in Section 3. Also in this section we outline the main steps of the proof of our homogenization theorem.

Then in Section 4 we construct the principal corrector in the asymptotic expansion and study the properties of this corrector.

Section 5 is devoted to constructing two additional terms of the expansion of u^ε. Then we introduce the effective matrix and prove its positive definiteness.

Estimates for the remainder in the asymptotic expansion are obtained in Section 6.

Finally, in Section 7 we complete the proof of the homogenization statements.

2. Problem setup and main result

We consider a homogenization problem for a random convolution type operator of the form

$$(L_\omega u)(x) \ = \ \mu(x,\omega) \int\limits_{\mathbb{R}^d} a(x-y)\mu(y,\omega)(u(y)-u(x))dy. \tag{2}$$

For the function $a(z)$ we assume the following:

$$a(z) \in L^1(\mathbb{R}^d) \cap L^2_{\rm loc}(\mathbb{R}^d), \quad a(z) \ge 0; \quad a(-z) = a(z), \tag{3}$$

and

$$\|a\|_{L^1(\mathbb{R}^d)} = \int\limits_{\mathbb{R}^d} a(z)\ dz = a_1 < \infty; \quad \sigma^2 = \int\limits_{\mathbb{R}^d} |z|^2 a(z)\ dz < \infty. \tag{4}$$

We also assume that

$$\begin{aligned}&\text{there exists a constant } c_0>0 \text{ and a cube } \mathbf{B}\subset\mathbb{R}^d,\\ &\text{such that } a(z)\ge c_0 \quad \text{for all } z\in\mathbf{B}.\end{aligned} \tag{5}$$

This additional condition on $a(z)$ is naturally satisfied for regular kernels, and we introduced (5) for a presentation simplicity. Assumption (5) essentially simplifies derivation of inequality (49), on which the proof of the smallness of the first corrector is based, see Proposition 4.4 below. We notice that inequality (49) can also be derived without assumption (5), however in this case additional arguments of measure theory are required.

Let $(\Omega,\mathcal{F},\mathbf{P})$ be a standard probability space. We assume that the random field $\mu(x,\omega)=\boldsymbol{\mu}(T_x\omega)$ is stationary and bounded from above and from below:

$$0<\alpha_1\le\mu(x,\omega)\le\alpha_2<\infty; \tag{6}$$

here $\boldsymbol{\mu}(\omega)$ is a random variable, and T_x, $x\in\mathbb{R}^d$, is an ergodic group of measurable transformations acting in ω-space Ω, $T_x:\Omega\mapsto\Omega$, and possessing the following properties:

- $T_{x+y}=T_x\circ T_y$ for all $x,\,y\in\mathbb{R}^d$, $T_0=\mathrm{Id}$,
- $\mathbf{P}(A)=\mathbf{P}(T_xA)$ for any $A\in\mathcal{F}$ and any $x\in\mathbb{R}^d$,
- T_x is a measurable map from $\mathbb{R}^d\times\Omega$ to Ω, where $\mathbb{R}^d$ is equipped with the Borel σ-algebra.

For each realization $\mu(\cdot,\omega)$ let us consider the following family of operators

$$(L^\varepsilon_\omega u)(x) \;=\; \frac{1}{\varepsilon^{d+2}}\int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon},\omega\Big)\mu\Big(\frac{y}{\varepsilon},\omega\Big)\Big(u(y)-u(x)\Big)dy, \tag{7}$$

with a parameter $\varepsilon>0$. We are interested in the limit behavior of the operators L^ε_ω as $\varepsilon\to0$. We are going to show that for a.e. ω the operators L^ε_ω converge to a differential operator with constant coefficients in the topology of resolvent convergence. Let us fix $m>0$, any $f\in L^2(\mathbb{R}^d)$, and define u^ε as the solution of equation:

$$(L^\varepsilon_\omega-m)u^\varepsilon \;=\; f, \qquad \text{i.e. } u^\varepsilon \;=\; (L^\varepsilon_\omega-m)^{-1}f \tag{8}$$

with $f\in L^2(\mathbb{R}^d)$. Denote by $\hat L$ the following operator in $L^2(\mathbb{R}^d)$:

$$\hat L u \;=\; \sum_{i,j=1}^d \Theta_{ij}\frac{\partial^2 u}{\partial x_i\,\partial x_j}, \quad \mathcal{D}(\hat L)=H^2(\mathbb{R}^d) \tag{9}$$

with a positive definite matrix $\Theta=\{\Theta_{ij}\},\ i,j=1,\dots,d$, defined below, see (103). Let $u_0(x)$ be the solution of equation

$$\sum_{i,j=1}^d \Theta_{ij}\frac{\partial^2 u_0}{\partial x_i\,\partial x_j}-mu_0=f, \qquad \text{i.e. } u_0 \;=\; (\hat L-m)^{-1}f \tag{10}$$

with the same right-hand side f as in (8).

Theorem 2.1. *Almost surely for any $f\in L^2(\mathbb{R}^d)$ and any $m>0$ the convergence holds:*

$$\|(L^\varepsilon_\omega-m)^{-1}f-(\hat L-m)^{-1}f\|_{L^2(\mathbb{R}^d)} \;\to 0 \quad \textit{as } \varepsilon\to0. \tag{11}$$

The statement of Theorem 2.1 remains valid in the case of non-symmetric operators L^ε of the form

$$(L_\omega^{\varepsilon,\mathrm{ns}} u)(x) \;=\; \frac{1}{\varepsilon^{d+2}} \int\limits_{\mathbb{R}^d} a\Big(\frac{x-y}{\varepsilon}\Big)\lambda\Big(\frac{x}{\varepsilon},\omega\Big)\mu\Big(\frac{y}{\varepsilon},\omega\Big)\Big(u(y)-u(x)\Big)dy \tag{12}$$

with $\lambda(z,\omega)=\boldsymbol{\lambda}(T_z\omega)$ such that $0<\alpha_1\le\lambda(x,\omega)\le\alpha_2<\infty$. In this case the equation (8) reads

$$(L_\omega^{\varepsilon,\mathrm{ns}} - m)u^\varepsilon \;=\; f. \tag{13}$$

Corollary 2.1. *Let $\lambda(z,\omega)$ and $\mu(z,\omega)$ satisfy condition (6). Then a.s. for any $f\in L^2(\mathbb{R}^d)$ and any $m>0$ the limit relation in (11) holds true with $\hat L^{\mathrm{ns}}u = \sum_{i,j=1}^d \Theta_{ij}^{\mathrm{ns}}\frac{\partial^2 u}{\partial x_i\,\partial x_j}$, $\Theta^{\mathrm{ns}} = \big(\mathbf{E}\{\frac{\boldsymbol{\mu}}{\boldsymbol{\lambda}}\}\big)^{-1}\Theta$, and Θ defined in (103).*

3. Asymptotic expansion for u^ε

We begin this section by introducing a set of functions $f\in C_0^\infty(\mathbb{R}^d)$ such that $u_0 = (\hat L - m)^{-1}f \in C_0^\infty(\mathbb{R}^d)$. We denote this set by $\mathcal{S}_0(\mathbb{R}^d)$. Observe that this set is dense in $L^2(\mathbb{R}^d)$. Indeed, if we take $\varphi(x)\in C^\infty(\mathbb{R})$ such that $0\le\varphi\le1$, $\varphi=1$ for $x\le0$ and $\varphi=0$ for $x\ge1$, then letting $f_n=(\hat L-m)\big(\varphi(|x|-n)(\hat L-m)^{-1}f(x)\big)$ one can easily check that $f_n\in C_0^\infty(\mathbb{R}^d)$ and $\|f_n-f\|_{L^2(\mathbb{R}^d)}\to0$, as $n\to\infty$.

We consider first the case when $f\in\mathcal{S}_0(\mathbb{R}^d)$ and denote by Q a cube centered at the origin and such that $\mathrm{supp}(u_0)\subset Q$. We want to prove the convergence

$$\|u^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \;\to 0, \qquad \text{as } \ \varepsilon\to0, \tag{14}$$

where the functions u^ε and u_0 are defined in (8) and (10), respectively. To this end we approximate the function $u^\varepsilon(x,\omega)$ by means of the following ansatz

$$\begin{aligned} w^\varepsilon(x,\omega) \;&=\; v^\varepsilon(x,\omega)+u_2^\varepsilon(x,\omega)+u_3^\varepsilon(x,\omega),\\ \text{with } \ v^\varepsilon(x,\omega) \;&=\; u_0(x)+\varepsilon\theta\big(\frac{x}{\varepsilon},\omega\big)\nabla u_0(x), \end{aligned} \tag{15}$$

where $\theta\big(z,\omega\big)$ is a vector function which is often called a corrector. It will be introduced later on as a solution of an auxiliary problem that does not depend on ε, see (22). A solution of this auxiliary problem is defined up to an additive constant vector, and we construct $\theta\big(z,\omega\big)$ in such a way that a.s. $\theta\big(0,\omega\big)=0$. We also set

$$\chi^\varepsilon(z,\omega)=\theta(z,\omega)+c^\varepsilon(\omega), \quad c^\varepsilon(\omega)=-\frac{1}{|Q|}\int\limits_Q\theta\big(\frac{x}{\varepsilon},\omega\big)dx. \tag{16}$$

Observe that under such a choice of the vector c^ε the function $\chi^\varepsilon\big(\frac{x}{\varepsilon},\omega\big)$ has zero average in Q. We show in Proposition 4.4 that $\varepsilon c^\varepsilon\to0$ a.s. It should be emphasized that $\theta(y,\omega)$ need not be a stationary field, that is we do not claim that $\theta(y,\omega)=\boldsymbol{\theta}(T_y\omega)$ for some random vector $\boldsymbol{\theta}(\omega)$.

Two other functions, u_2^ε and u_3^ε, that appear in the ansatz in (15) will be introduced in (81), (91), respectively.

After substitution v_ε for u to (7) we get

$$\begin{aligned}&(L^\varepsilon v^\varepsilon)(x) \;=\\ &\frac{1}{\varepsilon^{d+2}}\int\limits_{\mathbb{R}^d} a\big(\frac{x-y}{\varepsilon}\big)\mu\big(\frac{x}{\varepsilon}\big)\mu\big(\frac{y}{\varepsilon}\big)\Big(u_0(y)+\varepsilon\theta\big(\frac{y}{\varepsilon}\big)\nabla u_0(y)-u_0(x)-\varepsilon\theta\big(\frac{x}{\varepsilon}\big)\nabla u_0(x)\Big)dy;\end{aligned}$$

here and in what follows we drop the argument ω in the random fields $\mu(y,\omega)$, $\theta(y,\omega)$, etc., if it does not lead to ambiguity. After change of variables $\frac{x-y}{\varepsilon}=z$ we get

$$
\begin{aligned}
(L^\varepsilon v^\varepsilon)(x) &= \frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon})\mu(\frac{x}{\varepsilon}-z)\Big(u_0(x-\varepsilon z)-u_0(x)\Big)dz \\
&+\frac{1}{\varepsilon}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon})\mu(\frac{x}{\varepsilon}-z)\Big(\theta(\frac{x}{\varepsilon}-z)\nabla u_0(x-\varepsilon z)-\theta(\frac{x}{\varepsilon})\nabla u_0(x)\Big)dz.
\end{aligned}
\tag{17}
$$

The Taylor expansion of a function $u(y)$ with a remainder in the integral form reads

$$
\begin{aligned}
u(y) \;&=\; u(x)+\int\limits_0^1 \nabla u(x+(y-x)t)\cdot(y-x)\;dt \\
&=\; u(x)+\nabla u(x)\cdot(y-x)+\int\limits_0^1 \nabla\nabla u(x+(y-x)t)(y-x)(y-x)(1-t)\;dt
\end{aligned}
$$

and is valid for any $x,y\in\mathbb{R}^d$. Thus we can rewrite (17) as follows

$$
\begin{aligned}
&(L^\varepsilon v^\varepsilon)(x) = \\
&\frac{1}{\varepsilon}\mu\Big(\frac{x}{\varepsilon}\Big)\nabla u_0(x)\cdot\int\limits_{\mathbb{R}^d}\Big[-z+\theta\Big(\frac{x}{\varepsilon}-z\Big)-\theta\Big(\frac{x}{\varepsilon}\Big)\Big]a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)dz \\
&+\mu\Big(\frac{x}{\varepsilon}\Big)\nabla\nabla u_0(x)\cdot\int\limits_{\mathbb{R}^d}\Big[\frac{1}{2}z\otimes z-z\otimes\theta\Big(\frac{x}{\varepsilon}-z\Big)\Big]a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)\,dz+\phi_\varepsilon(x) \\
&=:\frac{1}{\varepsilon}I^\varepsilon_{-1}+\varepsilon^0 I^\varepsilon_0+\phi_\varepsilon
\end{aligned}
\tag{18}
$$

with

$$
\begin{aligned}
&\phi_\varepsilon(x,\omega) = \\
&\int\limits_{\mathbb{R}^d} a(z)\mu\Big(\frac{x}{\varepsilon}\Big)\mu\Big(\frac{x}{\varepsilon}-z\Big)\Big(\int\limits_0^1 \nabla\nabla u_0(x-\varepsilon zt)\cdot z\otimes z\,(1-t)\;dt-\frac{1}{2}\nabla\nabla u_0(x)\cdot z\otimes z\Big)dz \\
&+\frac{1}{\varepsilon}\mu\Big(\frac{x}{\varepsilon}\Big)\int\limits_{\mathbb{R}^d} a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)\theta\Big(\frac{x}{\varepsilon}-z\Big)\Big(\nabla u_0(x-\varepsilon z)-\nabla u_0(x)\Big)dz \\
&+\mu\Big(\frac{x}{\varepsilon}\Big)\nabla\nabla u_0(x)\int\limits_{\mathbb{R}^d} a(z)\mu\Big(\frac{x}{\varepsilon}-z\Big)z\otimes\theta\Big(\frac{x}{\varepsilon}-z\Big)\,dz.
\end{aligned}
\tag{19}
$$

Here and in what follows $z\otimes z$ stands for the matrix $\{z_iz_j\}_{i,j=1}^d$.

Let us outline the main steps of the proof of relation (14). In order to make the term I^ε_{-1} in (18) equal to zero, we should construct a random field $\theta\big(z,\omega\big)$ that satisfies the following equation

$$
\int\limits_{\mathbb{R}^d}\Big(-z+\theta\big(\frac{x}{\varepsilon}-z,\omega\big)-\theta\big(\frac{x}{\varepsilon},\omega\big)\Big)\,a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\;dz\;=\;0. \tag{20}
$$

The goal of the first step is to construct such a random field $\theta(z,\omega)$. Next we show that the second term I_0^ε can be represented as a sum

$$I_0^\varepsilon = \hat{L}u_0 + S\Big(\frac{x}{\varepsilon},\omega\Big)\nabla\nabla u_0 + f_2^\varepsilon(x,\omega),$$

where $S(z,\omega)$ is a stationary matrix-field with zero average, and $f_2^\varepsilon(x,\omega)$ is a non-stationary term; both of them are introduced below. We define u_2^ε and u_3^ε by

$$(L^\varepsilon - m)u_2^\varepsilon = -S\Big(\frac{x}{\varepsilon},\omega\Big)\nabla\nabla u_0, \quad (L^\varepsilon - m)u_3^\varepsilon = -f_2^\varepsilon(x,\omega),$$

and prove that $\|u_2^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0$, $\|u_3^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0$. Then considering the properties of the corrector θ, see Theorem 4.1, we derive the limit relation

$$\|\varepsilon\theta\big(\frac{x}{\varepsilon}\big)\nabla u_0(x)\|_{L^2(\mathbb{R}^d)} \to 0, \quad \text{as } \varepsilon \to 0.$$

This yields $\|w^\varepsilon - u_0\| \to 0$.

With this choice of θ, u_2^ε and u_3^ε the expression $(L^\varepsilon - m)w^\varepsilon$ can be rearranged as follows:

$$\begin{aligned}(L^\varepsilon - m)w^\varepsilon &= (L^\varepsilon - m)v^\varepsilon + (L^\varepsilon - m)(u_2^\varepsilon + u_3^\varepsilon) = (\hat{L} - m)u_0 + \phi_\varepsilon - m\varepsilon\theta\nabla u_0\\ &= f + \phi_\varepsilon - m\varepsilon\theta\nabla u_0 = (L^\varepsilon - m)u^\varepsilon + \phi_\varepsilon - m\varepsilon\theta\nabla u_0.\end{aligned}$$

We prove below in Lemma 6.1 that $\|\phi_\varepsilon\|_{L^2(\mathbb{R}^d)}$ is vanishing as $\varepsilon \to 0$. This implies the convergence $\|w^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0$ and, by the triangle inequality, the required relation in (14).

4. First corrector

In this Section we construct a solution of equation (20). Denote

$$r\big(\frac{x}{\varepsilon},\omega\big) = \int\limits_{\mathbb{R}^d} z\, a(z)\, \mu\big(\frac{x}{\varepsilon} - z,\omega\big)\, dz, \tag{21}$$

then $r(\xi,\omega) = \mathbf{r}(T_\xi\omega)$, $\xi = \frac{x}{\varepsilon}$, is a stationary field. Moreover, since $\mathbf{E}\mu(\xi - z,\omega) = \mathbf{E}\boldsymbol{\mu}(T_{\xi-z}\omega) = const$ for all z, then

$$\mathbf{E}r(\xi,\omega) = \int\limits_{\mathbb{R}^d} z\, a(z)\, \mathbf{E}\mu(\xi - z,\omega)\, dz \;=\; 0.$$

Equation (20) takes the form

$$r(\xi,\omega) \;=\; \int\limits_{\mathbb{R}^d} a(z)\mu(\xi - z,\omega)\,\big(\theta(\xi - z,\omega) - \theta(\xi,\omega)\big)\, dz. \tag{22}$$

We are going to show now that equation (22) has a solution that possesses the following properties:

A) the increments $\zeta_z(\xi,\omega) = \theta(z+\xi,\omega) - \theta(\xi,\omega)$ are stationary for any given z, i.e.

$$\zeta_z(\xi,\omega) = \zeta_z(0,T_\xi\omega);$$

B) $\varepsilon\theta\big(\frac{x}{\varepsilon},\omega\big)$ is a function of sub-linear growth in $L^2_{\rm loc}(\mathbb{R}^d)$: for any bounded Lipschitz domain $Q\subset\mathbb{R}^d$

$$\Big\|\varepsilon\,\theta\big(\frac{x}{\varepsilon},\omega\big)\Big\|_{L^2(Q)}\to 0 \quad \text{a.s. } \omega\in\Omega.$$

Here and in the sequel for presentation simplicity we write for the L^2 norm of a vector-function just $L^2(Q)$ instead of $L^2(Q\,;\,\mathbb{R}^d)$.

Theorem 4.1. *There exists a unique (up to an additive constant vector) solution $\theta\in L^2_{\rm loc}(\mathbb{R}^d)$ of equation (22) that satisfies conditions* **A**) *–* **B**).

Proof of Theorem 4.1. We divide the proof into several steps.

Step 1. Consider the following operator acting in $L^2(\Omega)$:

$$(A\varphi)(\omega)=\int\limits_{\mathbb{R}^d}a(z)\boldsymbol{\mu}(T_z\omega)\big(\varphi(T_z\omega)-\varphi(\omega)\big)dz. \tag{23}$$

Proposition 4.1. *The spectrum $\sigma(A)\subset(-\infty,0]$.*

Proof. It is straightforward to check that the operator A is bounded and symmetric in the weighted space $L^2(\Omega,\mathbf{P}_\mu)=L^2_\mu(\Omega)$ with $d\mathbf{P}_\mu(\omega)=\boldsymbol{\mu}(\omega)d\mathbf{P}(\omega)$. Denoting $\tilde\omega=T_z\omega,\ s=-z$, using stationarity of μ and considering the relation $a(-z)=a(z)$ we get

$$\begin{aligned}&\int\limits_{\Omega}\int\limits_{\mathbb{R}^d}a(z)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\varphi^2(T_z\omega)\,dz\,d\mathbf{P}(\omega)=\\ &\qquad\int\limits_{\Omega}\int\limits_{\mathbb{R}^d}a(z)\boldsymbol{\mu}(\tilde\omega)\boldsymbol{\mu}(T_{-z}\tilde\omega)\varphi^2(\tilde\omega)\,dz\,d\mathbf{P}(\tilde\omega)\\ &\quad=\int\limits_{\Omega}\int\limits_{\mathbb{R}^d}a(s)\boldsymbol{\mu}(\omega)\boldsymbol{\mu}(T_s\omega)\varphi^2(\omega)\,ds\,d\mathbf{P}(\omega).\end{aligned} \tag{24}$$

Thus

$$\begin{aligned}\big(A\varphi,\varphi\big)_{L^2_\mu}&=\int\limits_{\Omega}\int\limits_{\mathbb{R}^d}a(z)\boldsymbol{\mu}(T_z\omega)\big(\varphi(T_z\omega)-\varphi(\omega)\big)\varphi(\omega)\boldsymbol{\mu}(\omega)dzd\mathbf{P}(\omega)\\ &=-\frac12\int\limits_{\Omega}\int\limits_{\mathbb{R}^d}a(z)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\big(\varphi(T_z\omega)-\varphi(\omega)\big)^2dzd\mathbf{P}(\omega)<0.\end{aligned} \tag{25}$$

Since the norms in $L^2(\Omega)$ and $L^2_\mu(\Omega)$ are equivalent, the desired statement follows. □

Let us consider for any $\delta>0$ the equation

$$\delta\varphi(\omega)-\int\limits_{\mathbb{R}^d}a(z)\boldsymbol{\mu}(T_z\omega)(\varphi(T_z\omega)-\varphi(\omega))\,dz=r(\omega),\quad r(\omega)=\int\limits_{\mathbb{R}^d}za(z)\boldsymbol{\mu}(T_z\omega)\,dz. \tag{26}$$

By Proposition 4.1 the operator $(\delta I-A)^{-1}$ is bounded, then there exists a unique solution $\varkappa^\delta(\omega)=-(\delta I-A)^{-1}r(\omega)$ of (26). For any given $z\in R^d$ we set

$$u^\delta(z,\omega)=\varkappa^\delta(T_z\omega)-\varkappa^\delta(\omega).$$

Then

$$u^\delta(z_1+z_2,\omega) = u^\delta(z_2,\omega) + u^\delta(z_1, T_{z_2}\omega) \quad \forall\, z_1, z_2 \in \mathbb{R}^d. \tag{27}$$

For any $\xi \in \mathbb{R}^d$ as an immediate consequence of (26) we have

$$\begin{aligned} &\delta \varkappa^\delta(T_\xi\omega) - \int\limits_{\mathbb{R}^d} a(z)\boldsymbol{\mu}(T_{\xi+z}\omega)(\varkappa^\delta(T_{\xi+z}\omega) - \varkappa^\delta(T_\xi\omega))\, dz \\ &\qquad = \int\limits_{\mathbb{R}^d} z a(z)\boldsymbol{\mu}(T_{\xi+z}\omega)\, dz. \end{aligned} \tag{28}$$

Next we obtain a priori estimates for $\|\varkappa^\delta(T_z\omega) - \varkappa^\delta(\omega)\|_{L^2_M}$ with $dM(z,\omega) = a(z)dzd\mathbf{P}(\omega)$.

Proposition 4.2. *The following estimate holds:*

$$\|u^\delta(z,\omega)\|_{L^2_M} = \|\varkappa^\delta(T_z\omega) - \varkappa^\delta(\omega)\|_{L^2_M} \ \le\ C \tag{29}$$

with a constant C that does not depend on δ.

Proof. Multiplying equation (26) by $\varphi(\omega) = \boldsymbol{\mu}(\omega)\varkappa^\delta(\omega)$ and integrating the resulting relation over Ω yields

$$\begin{aligned} &\delta \int\limits_\Omega \big(\varkappa^\delta(\omega)\big)^2 \boldsymbol{\mu}(\omega)\, d\mathbf{P}(\omega) \\ &\quad - \int\limits_{\mathbb{R}^d}\int\limits_\Omega a(z)\boldsymbol{\mu}(T_z\omega)\big(\varkappa^\delta(T_z\omega) - \varkappa^\delta(\omega)\big)\varkappa^\delta(\omega)\boldsymbol{\mu}(\omega)\, dz\, d\mathbf{P}(\omega) \\ &\quad = \int\limits_{\mathbb{R}^d}\int\limits_\Omega z a(z)\varkappa^\delta(\omega)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\, dz\, d\mathbf{P}(\omega). \end{aligned} \tag{30}$$

The same change of variables as in (24) results in the relation

$$\begin{aligned} &\int\limits_{\mathbb{R}^d}\int\limits_\Omega z a(z)\varkappa^\delta(\omega)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\, dz\, d\mathbf{P}(\omega) \\ &= -\int\limits_{\mathbb{R}^d}\int\limits_\Omega z a(z)\varkappa^\delta(T_z\omega)\boldsymbol{\mu}(\omega)\boldsymbol{\mu}(T_z\omega)\, dz\, d\mathbf{P}(\omega), \end{aligned} \tag{31}$$

therefore, the right-hand side of (30) takes the form

$$\begin{aligned} &\int\limits_{\mathbb{R}^d}\int\limits_\Omega z a(z)\varkappa^\delta(\omega)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega) dz d\mathbf{P}(\omega) \\ &= -\frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_\Omega z a(z)\big(\varkappa^\delta(T_z\omega) - \varkappa^\delta(\omega)\big)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega) dz d\mathbf{P}(\omega). \end{aligned} \tag{32}$$

Equality (25) implies that the second term on the left-hand side of (30) can be rearranged in the following way

$$
\begin{aligned}
&-\int_{\mathbb{R}^d}\int_{\Omega} a(z)\boldsymbol{\mu}(T_z\omega)\big(\varkappa^\delta(T_z\omega)-\varkappa^\delta(\omega)\big)\varkappa^\delta(\omega)\boldsymbol{\mu}(\omega)\,dz\,d\mathbf{P}(\omega)\\
&=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\Omega} a(z)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\big(\varkappa^\delta(T_z\omega)-\varkappa^\delta(\omega)\big)^2 dz\,d\mathbf{P}(\omega).
\end{aligned}
\tag{33}
$$

Let us denote

$$
\begin{aligned}
J^\delta &= \int_{\mathbb{R}^d}\int_{\Omega} \boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\big(\varkappa^\delta(T_z\omega)-\varkappa^\delta(\omega)\big)^2 a(z)dz\,d\mathbf{P}(\omega)\\
&= \int_{\mathbb{R}^d}\int_{\Omega} \boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)(u^\delta(z,\omega))^2 dM(z,\omega)
\end{aligned}
$$

and

$$
\int_{\mathbb{R}^d}\int_{\Omega} \big(\varkappa^\delta(T_z\omega)-\varkappa^\delta(\omega)\big)^2 a(z)dz\,d\mathbf{P}(\omega) = \int_{\mathbb{R}^d}\int_{\Omega}(u^\delta(z,\omega))^2 dM(z,\omega) = \|u^\delta\|^2_{L^2_M},
$$

where $dM(z,\omega)=a(z)dzd\mathbf{P}(\omega)$. Then

$$
J^\delta = \int_{\mathbb{R}^d}\int_{\Omega} \boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)(u^\delta(z,\omega))^2 dM(z,\omega) \ge \alpha_1^2\|u^\delta\|^2_{L^2_M} \tag{34}
$$

and on the other hand, relations (30) - (33) imply the following upper bound on J^δ:

$$
J^\delta = \int_{\mathbb{R}^d}\int_{\Omega} \boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)(u^\delta(z,\omega))^2 dM(z,\omega) \le \frac{1}{2}\alpha_2^2\sigma\|u^\delta\|_{L^2_M}. \tag{35}
$$

Bounds (34) - (35) together yield

$$
\alpha_1^2\|u^\delta\|^2_{L^2_M} \le J^\delta \le \frac{1}{2}\alpha_2^2\sigma\|u^\delta\|_{L^2_M}.
$$

Consequently we obtain the estimate (29) with $C=\frac{\alpha_2^2}{2\alpha_1^2}\sigma$, and this estimate is uniform in δ. □

Corollary 4.1. *For any $\delta>0$ the following upper bound holds:*

$$
\sqrt{\delta}\,\|\varkappa^\delta\|_{L^2_\mu} \le C. \tag{36}
$$

Proof. From (30) we have

$$
\begin{aligned}
&\delta\int_{\Omega}\big(\varkappa^\delta(\omega)\big)^2\boldsymbol{\mu}(\omega)\,d\mathbf{P}(\omega)\\
&\quad=\int_{\mathbb{R}^d}\int_{\Omega} a(z)\boldsymbol{\mu}(T_z\omega)\big(\varkappa^\delta(T_z\omega)-\varkappa^\delta(\omega)\big)\varkappa^\delta(\omega)\boldsymbol{\mu}(\omega)\,dz\,d\mathbf{P}(\omega)\\
&\quad+\int_{\mathbb{R}^d}\int_{\Omega} z a(z)\varkappa^\delta(\omega)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\,dz\,d\mathbf{P}(\omega).
\end{aligned}
\tag{37}
$$

Then using (32), (33), (35) together with the Cauchy-Swartz inequality and bound (29), we obtain that the expression on the right-hand side of (37) is uniformly bounded in δ. $\square$

Proposition 4.2 implies that the family $\{u^\delta(z,\omega)\}_{\delta>0}$ is bounded in L^2_M. Consequently there exists a subsequence $u_j(z,\omega) = u^{\delta_j}(z,\omega)$, $j = 1, 2, \dots$, that converges in a weak topology of L^2_M as $\delta_j \to 0$. We denote this limit by $\theta(z,\omega)$:

$$w\text{-}\lim_{j\to\infty} u_j(z,\omega) = w\text{-}\lim_{\delta_j\to 0} \left(\varkappa^{\delta_j}(T_z\omega) - \varkappa^{\delta_j}(\omega)\right) = \theta(z,\omega). \tag{38}$$

Clearly, $\theta(z,\omega) \in L^2_M$, i.e.

$$\int_{\mathbb{R}^d}\int_{\Omega} \theta^2(z,\omega) a(z) dz d\mathbf{P}(\omega) < \infty, \tag{39}$$

and by the Fubini theorem $\theta(z,\omega) \in L^2(\Omega)$ for almost all z from the support of the function $a(z)$. In addition $\theta(0,\omega) \equiv 0$ and for any z

$$\mathbf{E}\theta(z,\omega) = \lim_{\delta_j\to 0} \Big(\mathbf{E}\varkappa^{\delta_j}(T_z\omega) - \mathbf{E}\varkappa^{\delta_j}(\omega)\Big) = 0. \tag{40}$$

Step 2. Property A. The function $\theta(z,\omega)$ introduced in (38) is not originally defined on the set $\{z \in \mathbb{R}^d : a(z) = 0\}$.

Proposition 4.3. *The function $\theta(z,\omega)$, given by (38), can be extended to $\mathbb{R}^d \times \Omega$ in such a way that $\theta(z,\omega)$ satisfies relation (27), i.e. $\theta(z,\omega)$ has stationary increments:*

$$\theta(z+\xi,\omega) - \theta(\xi,\omega) = \theta(z, T_\xi\omega) = \theta(z,T_\xi\omega) - \theta(0,T_\xi\omega). \tag{41}$$

Proof. Applying Mazur's theorem [21, Section V.1] we conclude that $\theta(z,\omega) = s\text{-}\lim\limits_{n\to\infty} w_n$ is the strong limit of a sequence w_n of convex combinations of elements $u_j(z,\omega) = u^{\delta_j}(z,\omega)$. The strong convergence implies that there exists a subsequence of $\{w_n\}$ that converges a.s. to the same limit $\theta(z,\omega)$:

$$\lim_{n_k\to\infty} w_{n_k}(z,\omega) = \theta(z,\omega) \quad \text{for a.e. } z \text{ and a.e. } \omega.$$

Since equality (27) holds for all u_j, it also holds for any convex linear combination w_n of u_j:

$$w_n(z_1+z_2,\omega) = w_n(z_2,\omega) + w_n(z_1, T_{z_2}\omega) \quad \forall\ n. \tag{42}$$

Thus taking the subsequence $\{w_{n_k}\}$ in equality (42) and passing to the point-wise limit $n_k \to \infty$ in any term of this equality we obtain (41) first only for such z_1, z_2 that z_1, z_2, z_1+z_2 belong to supp(a). Then we extend function $\theta(z,\omega)$ to a.e. $z \in \mathbb{R}^d$ using relation (41):

$$\theta(z_1+z_2,\omega) = \theta(z_2,\omega) + \theta(z_1, T_{z_2}\omega). \tag{43}$$

Observe that this extension is well-defined because relation (41) holds on the support of a.

Let us show that $\theta(z,\omega)$ is defined for all $z \in \mathbb{Z}^d$. To this end we observe that, due to the properties of the dynamical system T_z, the function $\theta(z_1, T_{z_2}\omega)$ is well-defined measurable function of z_1 and ω for all $z_2 \in \mathbb{R}^d$. The function $\theta(z_1+z_2,\omega)$ possesses the same property due to its particular structure. Then according to (43) the function $\theta(z_2,\omega)$ is defined for all $z_2 \in \mathbb{R}^d$. $\square$

Denote $\zeta_z(\xi,\omega) = \theta(z+\xi,\omega) - \theta(\xi,\omega)$, then for $z \in \mathbb{R}^d$ relation (41) yields

$$\zeta_z(\xi,\omega) = \zeta_z(0, T_\xi\omega), \tag{44}$$

i.e. for all $z \in \mathbb{R}^d$ the field $\zeta_z(\xi,\omega)$ is statistically homogeneous in ξ, and

$$\zeta_z(0,\omega) = \theta(z,\omega). \tag{45}$$

Thus by (38), (41) – (44) the random function $\theta(z,\omega)$ is not stationary, but its increments $\zeta_z(\xi,\omega) = \theta(z+\xi,\omega) - \theta(\xi,\omega)$ form a stationary field for any given z.

Step 3. At this step we show that θ satisfies equation (22).

Let us prove now that $\theta(z,\omega)$ defined by (38) is a solution of equation (20) (or (22)). To this end for an arbitrary function $\psi(\omega) \in L^2(\Omega)$ we multiply equality (28) by a function $\psi(\omega)\boldsymbol{\mu}(\omega)$ and integrate the resulting relation over Ω, then we have

$$\begin{aligned} &\delta \int\limits_{\Omega} \varkappa^\delta(T_\xi\omega)\psi(\omega)\boldsymbol{\mu}(\omega)\, d\mathbf{P}(\omega) \\ &\quad = \int\limits_{\mathbb{R}^d}\int\limits_{\Omega} a(z)\boldsymbol{\mu}(T_{\xi+z}\omega)\big(\varkappa^\delta(T_{\xi+z}\omega) - \varkappa^\delta(T_\xi\omega)\big)dz\psi(\omega)\boldsymbol{\mu}(\omega)d\mathbf{P}(\omega) \\ &\quad + \int\limits_{\mathbb{R}^d}\int\limits_{\Omega} za(z)\boldsymbol{\mu}(T_{\xi+z}\omega)dz\,\psi(\omega)\boldsymbol{\mu}(\omega)\, d\mathbf{P}(\omega). \end{aligned} \tag{46}$$

By estimate (36) and the Cauchy-Swartz inequality for any $\psi \in L^2(\Omega)$ we get

$$\delta \int\limits_{\Omega} \varkappa^\delta(T_\xi\omega)\psi(\omega)\boldsymbol{\mu}(\omega)\, d\mathbf{P}(\omega) \to 0 \quad \text{as } \delta \to 0. \tag{47}$$

Passing to the limit $\delta \to 0$ in equation (46) and taking into account (38) and (47), we obtain that for a.e. ω the function $\theta(z, T_\xi\omega)$ satisfies the equation

$$\int\limits_{\mathbb{R}^d} a(z)\boldsymbol{\mu}(T_{\xi+z}\omega)\theta(z,T_\xi\omega))\; dz = -\int\limits_{\mathbb{R}^d} za(z)\boldsymbol{\mu}(T_{\xi+z}\omega)\; dz.$$

Using (41) we get after the change of variables $z \to -z$

$$-\int\limits_{\mathbb{R}^d} a(z)\boldsymbol{\mu}(T_{\xi-z}\omega)(\theta(\xi-z,\omega) - \theta(\xi,\omega))\; dz + \int\limits_{\mathbb{R}^d} za(z)\boldsymbol{\mu}(T_{\xi-z}\omega)\; dz = 0, \tag{48}$$

and it is the same as (20). Thus we have proved that $\theta(z,\omega)$ is a solution of (22).

Step 4. Property B.

Assumption (5) and inequality (39) imply that

$$c_0 \int\limits_{\mathbf{B}}\int\limits_{\Omega} \theta^2(z,\omega)dzd\mathbf{P}(\omega) < \int\limits_{\mathbb{R}^d}\int\limits_{\Omega} \theta^2(z,\omega)a(z)dzd\mathbf{P}(\omega) < \infty,$$

and by the Fubini theorem we conclude that a.s.

$$\int_{\mathbf{B}} \theta^2(z,\omega)dz < \infty. \tag{49}$$

Thus $\theta(z,\omega) \in L^2(\mathbf{B})$ with $\|\theta(z,\omega)\|_{L^2(\mathbf{B})} = K(\omega)$ for a.e. ω, and $\mathbf{E}(K(\omega))^2 < \infty$.

Proposition 4.4 *(Sublinear growing of $\varepsilon\theta(\frac{x}{\varepsilon})$ in $L^2_{\rm loc}(\mathbb{R}^d)$). Denote by $\varphi_\varepsilon(z,\omega) = \varepsilon\,\theta\big(\frac{z}{\varepsilon},\omega\big)$. Then a.s.*

$$\|\varphi_\varepsilon(\cdot,\omega)\|_{L^2(Q)} \ \to\ 0 \quad \text{as } \ \varepsilon \to 0 \tag{50}$$

for any bounded Lipschitz domain $Q \subset \mathbb{R}^d$.

Proof. The proof relies on inequality (49). In what follows we assume without loss of the generality that $\mathbf{B} = [0,1]^d$.

Lemma 4.1. *The family of functions $\varphi_\varepsilon(z,\omega) = \varepsilon\,\theta\big(\frac{z}{\varepsilon},\omega\big)$ is bounded and compact in $L^2(Q)$.*

Proof. Introducing new variables $\frac{z}{\varepsilon} = y$ we have

$$\|\varphi_\varepsilon\|^2_{L^2(Q)} = \|\varepsilon\,\theta\big(\frac{z}{\varepsilon},\omega\big)\|^2_{L^2(Q)} = \int_Q \varepsilon^2\,\theta^2\big(\frac{z}{\varepsilon},\omega\big)dz = \int_{\varepsilon^{-1}Q} \varepsilon^{d+2}\,\theta^2(y,\omega)dy$$

$$= \varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{B_j} \theta^2(y,\omega)dy = \varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{B_j} (\theta(y,\omega) - \theta(j,\omega) + \theta(j,\omega))^2 dy$$

$$\le 2\varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{B_j} (\theta(y,\omega) - \theta(j,\omega))^2 dy \ + \ 2\varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \theta^2(j,\omega)\,|B_j|. \tag{51}$$

Here $j \in \mathbb{Z}^d \cap \frac{1}{\varepsilon}Q = \mathbb{Z}_{Q/\varepsilon}$, $B_j = j + [0,1)^d$. Then if $y \in B_j$, then $y = j + z,\ z \in \mathbf{B} = [0,1)^d$, and we can rewrite the first term on the right-hand side of (51) as follows

$$2\,\varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{\mathbf{B}} (\theta(j+z,\omega) - \theta(j,\omega))^2 dz = 2\,\varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{\mathbf{B}} \theta^2(z,T_j\omega)dz.$$

Using the fact that $\theta_B(j,\omega) := \int_{\mathbf{B}} \theta^2(z,T_j\omega)dz$ is a stationary field and $\theta(z,\omega) \in L^2(\mathbf{B})$, by the Birkhoff ergodic theorem we obtain that there exists a random variable $\mathcal{J}(\omega)$ such that

$$2\,\varepsilon^d \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{\mathbf{B}} \theta^2(z,T_j\omega)dz \ \to\ 2|Q|\mathcal{J}, \qquad \mathbf{E}\mathcal{J}(\omega) = \mathbf{E}\int_{\mathbf{B}} \theta^2(z,\omega)dz < \infty.$$

Consequently, the first term on the right-hand side in (51) is vanishing as $\varepsilon \to 0$:

$$2\varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \int_{\mathbf{B}} \theta^2(z,T_j\omega)dz \ \to\ 0. \tag{52}$$

Let us prove now that a.s. the second term on the right-hand side of (51) is bounded. Denoting

$$\widehat{\varphi}_\varepsilon(z) = \varepsilon\,\widehat{\theta}\big(\frac{z}{\varepsilon},\omega\big),$$

where $\widehat{\theta}$ is a piecewise constant function: $\widehat{\theta}\big(\frac{z}{\varepsilon},\omega\big) = \theta\big([\frac{z}{\varepsilon}],\omega\big) = \theta(j,\omega)$ as $z \in \varepsilon B_j$, we rewrite this term as follows:

$$2\,\varepsilon^{d+2} \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \theta^2(j,\omega) = 2\,\|\varepsilon\,\widehat{\theta}\big(\frac{z}{\varepsilon},\omega\big)\|^2_{L^2(Q)} = 2\|\widehat{\varphi}_\varepsilon(z)\|^2_{L^2(Q)}. \tag{53}$$

Let us estimate the difference gradient of $\widehat{\varphi}_\varepsilon$:

$$\|\mathrm{grad}\,\widehat{\varphi}_\varepsilon\|^2_{(L^2(Q))^d} = \varepsilon^2 \int\limits_Q \sum_{k=1}^d \frac{\big(\theta\big([\frac{1}{\varepsilon}(z+\varepsilon e_k)],\omega\big) - \theta\big([\frac{z}{\varepsilon}],\omega\big)\big)^2}{\varepsilon^2}\,dz$$

$$= \int\limits_Q \sum_{k=1}^d \big(\theta\big([\frac{z}{\varepsilon}] + e_k,\omega\big) - \theta\big([\frac{z}{\varepsilon}],\omega\big)\big)^2\,dz = \varepsilon^d \sum_{k=1}^d \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \big(\theta(j+e_k,\omega) - \theta(j,\omega)\big)^2.$$

Since $\theta(j+e_k,\omega) - \theta(j,\omega) = \theta(e_k, T_j\omega)$ is stationary for each e_k, then by the Birkhoff ergodic theorem

$$\|\mathrm{grad}\,\widehat{\varphi}_\varepsilon\|^2_{(L^2(Q))^d} = \varepsilon^d \sum_{k=1}^d \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \big(\theta(j+e_k,\omega) - \theta(j,\omega)\big)^2 \;\to\; |Q| \sum_{k=1}^d C_k(\omega), \tag{54}$$

where $\mathbf{E}C_k = \mathbf{E}\theta^2(e_k,\omega)$.

Next we prove that a.s. the following estimate holds:

$$\bar{\theta}_\varepsilon(\omega) = \int\limits_Q \widehat{\varphi}_\varepsilon(z,\omega)dz = \varepsilon^d \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \varepsilon\,\theta(j,\omega) \le \widetilde{C}(\omega). \tag{55}$$

We apply the induction and start with $d = 1$. Using stationarity of $\theta(j+1,\omega) - \theta(j,\omega)$ we have by the ergodic theorem

$$\varepsilon^2 \Big| \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \theta(j,\omega)\Big| \le \varepsilon^2 \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \sum_{k=0}^{j-1} |\theta(k+1,\omega) - \theta(k,\omega)|$$

$$\le \varepsilon^2 \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \sum_{k\in\mathbb{Z}_{Q/\varepsilon}} |\theta(k+1,\omega) - \theta(k,\omega)|$$

$$= \varepsilon^2 \frac{|Q|}{\varepsilon} \sum_{k\in\mathbb{Z}_{Q/\varepsilon}} |\theta(e_1, T_k\omega)| \;\to\; |Q|^2 \mathbf{E}|\theta(e_1,\omega)| = \bar{C}_1.$$

Thus

$$\varlimsup_{\varepsilon\to 0} \varepsilon^2 \Big| \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \theta(j,\omega)\Big| \le \bar{C}_1,$$

and this implies that for a.e. ω

$$\sup_\varepsilon \Big|\varepsilon^2 \sum_{j\in\mathbb{Z}_{Q/\varepsilon}} \theta(j,\omega)\Big| \le \widetilde{C}_1(\omega), \tag{56}$$

where the constant $\widetilde{C}_1(\omega)$ depends only on ω.

Let us show how to derive the required upper bound in the dimension $d = 2$ using (56). In this case $j \in \mathbb{Z}_{Q/\varepsilon}$, $j = (j_1, j_2)$, and we assume without loss of generality that $Q \subset [-q, q]^2$. Then

$$\theta((j_1,j_2),\omega) = \sum_{k=0}^{j_2-1} \big(\theta((j_1,k+1),\omega) - \theta((j_1,k),\omega)\big) \;+\; \theta((j_1,0),\omega),$$

and for any $j = (j_1, j_2) \in \mathbb{Z}_{Q/\varepsilon}$ we get

$$|\theta((j_1,j_2),\omega)| \le \sum_{k=-q/\varepsilon}^{q/\varepsilon} \big|\theta((j_1,k+1),\omega) - \theta((j_1,k),\omega)\big| \;+\; |\theta((j_1,0),\omega)|.$$

Using (56) and the ergodic property of the field $|\theta(e_2, T_j\omega)|$ we obtain the following upper bound

$$\varepsilon^3 \Big| \sum_{(j_1,j_2)\in\mathbb{Z}_{Q/\varepsilon}} \theta((j_1,j_2),\omega)\Big|$$
$$\le \varepsilon^3 \sum_{j_1=-q/\varepsilon}^{q/\varepsilon} \frac{2q}{\varepsilon} \sum_{k=-q/\varepsilon}^{q/\varepsilon} |\theta(e_2, T_{(j_1,k)}\omega)| \;+\; \varepsilon^3 \sum_{j_1=-q/\varepsilon}^{q/\varepsilon} \frac{2q}{\varepsilon} |\theta((j_1,0),\omega)|$$
$$= 2q\varepsilon^2 \sum_{(j_1,k)\in\mathbb{Z}_{Q/\varepsilon}} |\theta(e_2, T_{(j_1,k)}\omega)| + 2q\varepsilon^2 \sum_{j_1=-q/\varepsilon}^{q/\varepsilon} |\theta((j_1,0),\omega)| \le \widetilde{C}_2(\omega) + 2q\widetilde{C}_1(\omega),$$

where $2q$ is the 1-d volume of slices of Q that are orthogonal to e_1. The case of $d > 2$ is considered in the same way.

Applying the standard discrete Poincaré inequality or the Poincaré inequality for piece-wise linear approximations of discrete functions we obtain from (54) - (55) that a.s.

$$\|\widehat{\varphi}_\varepsilon\|^2_{L^2(Q)} \le g_1\Big(\int_Q \widehat{\varphi}_\varepsilon(z,\omega)dz\Big)^2 + g_2\|\mathrm{grad}\,\widehat{\varphi}_\varepsilon\|^2_{(L^2(Q))^d} \le K(\omega), \tag{57}$$

where the constants g_1, g_2, and $K(\omega)$ do not depend on n.

Thus using the same piece-wise linear approximations and considering the compactness of embedding of $H^1(Q)$ to $L^2(Q)$ we derive from (54) and (57) that the set of functions $\{\widehat{\varphi}_\varepsilon\}$ is compact in $L^2(Q)$. As follows from (51) – (52)

$$\varphi_\varepsilon = \widehat{\varphi}_\varepsilon + \breve{\varphi}_\varepsilon, \quad \text{where} \;\; \breve{\varphi}_\varepsilon(x) = \varepsilon\big(\theta(\frac{x}{\varepsilon}) - \widehat{\theta}(\frac{x}{\varepsilon})\big), \quad \|\breve{\varphi}_\varepsilon\|_{L^2(Q)} \to 0 \;(\varepsilon \to 0).$$

This together with compactness of $\{\widehat{\varphi}_\varepsilon\}$ implies the compactness of the family $\{\varphi_\varepsilon\}$. Lemma is proved. □

Next we show that any limit point of the family $\{\varphi_\varepsilon\}$ as $\varepsilon \to 0$ is a constant function.

Lemma 4.2. *Let $\{\varphi_\varepsilon\}$ converge for a subsequence to φ_0 in $L^2(Q)$. Then $\varphi_0 = const$.*

Proof. According to [15] the set $\{\mathrm{div}\phi : \phi \in (C_0^\infty(Q))^d\}$ is dense in the subspace of functions from $L^2(Q)$ with zero average. It suffices to show that

$$\int_Q \mathrm{div}\phi(x)\varphi_\varepsilon(x)\,dx \longrightarrow 0, \quad \text{as } \varepsilon \to 0, \tag{58}$$

for any $\phi = (\phi^1, \phi^2, \dots, \phi^d) \in (C_0^\infty(Q))^d$. Clearly,

$$\frac{1}{\varepsilon}(\phi^j(x+\varepsilon e_j) - \phi^j(x)) = \partial_{x_j}\phi^j(x) + \varepsilon v_\varepsilon,$$

where $\|v_\varepsilon\|_{L^\infty(Q)} \le C$. Then, for sufficiently small ε, we have

$$\int\limits_Q \mathrm{div}\phi(x)\varphi_\varepsilon(x)\,dx = \int\limits_Q (\phi^j(x+\varepsilon e_j) - \phi^j(x))\theta\big(\frac{x}{\varepsilon},\omega\big)\,dx \,+\, o(1)$$

$$= \int\limits_Q \phi^j(x)\big(\theta\big(\frac{x}{\varepsilon} - e_j,\omega\big) - \theta\big(\frac{x}{\varepsilon},\omega\big)\big)\,dx \,+\, o(1),$$

where $o(1)$ tends to zero as $\varepsilon \to 0$ by Lemma 4.1. Since $\theta(z - e_j, \omega) - (\theta(z,\omega)$ is a stationary function, by the Birkhoff ergodic theorem the integral on the right-hand side converges to zero a.s. as $\varepsilon \to 0$, and the desired statement follows. □

Our next goal is to show that almost surely the limit relation in (50) holds. By Lemma 4.1 the constants $\varepsilon c^\varepsilon$ with c^ε defined in (16) are a.s. uniformly in ε bounded, that is

$$|\varepsilon c^\varepsilon| \le K(\omega) \tag{59}$$

for all sufficiently small $\varepsilon > 0$.

Consider a convergent subsequence $\{\varphi_{\varepsilon_n}\}_{n=1}^\infty$. By Lemma 4.2 the limit function is a constant, denote this constant by φ_0. Assume that $\varphi_0 \ne 0$. Then

$$\varphi_{\varepsilon_n}(z) = \varphi_0 + \rho_{\varepsilon_n}(z),$$

where $\|\rho_{\varepsilon_n}\|_{L^2(Q)} \to 0$ as $\varepsilon_n \to 0$. Clearly, we have

$$\varphi_{2\varepsilon_n}(z) = 2\varepsilon_n\theta\Big(\frac{z}{2\varepsilon_n}\Big) = 2\varepsilon_n\theta\Big(\frac{z/2}{\varepsilon_n}\Big) = 2\varphi_0 + 2\rho_{\varepsilon_n}\Big(\frac{z}{2}\Big) \to 2\varphi_0,$$

because $\|\rho_{\varepsilon_n}(\cdot/2)\|_{L^2(Q)} \to 0$ as $\varepsilon_n \to 0$. Similarly, for any $N \in \mathbb{Z}^+$ we have

$$\varphi_{N\varepsilon_n}(z) \;\to\; N\varphi_0 \qquad \text{in } L^2(Q).$$

Choosing N in such a way that $N|\varphi_0| > K(\omega)$ we arrive at a contradiction with (59). Therefore, $\varphi_0 = 0$ for any convergent subsequence. This yields the desired convergence in (50) and completes the proof of Proposition 4.4. □

Step 5. Uniqueness of θ.

Proposition 4.5 *(Uniqueness). Problem* (22) *has a unique up to an additive constant solution* $\theta(z,\omega)$, $\theta \in L^2_M$, *with statistically homogeneous increments such that* (50) *holds true.*

Proof. Consider two arbitrary solutions $\theta_1(z,\omega)$ and $\theta_2(z,\omega)$ of problem (22). Then the difference $\Delta(z,\omega) = \theta_1(z,\omega) - \theta_2(z,\omega)$ satisfies the equation

$$\int_{\mathbb{R}^d} a(z)\mu(\xi+z,\omega)\big(\Delta(\xi+z,\omega)-\Delta(\xi,\omega)\big)\ dz=0 \tag{60}$$

for a.e. ω and for all $\xi\in\mathbb{R}^d$.

Let us remark that the function $\Delta(z,\omega)$ inherits properties **A)** and **B)** of $\theta_1(z,\omega)$ and $\theta_2(z,\omega)$. Consider a cut-off function $\varphi(\frac{|\xi|}{R})$ parameterized by $R>0$, where $\varphi(r)$, $r\in\mathbb{R}$, is a function defined by

$$\varphi(r)=\begin{cases}1, & r\le 1,\\ 2-r, & 1<r<2,\\ 0, & r\ge 2.\end{cases}$$

For any $R>0$, multiplying equation (60) by $\mu(\xi,\omega)\Delta(\xi,\omega)\varphi(\frac{|\xi|}{R})$ and integrating the resulting relation in ξ over $\mathbb{R}^d$, we obtain the following equality

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\big(\Delta(\xi+z,\omega)-\Delta(\xi,\omega)\big)\Delta(\xi,\omega)\varphi(\frac{|\xi|}{R})dzd\xi=0. \tag{61}$$

Using the relation $a(-z)=a(z)$, after change of variables $z\to -z$, $\xi-z=\xi'$, we get

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)\mu(\xi'+z,\omega)\mu(\xi',\omega)\big(\Delta(\xi',\omega)-\Delta(\xi'+z,\omega)\big)\\ \times\Delta(\xi'+z,\omega)\varphi\big(\frac{|\xi'+z|}{R}\big)dzd\xi'=0.\end{aligned} \tag{62}$$

Renaming ξ' back to ξ in the last equation and taking the sum of (61) and (62) we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\big(\Delta(\xi+z,\omega)-\Delta(\xi,\omega)\big)\\ &\qquad\times\Big(\Delta(\xi+z,\omega)\varphi(\frac{|\xi+z|}{R})-\Delta(\xi,\omega)\varphi(\frac{|\xi|}{R})\Big)dz\,d\xi\\ &=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Big(\Delta(\xi+z,\omega)-\Delta(\xi,\omega)\Big)^2\varphi(\frac{|\xi|}{R})\,dz\,d\xi\\ &+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)\ \mu(\xi+z,\omega)\mu(\xi,\omega)\big(\Delta(\xi+z,\omega)-\Delta(\xi,\omega)\big)\\ &\qquad\times\Delta(\xi+z,\omega)\big(\varphi(\frac{|\xi+z|}{R})-\varphi(\frac{|\xi|}{R})\big)dz\,d\xi\\ &=J_1^R\ +\ J_2^R=0.\end{aligned} \tag{63}$$

Letting $R=\varepsilon^{-1}$, we first estimate the contribution of J_2^R.

Lemma 4.3. *The following limit relation holds a.s.:*

$$\frac{1}{R^d}|J_2^R|\ \to\ 0\qquad as\ \ R\to\infty. \tag{64}$$

Proof. Denote $\Delta_z(T_\xi\omega) = \Delta(\xi+z,\omega) - \Delta(\xi,\omega)$, then $\Delta_z(T_\xi\omega)$ is stationary in ξ for any given z.

We consider separately the integration over $|\xi| > 3R$ and $|\xi| \le 3R$ in the integral J_2^R:

$$J_2^R = \int\limits_{\mathbb{R}^d}\int\limits_{|\xi|>3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z(T_\xi\omega)\Delta(\xi+z,\omega)\big(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})\big)dz\,d\xi$$
$$+\int\limits_{\mathbb{R}^d}\int\limits_{|\xi|\le 3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z(T_\xi\omega)\Delta(\xi+z,\omega)\big(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})\big)dz\,d\xi.$$

If $|\xi| > 3R$, then $\varphi(\frac{|\xi|}{R}) = 0$. Also, $\varphi(\frac{|\xi+z|}{R}) = 0$ if $|\xi| > 3R$ and $|z| > R$. Then we obtain the following upper bound

$$\frac{1}{R^d}\int\limits_{\mathbb{R}^d}\int\limits_{|\xi|>3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)|\Delta_z(T_\xi\omega)||\Delta(\xi+z,\omega)|\varphi(\frac{|\xi+z|}{R})d\xi\,dz$$
$$\le \frac{\alpha_2^2}{R^d}\int\limits_{|\eta|\le 2R}\Big(\int\limits_{|z|>R} |z|a(z)|\Delta_z(T_{\eta-z}\omega)|\,dz\Big)\frac{1}{R}|\Delta(\eta,\omega)|\varphi(\frac{|\eta|}{R})d\eta \tag{65}$$
$$\le \frac{\alpha_2^2}{R^d}\int\limits_{|\eta|\le 2R} \phi(T_\eta\omega)\frac{1}{R}|\Delta(\eta,\omega)|\varphi(\frac{|\eta|}{R})\,d\eta,$$

where $\eta = \xi + z$,

$$\phi(T_\eta\omega) = \int\limits_{\mathbb{R}^d} |z|a(z)|\Delta_z(T_{\eta-z}\omega)|\,dz,$$

and in the first inequality we have used the fact that $1 < \frac{|z|}{R}$ if $|z| > R$. Since $\Delta_z(\omega) \in L^2_M$, then $\phi(\omega) \in L^2(\Omega)$. Applying the Cauchy-Swartz inequality to the last integral in (65) and recalling the relation $R = \varepsilon^{-1}$ we have

$$\frac{\alpha_2^2}{R^d}\int\limits_{|\eta|\le 2R} \phi(T_\eta\omega)\frac{|\Delta(\eta,\omega)|}{R}\varphi(\frac{|\eta|}{R})\,d\eta$$
$$\le \alpha_2^2\Big(\frac{1}{R^d}\int\limits_{|\eta|\le 2R}\phi^2(T_\eta\omega)d\eta\Big)^{\frac12}\Big(\frac{1}{R^d}\int\limits_{|\eta|\le 2R}\big(\frac{|\Delta(\eta,\omega)|}{R}\big)^2d\eta\Big)^{\frac12} \to 0, \tag{66}$$

as $R \to \infty$, because the first integral on the right hand side is bounded due to the stationarity of $\phi(T_\eta\omega)$, and the second integral tends to 0 due to sublinear growth of $\Delta(\eta,\omega)$, see (50).

If $|\xi| \le 3R$, then the corresponding part of $R^{-d}J_2^R$ can be rewritten as a sum of two terms

$$\frac{1}{R^d}\int\limits_{\mathbb{R}^d}\int\limits_{|\xi|\le 3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z(T_\xi\omega)(\Delta(\xi+z,\omega)-\Delta(\xi,\omega))$$
$$\times\big(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})\big)d\xi\,dz$$
$$+\frac{1}{R^d}\int\limits_{\mathbb{R}^d}\int\limits_{|\xi|\le 3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z(T_\xi\omega)\Delta(\xi,\omega)\big(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})\big)d\xi\,dz$$
$$= I_1 + I_2.$$

We estimate $|I_1|$ and $|I_2|$ separately. Using the inequality $|\varphi(\frac{|x|}{R}) - \varphi(\frac{|y|}{R})| \le \frac{|x-y|}{R}$ by the same arguments as above we get

$$|I_2| \le \frac{\alpha_2^2}{R^d} \int\limits_{\mathbb{R}^d} \int\limits_{|\xi|\le 3R} a(z)|\Delta_z(T_\xi\omega)||\Delta(\xi,\omega)|\frac{|z|}{R} d\xi\, dz$$
$$\le \alpha_2^2 \Big(\frac{1}{R^d} \int\limits_{|\xi|\le 3R} \phi^2(T_\xi\omega) d\xi\Big)^{\frac12} \Big(\frac{1}{R^d} \int\limits_{|\xi|\le 3R} (\frac{|\Delta(\xi,\omega)|}{R})^2 d\xi\Big)^{\frac12} \to 0.$$

To estimate I_1 we divide the area of integration in z into two parts: $|z| < \sqrt{R}$ and $|z| \ge \sqrt{R}$, and first consider the integral

$$I_1^{(<)} = \frac{1}{R^d} \int\limits_{|z|<\sqrt{R}} \int\limits_{|\xi|\le 3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z^2(T_\xi\omega)\big(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})\big) d\xi\, dz$$

Since $|z| \le \sqrt{R}$, we have $|\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})| \le \frac{1}{\sqrt{R}}$. Therefore,

$$|I_1^{(<)}| \le \alpha_2^2 \frac{1}{\sqrt{R}}\ \frac{1}{R^d} \int\limits_{|\xi|\le 3R} \int\limits_{\mathbb{R}^d} a(z)\Delta_z^2(T_\xi\omega) dz\, d\xi \to 0,$$

as $R \to \infty$; here we have used the fact that

$$\frac{1}{R^d} \int\limits_{|\xi|\le 3R} \int\limits_{\mathbb{R}^d} a(z)\Delta_z^2(T_\xi\omega) dz\, d\xi \to c_0 \mathbf{E}\Big(\int\limits_{\mathbb{R}^d} a(z)\Delta_z^2(\omega) dz\Big)$$

with a constant c_0 equal to the volume of a ball of radius 3 in $\mathbb{R}^d$. We turn to the second integral

$$I_1^{(>)} = \frac{1}{R^d} \int\limits_{|z|\ge\sqrt{R}} \int\limits_{|\xi|\le 3R} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z^2(T_\xi\omega)\big(\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})\big) d\xi\, dz.$$

Considering the inequality $|\varphi(\frac{|\xi+z|}{R}) - \varphi(\frac{|\xi|}{R})| \le 1$ we obtain

$$|I_1^{(>)}| \le \alpha_2^2 \frac{1}{R^d} \int\limits_{|\xi|\le 3R} \int\limits_{|z|\ge\sqrt{R}} a(z)\Delta_z^2(T_\xi\omega)\, dz\, d\xi. \tag{67}$$

Denote by $\psi_R(\omega)$ the stationary function defined by

$$\psi_R(\omega) = \int\limits_{|z|\ge\sqrt{R}} a(z)\Delta_z^2(\omega)\, dz.$$

Since $\Delta_z(\omega) \in L^2_M$, then

$$\mathbf{E}\psi_R(\omega) \to 0 \quad \text{as } R \to \infty. \tag{68}$$

Moreover, function $\psi_R(\omega)$ is a.s. decreasing in R. Using the ergodic theorem, (67) and (68), we conclude that $|I_1^{(>)}|$ tends to zero as $R \to \infty$. Thus we have proved that $|I_1| + |I_2| \to 0$ as $R \to \infty$ a.s. Together with (66) this implies (64). □

We proceed with the term J_1^R in (63):

$$J_1^R = \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z^2(\xi,\omega)\varphi(\frac{|\xi|}{R})\,dz\,d\xi.$$

Using the ergodic theorem we get as $R\to\infty$

$$\begin{aligned}\frac{1}{R^d}J_1^R &= \frac{1}{R^d}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\xi+z,\omega)\mu(\xi,\omega)\Delta_z^2(\xi,\omega)\varphi(\frac{|\xi|}{R})\,dz\,d\xi\\ &\longrightarrow \quad c_1\mathbf{E}\int\limits_{\mathbb{R}^d} a(z)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\Delta_z^2(\omega)dz,\end{aligned}\tag{69}$$

where $c_1=\int_{\mathbb{R}^d}\varphi(|\xi|)d\xi>0$. Consequently from (63) - (64) it follows that

$$\frac{1}{R^d}|J_1^R| \ \to\ 0 \qquad \text{as}\ \ R\to\infty,\tag{70}$$

and together with (69) this implies that

$$\mathbf{E}\int\limits_{\mathbb{R}^d} a(z)\boldsymbol{\mu}(T_z\omega)\boldsymbol{\mu}(\omega)\Delta_z^2(\omega)dz=0.\tag{71}$$

Using condition (5) we conclude from (71) that $\Delta_z(\omega)\equiv 0$ for a.e. z and a.e. ω, and hence $\theta_1(z,\omega)=\theta_2(z,\omega)$. Proposition is proved. □

This completes the proof of Theorem 4.1. □

5. Additional terms of the asymptotic expansion

Recall that I_0^ε stands for the sum of all terms of order ε^0 in (18) and that $u_0\in C_0^\infty(\mathbb{R}^d)$. Our first goal is to determine the coefficients of the effective elliptic operator $\hat L$. To this end we consider the following scalar product of I_0^ε with a function $\varphi\in L^2(\mathbb{R}^d)$:

$$(I_0^\varepsilon,\varphi)=\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\Big(\frac12 z\otimes z - z\otimes\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\Big)\ a(z)\mu\big(\frac{x}{\varepsilon},\omega\big)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\ dz\ \nabla\nabla u_0(x)\varphi(x)dx.\tag{72}$$

After change of variables $x=\varepsilon\eta$ we have

$$\begin{aligned}(I_0^\varepsilon,\varphi)&=\varepsilon^d\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}\frac12 a(z)\,z\otimes z\,\mu(\eta,\omega)\mu(\eta-z,\omega)\,dz\,\nabla\nabla u_0(\varepsilon\eta)\,\varphi(\varepsilon\eta)\,d\eta\\ &-\varepsilon^d\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\,z\otimes\theta(\eta-z,\omega)\mu(\eta,\omega)\mu(\eta-z,\omega)\,dz\,\nabla\nabla u_0(\varepsilon\eta)\,\varphi(\varepsilon\eta)\,d\eta\\ &=I_1^\varepsilon(\varphi)-I_2^\varepsilon(\varphi).\end{aligned}\tag{73}$$

We consider the integrals $I_1^\varepsilon(\varphi)$ and $I_2^\varepsilon(\varphi)$ separately. Since $\int_{\mathbb{R}^d}|z|^2a(z)ds\le\infty$, then

$$\int\limits_{\mathbb{R}^d} z\otimes z\,a(z)\mu(0,\omega)\mu(-z,\omega)\,dz\in\left(L^\infty(\Omega)\right)^{d^2}.$$

Therefore, by the Birkhoff ergodic theorem a.s.

$$\int_{\mathbb{R}^d} z\otimes z\, a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,dz \rightharpoonup D_1 \quad \text{weakly in} \quad (L^2_{\rm loc}(\mathbb{R}^d))^{d^2}$$

with

$$D_1=\int_{\mathbb{R}^d}\frac12\, z\otimes z\, a(z)\,\mathbf{E}\{\mu(0,\omega)\mu(-z,\omega)\}\,dz. \tag{74}$$

Recalling that $u_0\in C_0^\infty(\mathbb{R}^d)$, we obtain

$$I_1^\varepsilon(\varphi)\to\int_{\mathbb{R}^d} D_1\nabla\nabla u_0(x)\varphi(x)\,dx. \tag{75}$$

The second integral in (73) contains the non-stationary random field $\theta(z,\omega)$, and we rewrite $I_2(\varphi)$ as a sum of two terms, such that the first term contains the stationary field $\zeta_z(\eta,\omega)$ and the contribution of the second one is asymptotically negligible. In order to estimate the contribution of the second term we construct an additional corrector u_2^ε, see formula (81) below.

We have

$$\begin{aligned}
I_2^\varepsilon(\varphi)&=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\theta(\frac{x}{\varepsilon}-z,\omega)\nabla\nabla u_0(x)\varphi(x)\,dx\,dz\\
&=\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\theta(\frac{x}{\varepsilon}-z,\omega)\nabla\nabla u_0(x)\varphi(x)\,dx\,dz\\
&-\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)z\,\mu(\frac{y}{\varepsilon},\omega)\mu(\frac{y}{\varepsilon}-z,\omega)\theta(\frac{y}{\varepsilon},\omega)\nabla\nabla u_0(y-\varepsilon z)\varphi(y-\varepsilon z)\,dy\,dz\\
&=\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\\
&\qquad\times\Big(\theta(\frac{x}{\varepsilon}-z,\omega)\nabla\nabla u_0(x)\varphi(x)-\theta(\frac{x}{\varepsilon},\omega)\nabla\nabla u_0(x-\varepsilon z)\varphi(x-\varepsilon z)\Big)dxdz\\
&=\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\big(\theta(\frac{x}{\varepsilon}-z,\omega)-\theta(\frac{x}{\varepsilon},\omega)\big)\nabla\nabla u_0(x)\varphi(x)dx\,dz\\
&+\frac12\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\\
&\qquad\times\big(\nabla\nabla u_0(x)\varphi(x)-\nabla\nabla u_0(x-\varepsilon z)\varphi(x-\varepsilon z)\big)dx\,dz,
\end{aligned} \tag{76}$$

here and in what follows $z\theta(z)\nabla\nabla u_0(x)$ stands for $z^i\theta^j(z)\partial_{x_i}\partial_{x_j}u_0(x)$. The field $\zeta_{-z}(\eta,\omega)=\theta(\eta-z,\omega)-\theta(\eta,\omega)$ is stationary for any given z, and

$$\int_{\mathbb{R}^d} a(z)z\otimes\zeta_{-z}(0,\omega)\mu(0,\omega)\mu(-z,\omega)\,dz\in(L^2(\Omega))^{d^2}. \tag{77}$$

Indeed, in view of (39) and (45) by the Cauchy-Schwarz inequality we have

$$\int\limits_{\Omega}\Big(\int\limits_{\mathbb{R}^d}|a(z)z\otimes\zeta_{-z}(0,\omega)\mu(0,\omega)\mu(-z,\omega)|\,dz\Big)^2 d\mathbf{P}(\omega)\le$$

$$\alpha_2^2\Big(\int\limits_{\mathbb{R}^d}a(z)|z|^2dz\Big)\Big(\int\limits_{\mathbb{R}^d}\int\limits_{\Omega}a(z)\,|\theta(-z,\omega)|^2dzd\mathbf{P}(\omega)\Big)<\infty.$$

Consequently applying the ergodic theorem to the stationary field (77) we obtain for the first integral in (76) as $\varepsilon\to0$

$$\begin{aligned}&\frac12\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}a(z)z\zeta_{-z}(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\nabla\nabla u_0(x)\varphi(x)dx\,dz\ \to\\&\frac12\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}a(z)z\mathbf{E}\{\zeta_{-z}(0,\omega)\mu(0,\omega)\mu(-z,\omega)\}\nabla\nabla u_0(x)\varphi(x)dx\,dz\\&=\int\limits_{\mathbb{R}^d}D_2\,\nabla\nabla u_0(x)\varphi(x)\,dx,\end{aligned}\tag{78}$$

where we have used the notation

$$D_2=\frac12\int\limits_{\mathbb{R}^d}a(z)z\otimes\mathbf{E}\{\zeta_{-z}(0,\omega)\mu(0,\omega)\mu(-z,\omega)\}\,dz.\tag{79}$$

Denote the last integral on the right-hand side in (76) by $J_2^\varepsilon(\varphi)$:

$$\begin{aligned}J_2^\varepsilon(\varphi)=\frac12\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}&a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\\&\times\big(\nabla\nabla u_0(x)\varphi(x)-\nabla\nabla u_0(x-\varepsilon z)\varphi(x-\varepsilon z)\big)dx\,dz\end{aligned}\tag{80}$$

and consider this expression as a functional on $L^2(\mathbb{R}^d)$ acting on function φ. In order to show that for each $\varepsilon>0$ the functional J_2^ε is a bounded linear functional on $L^2(\mathbb{R}^d)$ we represent J_2^ε as a sum $J_2^\varepsilon=J_2^{1,\varepsilon}+J_2^{2,\varepsilon}+J_2^{3,\varepsilon}$ with $J_2^{1,\varepsilon}$, $J_2^{2,\varepsilon}$ and $J_2^{3,\varepsilon}$ introduced below and estimate each of these functionals separately. By Proposition 4.4 a.s. $\theta(\frac{x}{\varepsilon},\omega)\in L^2_{\rm loc}(\mathbb{R}^d)$ for all $\varepsilon>0$. Therefore,

$$J_2^{1,\varepsilon}(\varphi)=\frac12\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\nabla\nabla u_0(x)\varphi(x)dx\,dz$$

is a.s. a bounded linear functional on $L^2(\mathbb{R}^d)$. Similarly,

$$\begin{aligned}J_2^{2,\varepsilon}(\varphi)=\frac12\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d}&a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\\&\times\,\theta(\frac{x}{\varepsilon}-z,\omega)\nabla\nabla u_0(x-\varepsilon z)\varphi(x-\varepsilon z)dx\,dz\end{aligned}$$

is a.s. a bounded linear functional on $L^2(\mathbb{R}^d)$. Due to (39) and by the Birkhoff ergodic theorem the linear functional

$$J_2^{3,\varepsilon}(\varphi) = \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega) \times\Big(\theta(\frac{x}{\varepsilon},\omega)-\theta(\frac{x}{\varepsilon}-z,\omega)\Big)\nabla\nabla u_0(x-\varepsilon z)\varphi(x-\varepsilon z)dx\,dz$$
$$= \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon}+z,\omega)\mu(\frac{x}{\varepsilon},\omega)\,\Big(\theta(\frac{x}{\varepsilon}+z,\omega)-\theta(\frac{x}{\varepsilon},\omega)\Big)\nabla\nabla u_0(x)\varphi(x)dx\,dz$$
$$= \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon}+z,\omega)\mu(\frac{x}{\varepsilon},\omega)\,\theta(z,T_{\frac{x}{\varepsilon}}\omega)\nabla\nabla u_0(x)\varphi(x)dx\,dz$$

is a.s. bounded in $L^2(\mathbb{R}^d)$. Since $J_2^\varepsilon(\varphi)=J_2^{1,\varepsilon}(\varphi)+J_2^{2,\varepsilon}(\varphi)+J_2^{3,\varepsilon}(\varphi)$, the desired boundedness of J_2^ε follows. Then by the Riesz theorem for a.e. ω there exists a function $f_2^\varepsilon=f_2^\varepsilon(u_0)\in L^2(\mathbb{R}^d)$ such that $J_2^\varepsilon(\varphi)=(f_2^\varepsilon,\varphi)$. We emphasize that here we do not claim that the norm of J_2^ε admits a uniform in ε estimate.

Next we show that the contribution of f_2^ε to w^ε is vanishing. To this end consider the function (additional corrector)

$$u_2^\varepsilon(x,\omega)=(-L^\varepsilon+m)^{-1}f_2^\varepsilon(x,\omega). \tag{81}$$

Lemma 5.1. $\|u_2^\varepsilon\|_{L^2(\mathbb{R}^d)}\to 0$ *as* $\varepsilon\to 0$ *for a.e.* ω.

Proof. Taking $\varphi=u_2^\varepsilon$ we get

$$((-L^\varepsilon+m)u_2^\varepsilon,u_2^\varepsilon)=(f_2^\varepsilon,u_2^\varepsilon). \tag{82}$$

Considering (7) the left-hand side of (82) can be rearranged as follows:

$$\begin{aligned} &-\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_2^\varepsilon(x-\varepsilon z)-u_2^\varepsilon(x))dz\,u_2^\varepsilon(x)dx \\ &\qquad +m\int\limits_{\mathbb{R}^d}(u_2^\varepsilon)^2(x)dx \\ &=\frac{1}{2\varepsilon^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_2^\varepsilon(x-\varepsilon z)-u_2^\varepsilon(x))^2dzdx \\ &\qquad +m\int\limits_{\mathbb{R}^d}(u_2^\varepsilon)^2(x)dx. \end{aligned} \tag{83}$$

We denote

$$G_1^2=\frac{1}{2\varepsilon^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_2^\varepsilon(x-\varepsilon z)-u_2^\varepsilon(x))^2dzdx,$$
$$G_2^2=m\int\limits_{\mathbb{R}^d}(u_2^\varepsilon)^2(x)dx.$$

It follows from (80) that the right-hand side of (82) takes the form

$$
\begin{aligned}
J_2^\varepsilon(u_2^\varepsilon) &= \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\theta(\frac{x}{\varepsilon},\omega)\\
&\qquad\qquad \times\big(\nabla\nabla u_0(x)u_2^\varepsilon(x)-\nabla\nabla u_0(x-\varepsilon z)u_2^\varepsilon(x-\varepsilon z)\big)dxdz\\
&= \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\nabla\nabla u_0(x)\\
&\qquad\qquad \times\big(u_2^\varepsilon(x)-u_2^\varepsilon(x-\varepsilon z)\big)dx\,dz\\
&+ \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\\
&\qquad\qquad \times\big(\nabla\nabla u_0(x)-\nabla\nabla u_0(x-\varepsilon z)\big)u_2^\varepsilon(x-\varepsilon z)dxdz\\
&= \frac{1}{2}(I_1+I_2).
\end{aligned}
\tag{84}
$$

It is proved in Proposition 4.4 that a.s. $\|\varepsilon\theta(\frac{x}{\varepsilon},\omega)\|_{L^2(B)}\to 0$ as $\varepsilon\to 0$ for any ball $B\subset\mathbb{R}^d$. By the Cauchy-Schwartz inequality we obtain the following upper bounds for I_1:

$$
\begin{aligned}
I_1 &\le \Bigg(\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\big(u_2^\varepsilon(x)-u_2^\varepsilon(x-\varepsilon z)\big)^2dxdz\Bigg)^{\frac12}\\
&\Bigg(\frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)|z|^2\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\varepsilon^2\big|\theta(\frac{x}{\varepsilon},\omega)\big|^2(\nabla\nabla u_0(x))^2dxdz\Bigg)^{\frac12}\\
&\le \frac{o(1)}{\varepsilon}\;\Bigg(\frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\big(u_2^\varepsilon(x)-u_2^\varepsilon(x-\varepsilon z)\big)^2dxdz\Bigg)^{\frac12}\\
&= G_1\cdot o(1),
\end{aligned}
\tag{85}
$$

where $o(1)\to 0$ as $\varepsilon\to 0$. We turn to the second integral I_2. Let B be a ball centered at the origin and such that $\mathrm{supp}(u_0)\subset B$, $\mathrm{dist}(\mathrm{supp}(u_0),\partial B)>1$. Then

$$
\begin{aligned}
&\Bigg|\int\limits_{\mathbb{R}^d}\int\limits_{B} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\\
&\qquad\qquad \times\big(\nabla\nabla u_0(x)-\nabla\nabla u_0(x-\varepsilon z)\big)u_2^\varepsilon(x-\varepsilon z)dx\,dz\Bigg|\\
&\le C\int\limits_{\mathbb{R}^d}\int\limits_{B} a(z)|z|^2\,\big|\varepsilon\theta(\frac{x}{\varepsilon},\omega)\big|\,|u_2^\varepsilon(x-\varepsilon z)|dx\,dz\\
&\le \|u_2^\varepsilon\|_{L^2(\mathbb{R}^d)}\cdot o(1)=G_2\cdot o(1).
\end{aligned}
\tag{86}
$$

The integral over $B^c=\mathbb{R}^d\setminus B$ can be estimated in the following way:

$$
\Bigg|\int\limits_{\mathbb{R}^d}\int\limits_{B^c} a(z)z\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\big(\nabla\nabla u_0(x)-\nabla\nabla u_0(x-\varepsilon z)\big)u_2^\varepsilon(x-\varepsilon z)dxdz\Bigg|
$$

$$
\Bigg|\int\limits_{\mathbb{R}^d}\int\limits_{B^c} a(z)z\,\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\,\theta(\frac{x}{\varepsilon},\omega)\nabla\nabla u_0(x-\varepsilon z)u_2^\varepsilon(x-\varepsilon z)dx\,dz\Bigg|
$$

$$\begin{aligned}
&\le C \int\limits_{|z|\ge\frac{1}{\varepsilon}} \int\limits_{B^c} a(z)|z| \left|\theta(\frac{x}{\varepsilon},\omega)\right| |\nabla\nabla u_0(x-\varepsilon z)|\, |u_2^\varepsilon(x-\varepsilon z)|\, dx\, dz \\
&\le C \int\limits_{|z|\ge\frac{1}{\varepsilon}} \int\limits_{\mathbb{R}^d} a(z)|z| \left|\theta(\frac{x}{\varepsilon}+z,\omega)\right| |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx\, dz \\
&\le C \int\limits_{|z|\ge\frac{1}{\varepsilon}} \int\limits_{\mathbb{R}^d} a(z)|z| \Big[\left|\theta(\frac{x}{\varepsilon}+z,\omega)-\theta(\frac{x}{\varepsilon},\omega)\right| + \left|\theta(\frac{x}{\varepsilon},\omega)\right|\Big] |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx\, dz.
\end{aligned} \tag{87}$$

We have

$$\begin{aligned}
&\int\limits_{|z|\ge\frac{1}{\varepsilon}} \int\limits_{\mathbb{R}^d} a(z)|z| \left|\theta(\frac{x}{\varepsilon},\omega)\right| |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx\, dz \\
&\le \int\limits_{\mathbb{R}^d} \int\limits_{\mathbb{R}^d} a(z)|z|^2 \left|\varepsilon\theta(\frac{x}{\varepsilon},\omega)\right| |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx\, dz \le G_2\cdot o(1)
\end{aligned}$$

and

$$\begin{aligned}
&\int\limits_{|z|\ge\frac{1}{\varepsilon}} \int\limits_{\mathbb{R}^d} a(z)|z| \Big[\left|\theta(\frac{x}{\varepsilon}+z,\omega)-\theta(\frac{x}{\varepsilon},\omega)\right|\Big] |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx\, dz \\
&\le \int\limits_{|z|\ge\frac{1}{\varepsilon}} \int\limits_{\mathbb{R}^d} a(z)|z| \left|\zeta_z(T_{\frac{x}{\varepsilon}}\omega)\right| |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx\, dz \\
&\le \Bigg(\int\limits_{|z|\ge\frac{1}{\varepsilon}} a(z)z^2\, dz\Bigg)^{\frac12} \int\limits_{\mathbb{R}^d} \Bigg(\int\limits_{\mathbb{R}^d} a(z)\left|\zeta_z(T_{\frac{x}{\varepsilon}}\omega)\right|^2 dz\Bigg)^{\frac12} |\nabla\nabla u_0(x)|\, |u_2^\varepsilon(x)|\, dx \\
&\le o(1) \Bigg(\int\limits_{\mathbb{R}^d} |u_2^\varepsilon(x)|^2\, dx\Bigg)^{\frac12} \Bigg(\int\limits_{\mathbb{R}^d} \Bigg(\int\limits_{\mathbb{R}^d} a(z)\left|\zeta_z(T_{\frac{x}{\varepsilon}}\omega)\right|^2 dz\Bigg) |\nabla\nabla u_0(x)|^2\, dx\Bigg)^{\frac12} \\
&= G_2\cdot o(1).
\end{aligned}$$

Since $\zeta_z(\omega)\in L^2_M$, the second integral in the right hand side here converges to a constant by the ergodic theorem.

Combining the last two estimates we conclude that the term on the right-hand side in (87) does not exceed $G_2\cdot o(1)$. Therefore, considering (86), we obtain $I_1\le G_2\cdot o(1)$. This estimate and (85) imply that

$$G_1^2+G_2^2 = I_1+I_2 \le (G_1+G_2)\cdot o(1).$$

Consequently, $G_1\to 0$ and $G_2 = m^{1/2}\|u_2^\varepsilon\|_{L^2(\mathbb{R}^d)}\to 0$ as $\varepsilon\to 0$. Lemma is proved. □

Thus we can rewrite I_0^ε (all the terms of the order ε^0) as follows

$$\begin{aligned}
I_0^\varepsilon &= (D_1-D_2)\cdot\nabla\nabla u_0 + f_2^\varepsilon + S(\frac{x}{\varepsilon},\omega)\cdot\nabla\nabla u_0, \\
S(\frac{x}{\varepsilon},\omega) &= \Psi_1(\frac{x}{\varepsilon},\omega) - \Psi_2(\frac{x}{\varepsilon},\omega),
\end{aligned} \tag{88}$$

where the matrices D_1 and D_2 are defined in (74) and (79) respectively, and $S(\frac{x}{\varepsilon},\omega)$, $\Psi_1(\frac{x}{\varepsilon},\omega)$, $\Psi_2(\frac{x}{\varepsilon},\omega)$ are stationary fields with zero mean which are given by

$$\Psi_1(\frac{x}{\varepsilon},\omega) = \frac{1}{2}\int\limits_{\mathbb{R}^d} a(z)z^2\Big[\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega) - \mathbf{E}\{\mu(0,\omega)\mu(-z,\omega)\}\Big]dz, \tag{89}$$

$$\begin{aligned}\Psi_2(\frac{x}{\varepsilon},\omega) = \frac{1}{2}\int\limits_{\mathbb{R}^d} a(z)z\Big[\zeta_{-z}(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)\\ - \mathbf{E}\{\zeta_{-z}(0,\omega)\mu(0,\omega)\mu(-z,\omega)\}\Big]dz.\end{aligned} \tag{90}$$

Denote

$$u_3^\varepsilon(x,\omega) = (-L^\varepsilon+m)^{-1}F^\varepsilon(x,\omega), \quad \text{where } F^\varepsilon(x,\omega) = S(\frac{x}{\varepsilon},\omega)\cdot\nabla\nabla u_0(x). \tag{91}$$

Since $\operatorname{supp} u_0 \subset B$ is a bounded subset of $\mathbb{R}^d$ and

$$\int\limits_{\mathbb{R}^d} a(z)|z|\,\big|\zeta_{-z}(\omega)\big|\,dz \in L^2(\Omega),$$

then by the Birkhoff theorem $u_3^\varepsilon \in L^2(\mathbb{R}^d)$. Our goal is to prove that $\|u_3^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$. We first show that the family $\{u_3^\varepsilon\}$ is bounded in $L^2(\mathbb{R}^d)$.

Lemma 5.2. *The family of functions u_3^ε defined by (91) is uniformly bounded in $L^2(\mathbb{R}^d)$ for e.a. ω: $\|u_3^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C$ for any $0<\varepsilon<1$.*

Proof. Since the operator $(-L^\varepsilon+m)^{-1}$ is bounded ($\|(-L^\varepsilon+m)^{-1}\| \le \frac{1}{m}$), then it is sufficient to prove that $\|F^\varepsilon(x,\omega)\|_{L^2(\mathbb{R}^d)} \le C$ uniformly in ε. By the Birkhoff ergodic theorem the functions $\Psi_1(\frac{x}{\varepsilon},\omega)$ and $\Psi_2(\frac{x}{\varepsilon},\omega)$ a.s. converge to zero weakly in $L^2(B)$, so does $S(\frac{x}{\varepsilon},\omega)$. Then $S(\frac{x}{\varepsilon},\omega)\cdot\nabla\nabla u_0$ a.s. converges to zero weakly in $L^2(\mathbb{R}^d)$. This implies the desired boundedness. □

Lemma 5.3. *For any cube B centered at the origin $\|u_3^\varepsilon\|_{L^2(B)} \to 0$ as $\varepsilon\to 0$ for e.a. ω.*

Proof. The first step of the proof is to show that any sequence $\{u_3^{\varepsilon_j}\}$, $\varepsilon_j \to 0$, is compact in $L^2(B)$. Using definition (91) we have

$$((-L^\varepsilon+m)u_3^\varepsilon, u_3^\varepsilon) \ = \ (F^\varepsilon, u_3^\varepsilon).$$

The left-hand side of this relation can be rewritten as

$$\begin{aligned}
&\int\limits_{\mathbb{R}^d} (-L^\varepsilon + m) u_3^\varepsilon(x) u_3^\varepsilon(x) dx \\
&= m \int\limits_{\mathbb{R}^d} (u_3^\varepsilon(x))^2 dx \\
&\quad - \frac{1}{\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\, \mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_3^\varepsilon(x-\varepsilon z) - u_3^\varepsilon(x)) u_3^\varepsilon(x) dz dx \\
&= m \int\limits_{\mathbb{R}^d} (u_3^\varepsilon(x))^2 dx \\
&\quad + \frac{1}{2\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\, \mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_3^\varepsilon(x-\varepsilon z) - u_3^\varepsilon(x))^2 dz dx.
\end{aligned} \tag{92}$$

Consequently we obtain the following equality

$$\begin{aligned}
&m \int\limits_{\mathbb{R}^d} (u_3^\varepsilon(x))^2 dx \\
&+ \frac{1}{2\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_3^\varepsilon(x-\varepsilon z) - u_3^\varepsilon(x))^2 dz dx \\
&= (F^\varepsilon, u_3^\varepsilon).
\end{aligned} \tag{93}$$

Considering the uniform boundedness of F^ε and u_3^ε, see Lemma 5.2, we immediately conclude that

$$\frac{1}{\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)\, \mu(\frac{x}{\varepsilon},\omega)\mu(\frac{x}{\varepsilon}-z,\omega)(u_3^\varepsilon(x-\varepsilon z) - u_3^\varepsilon(x))^2 dz dx < K \tag{94}$$

uniformly in ε and for a.e. ω. Therefore,

$$m \int\limits_{\mathbb{R}^d} (u_3^\varepsilon(x))^2 dx + \frac{1}{\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)(u_3^\varepsilon(x-\varepsilon z) - u_3^\varepsilon(x))^2 dz dx < K \tag{95}$$

For the sake of definiteness assume that $B = [-1,1]^d$. The cubes of other size can be considered in exactly the same way. Let $\phi(s)$ be an even $C_0^\infty(\mathbb{R})$ function such that $0 \le \phi \le 1$, $\phi(s) = 1$ for $|s| \le 1$, $\phi(s) = 0$ for $|s| \ge 2$, and $|\phi'(s)| \le 2$. Denote $\tilde u_3^\varepsilon(x) = \phi(|x|) u_3^\varepsilon(x)$. It is straightforward to check that

$$m \int\limits_{\mathbb{R}^d} (\tilde u_3^\varepsilon(x))^2 dx + \frac{1}{\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)(\tilde u_3^\varepsilon(x-\varepsilon z) - \tilde u_3^\varepsilon(x))^2 dz dx < K \tag{96}$$

We also choose $\mathcal{R}$ in such a way that $\int_{|z|\le\mathcal{R}} a(z)dz \ge \frac{1}{2}$ and introduce

$$\tilde a(z) = \mathbf{1}_{\{|z|\le\mathcal{R}\}}\, a(z) \Big(\int\limits_{|z|\le\mathcal{R}} a(z) dz \Big)^{-1}.$$

Then

$$m \int\limits_{\mathbb{R}^d} (\tilde u_3^\varepsilon(x))^2 dx + \frac{1}{\varepsilon^2} \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \tilde a(z)(\tilde u_3^\varepsilon(x-\varepsilon z) - \tilde u_3^\varepsilon(x))^2 dz dx < K. \tag{97}$$

Letting $\tilde{B} = [-\pi, \pi]^d$, we denote by $\hat{u}_3^\varepsilon(x)$ the $\tilde{B}$ periodic extension of $\tilde{u}_3^\varepsilon(x)$. For the extended function we have

$$m \int_{\tilde{B}} (\hat{u}_3^\varepsilon(x))^2 dx + \frac{1}{\varepsilon^2} \int_{\tilde{B}} \int_{\mathbb{R}^d} \tilde{a}(z)(\hat{u}_3^\varepsilon(x-\varepsilon z) - \hat{u}_3^\varepsilon(x))^2 dz dx < K. \tag{98}$$

The functions $e_k(x) = \frac{1}{(2\pi)^{d/2}} e^{ikx}$, $k \in Z^d$, form an orthonormal basis in $L^2(B)$, and

$$\hat{u}_3^\varepsilon(x) = \sum_k \alpha_k^\varepsilon e_k(x), \quad \hat{u}_3^\varepsilon(x-\varepsilon z) = \sum_k \alpha_k^\varepsilon e^{-i\varepsilon kz} e_k(x);$$
$$\|\hat{u}_3^\varepsilon(x)\|^2 = \sum_k (\alpha_k^\varepsilon)^2, \quad \|\hat{u}_3^\varepsilon(x-\varepsilon z) - \hat{u}_3^\varepsilon(x)\|^2 = \sum_k (\alpha_k^\varepsilon)^2 |e^{-i\varepsilon kz} - 1|^2.$$

Then inequality (94) is equivalent to the following bound

$$\frac{1}{\varepsilon^2} \sum_k (\alpha_k^\varepsilon)^2 \int_{\mathbb{R}^d} \tilde{a}(z)|e^{-i\varepsilon kz} - 1|^2 dz < C. \tag{99}$$

Lemma 5.4. *For any $k \in Z^d$ and any $0 < \varepsilon < 1$ there exist constants C_1, C_2 (depending on d) such that*

$$\int_{\mathbb{R}^d} \tilde{a}(z)|e^{-i\varepsilon kz} - 1|^2 dz \ge \min\{C_1 k^2 \varepsilon^2, \ C_2\}. \tag{100}$$

Proof. For small ε, the lower bound by $C_1 k^2 \varepsilon^2$ follows from the expansion of $e^{-i\varepsilon kz}$ in the neighborhood of 0. For large enough $\varepsilon|k| \ge \varkappa_0 > 1$ we use the following inequality

$$\int_{\mathbb{R}^d} \tilde{a}(z)|e^{-i\varepsilon kz} - 1|^2 dz \ge c_0 \int_{[0,1]^d} |e^{-i\varepsilon kz} - 1|^2 dz \ge c_0 \big(2 - \frac{2}{\varkappa_0}\big)^d. \quad \square$$

Let us consider a sequence $\varepsilon_j \to 0$. Using inequalities (99)-(100) we will construct now for any $\delta > 0$ a finite 2δ-net covering all elements of the sequence $u_3^{\varepsilon_j}$. For any $\delta > 0$ we take $|k_0|$ and j_0 such that

$$\frac{C}{\delta} < C_1 |k_0|^2 < \frac{C_2}{\varepsilon_{j_0}^2}, \tag{101}$$

where C, C_1, C_2 are the same constants as in (99)-(100). Then it follows from (99)-(101) that

$$\sum_{k:|k|\ge|k_0|} C_1 |k_0|^2 (\alpha_k^{\varepsilon_j})^2 < \sum_{k:|k|\ge|k_0|} \min\Big\{C_1|k|^2, \frac{C_2}{\varepsilon_j^2}\Big\} (\alpha_k^{\varepsilon_j})^2 < C \quad \text{for any} \ \ j > j_0.$$

Consequently we obtain the uniform bound on the tails of $\hat{u}_3^{\varepsilon_j}$ for all $j > j_0$:

$$\sum_{k:|k|\ge|k_0|} (\alpha_k^{\varepsilon_j})^2 < \frac{C}{C_1|k_0|^2} < \delta. \tag{102}$$

Denote by $\mathcal{H}_{k_0} \subset L^2(\tilde{B})$ a linear span of basis vectors $\{e_k, \ |k| < |k_0|\}$. Evidently, it is a finite-dimensional subspace. Then we have

$$\hat{u}_3^\varepsilon = w_{k_0}^\varepsilon + \sum_{k:|k|\ge|k_0|} \alpha_k^\varepsilon e_k, \quad \text{where} \ \ w_{k_0}^\varepsilon = P_{\mathcal{H}_{k_0}} u_3^\varepsilon.$$

Since we already know from Lemma 5.2 that the functions $\hat u_3^{\varepsilon_j}$ are uniformly bounded in $L^2(\tilde B)$, then the functions $w_{k_0}^{\varepsilon_j}$ are also uniformly bounded. Therefore there exists in $\mathcal{H}_{k_0}$ a finite δ-net covering the functions $\{w_{k_0}^{\varepsilon_j},\, j>j_0\}$. Estimate (102) implies that the same net will be the 2δ-net for the functions $\{\hat u_3^{\varepsilon_j},\, j>j_0\}$. We need to add to this net j_0 elements to cover first j_0 functions $\hat u_3^{\varepsilon_j}$, $j=1,\dots,j_0$.

Thus we constructed the finite 2δ-net for any $\delta>0$ which proves the compactness of $\{\hat u_3^\varepsilon\}$ as $\varepsilon\to 0$ in $L^2(\tilde B)$.

Since $u_3^\varepsilon(x)=\hat u_3^\varepsilon(x)$ for $x\in B$, we conclude that the family $\{u_3^\varepsilon\}$ is compact in $L^2(B)$. In the same way one can show that this family is compact on any cube $B=[-L,L]^d$. This completes the proof of Lemma. □

Lemma 5.5. *The following limit relation holds:* $\|u_3^\varepsilon\|_{L^2(\mathbb{R}^d)}\to 0$, *as* $\varepsilon\to 0$.

Proof. We go back to formula (93). On the right-hand side of this equality we have the inner product of two sequences F^ε and u_3^ε Since the sequence $F^\varepsilon\rightharpoonup 0$ weakly in $L^2(B)$, and the sequence u_3^ε is compact in $L^2(B)$, the product $(F^\varepsilon,u_3^\varepsilon)\to 0$ as $\varepsilon\to 0$. Therefore, both integrals on the left-hand side of (93) also tend to zero as $\varepsilon\to 0$, and we obtain that $\|u_3^\varepsilon\|_{L^2(\mathbb{R}^d)}\to 0,\ \varepsilon\to 0$. □

Denote by Θ the matrix $\Theta=D_1-D_2$, where D_1, D_2 are defined by (74), (79). Our next goal is to show that D_1-D_2 is a positive definite matrix.

Proposition 5.1. *The matrix* $\Theta=D_1-D_2$ *is positive definite:*

$$\Theta \;=\; \frac12\int\limits_{\mathbb{R}^d}\int\limits_{\Omega}\big(z\otimes z - z\otimes\zeta_{-z}(0,\omega)\big)\,a(z)\,\mu(0,\omega)\mu(-z,\omega)\,dz\,d\mathbf{P}(\omega)>0. \tag{103}$$

Proof. We recall that $\varkappa^\delta(\omega)$ stands for a unique solution of equation (26). Letting $\varkappa^\delta_\eta(\omega)=\eta\cdot\varkappa^\delta(\omega)$, $\eta\in\mathbb{R}^d\setminus\{0\}$, one can easily obtain

$$\begin{aligned}&\delta\int\limits_\Omega\big(\varkappa^\delta_\eta(\omega)\big)^2\mu(\omega)\,d\mathbf{P}(\omega)\\ &\qquad-\int\limits_{\mathbb{R}^d}\int\limits_\Omega a(z)\mu(T_z\omega)\big(\varkappa^\delta_\eta(T_z\omega)-\varkappa^\delta_\eta(\omega)\big)\varkappa^\delta_\eta(\omega)\mu(\omega)\,dz\,d\mathbf{P}(\omega)\\ &\qquad=\int\limits_{\mathbb{R}^d}\int\limits_\Omega(\eta\cdot z)a(z)\varkappa^\delta_\eta(\omega)\mu(T_z\omega)\mu(\omega)\,dz\,d\mathbf{P}(\omega).\end{aligned}\tag{104}$$

In the same way as in the proof of Proposition 4.1, we derive the following relation:

$$\begin{aligned}&\delta\int\limits_\Omega\big(\varkappa^\delta_\eta(\omega)\big)^2\mu(\omega)\,d\mathbf{P}(\omega)\\ &\qquad+\frac12\int\limits_{\mathbb{R}^d}\int\limits_\Omega a(z)\mu(T_z\omega)\big(\varkappa^\delta_\eta(T_z\omega)-\varkappa^\delta_\eta(\omega)\big)^2\mu(\omega)\,dz\,d\mathbf{P}(\omega)\\ &\qquad=-\frac12\int\limits_{\mathbb{R}^d}\int\limits_\Omega(\eta\cdot z)a(z)\big(\varkappa^\delta_\eta(T_z\omega)-\varkappa^\delta_\eta(\omega)\big)\mu(T_z\omega)\mu(\omega)\,dz\,d\mathbf{P}(\omega).\end{aligned}\tag{105}$$

According to (38) the sequence $\eta\cdot(\varkappa^{\delta_j}_\eta(T_z\omega)-\varkappa^{\delta_j}_\eta(\omega))$ converges weakly in L^2_M as $\delta_j\to 0$ to $\eta\cdot\theta(z,\omega)$. Passing to the limit $\delta_j\to 0$ in relation (105) and considering the lower semicontinuity of the L^2_M norm with respect to the weak topology, we arrive at the following inequality

$$
\begin{aligned}
&\frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\Omega} a(z)\mu(T_z\omega)\big(\eta\cdot\theta(z,\omega)\big)^2\mu(\omega)\,dz\,d\mathbf{P}(\omega)\\
&\le -\frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\Omega}(\eta\cdot z)a(z)\big(\eta\cdot\theta(z,\omega)\big)\mu(T_z\omega)\mu(\omega)\,dz\,d\mathbf{P}(\omega).
\end{aligned}
\tag{106}
$$

One can easily check that

$$
\begin{aligned}
\Theta\eta\cdot\eta &= \frac{1}{2}\,\eta_i\eta_j\int\limits_{\mathbb{R}^d}\int\limits_{\Omega}\big(z^iz^j - z^i\zeta^j_{-z}(0,\omega)\big)\,a(z)\,\mu(0,\omega)\mu(-z,\omega)\,dz\,d\mathbf{P}(\omega)\\
&= \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\Omega}\big((\eta\cdot z)^2 + (\eta\cdot z)(\eta\cdot\theta(z,\omega))\big)\,a(z)\,\mu(0,\omega)\mu(z,\omega)\,dz\,d\mathbf{P}(\omega).
\end{aligned}
$$

Combining the latter relation with (106) we obtain

$$
\Theta\eta\cdot\eta \ge \frac{1}{2}\int\limits_{\mathbb{R}^d}\int\limits_{\Omega}\big((\eta\cdot z) + (\eta\cdot z)(\eta\cdot\theta(z,\omega))\big)^2\,a(z)\,\mu(0,\omega)\mu(z,\omega)\,dz\,d\mathbf{P}(\omega).
$$

Since $\theta(z,\omega)$ is a.s. a function of sublinear growth in z, we conclude that $\eta\cdot\theta(z,\omega)\not\equiv\eta\cdot z$, consequently the integral on the right-hand side here is strictly positive. This yields the desired positive definiteness. □

6. Estimation of the remainder ϕ_ε

In this section we consider the remainder $\phi_\varepsilon(x,\omega)$ given by (19) and prove that $\|\phi_\varepsilon\|_{L^2(\mathbb{R}^d)}$ vanishes a.s. as $\varepsilon\to 0$.

Lemma 6.1. *Let $u_0\in C_0^\infty(\mathbb{R}^d)$. Then a.s.*

$$
\|\phi_\varepsilon(\cdot,\omega)\|_{L^2(\mathbb{R}^d)}\ \to\ 0 \qquad \text{as}\ \ \varepsilon\to 0. \tag{107}
$$

Proof. The first term in (19) can be written as

$$
\begin{aligned}
&\phi_\varepsilon^{(1)}(x,\omega)\\
&=\int\limits_{\mathbb{R}^d} dz\ a(z)\mu\Big(\frac{x}{\varepsilon},\omega\Big)\mu\Big(\frac{x}{\varepsilon}-z,\omega\Big)\\
&\qquad\qquad\times\int\limits_0^1\ \Big(\nabla\nabla u_0(x-\varepsilon zt)-\nabla\nabla u_0(x)\Big)z\otimes z(1-t)\ dt.
\end{aligned}
$$

It doesn't depend on the random corrector θ and can be considered exactly in the same way as in [18, Proposition 5]. Thus we have

$$
\|\phi_\varepsilon^{(1)}\|_{L^2(\mathbb{R}^d)}\to 0 \qquad \text{as}\ \ \varepsilon\to 0. \tag{108}
$$

Let us denote by $\phi_\varepsilon^{(2)}$ the sum of the second and the third terms in (19):

$$
\begin{aligned}
&\phi_\varepsilon^{(2)}(x,\omega)\\
&= \mu\big(\frac{x}{\varepsilon},\omega\big)\int\limits_{\mathbb{R}^d} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\\
&\qquad\times\Big(\frac{1}{\varepsilon}\big(\nabla u_0(x-\varepsilon z)-\nabla u_0(x)\big)+z\,\nabla\nabla u_0(x)\Big)dz.
\end{aligned}
\tag{109}
$$

We take sufficiently large $L>0$ such that supp $u_0\subset\{|x|<\frac{1}{2}L\}$ and estimate $\phi_\varepsilon^{(2)}(x,\omega)$ separately in the sets $\{|x|<L\}$ and $\{|x|>L\}$. If $|x|>L$, then $u_0(x)=0$. Since $a(z)$ has a finite second moment in $\mathbb{R}^d$, for any $c>0$ we have

$$
\frac{1}{\varepsilon^2}\int\limits_{|z|>\frac{c}{\varepsilon}} a(z)dz=\frac{1}{\varepsilon^2}\int\limits_{|z|>\frac{c}{\varepsilon}} a(z)\frac{z^2}{z^2}dz\le\frac{1}{c^2}\int\limits_{|z|>\frac{c}{\varepsilon}} a(z)z^2dz\to 0,\quad\text{as } \varepsilon\to 0.
\tag{110}
$$

Therefore,

$$
\begin{aligned}
&\|\phi_\varepsilon^{(2)}\,\chi_{|x|>L}\|^2_{L^2(\mathbb{R}^d)}=\\
&\int\limits_{|x|>L}\Big(\int\limits_{|x-\varepsilon z|<\frac{L}{2}}\frac{1}{\varepsilon}\mu\big(\frac{x}{\varepsilon},\omega\big)a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\nabla u_0(x-\varepsilon z)dz\Big)^2dx\\
&<\alpha_2^4\Big(\frac{1}{\varepsilon^2}\int\limits_{|z|>\frac{L}{2\varepsilon}} a(z)\,dz\Big)^2\|\varepsilon\theta\big(\frac{y}{\varepsilon},\omega\big)\nabla u_0(y)\|^2_{L^2(\mathbb{R}^d)}\to 0;
\end{aligned}
\tag{111}
$$

Here we have also used the limit relation $\|\varepsilon\theta(\frac{y}{\varepsilon},\omega)\nabla u_0(y)\|_{L^2(\mathbb{R}^d)}\to 0$ that is ensured by Proposition 4.4. Denote $\chi_{<L}(x)=\chi_{\{|x|<L\}}(x)$ and represent the function $\phi_\varepsilon^{(2)}(x,\omega)\,\chi_{<L}(x)$ as follows:

$$
\phi_\varepsilon^{(2)}(x,\omega)\,\chi_{<L}(x)=\gamma_\varepsilon^{<}(x,\omega)+\gamma_\varepsilon^{>}(x,\omega),
\tag{112}
$$

where

$$
\begin{aligned}
\gamma_\varepsilon^{<}(x,\omega)&=\mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x)\int\limits_{|\varepsilon z|<2L} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\\
&\qquad\times\Big(\frac{1}{\varepsilon}\big(\nabla u_0(x-\varepsilon z)-\nabla u_0(x)\big)+z\nabla\nabla u_0(x)\Big)dz;\\
\gamma_\varepsilon^{>}(x,\omega)&=\mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x)\int\limits_{|\varepsilon z|>2L} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\\
&\qquad\times\Big(\frac{1}{\varepsilon}\big(\nabla u_0(x-\varepsilon z)-\nabla u_0(x)\big)+z\nabla\nabla u_0(x)\Big)dz.
\end{aligned}
\tag{113}
$$

Since $u_0\in C_0^\infty(\mathbb{R}^d)$, the Teylor decomposition applies to $\nabla u_0(x-\varepsilon z)$, and we get

$$
\frac{1}{\varepsilon}\big(\nabla u_0(x-\varepsilon z)-\nabla u_0(x)\big)+z\,\nabla\nabla u_0(x)=\frac{\varepsilon}{2}\nabla\nabla\nabla u_0(\xi)\,z\otimes z
$$

with some $\xi \in \operatorname{supp} u_0$, here the notation $\nabla\nabla\nabla u_0(\xi)\, z\otimes z$ is used for the vector function $(\nabla\nabla\nabla u_0(\xi)\, z\otimes z)^i = \partial_{x^j}\partial_{x^k}\partial_{x^i} u_0(\xi) z^j z^k$. Then the right-hand side of the first formula in (113) admits the estimate

$$
\begin{aligned}
&\mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x)\Big| \int\limits_{|\varepsilon z|<2L} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon}-z,\omega\big) \\
&\qquad\qquad \times \Big(\frac{1}{\varepsilon}\big(\nabla u_0(x-\varepsilon z)-\nabla u_0(x)\big)+z\nabla\nabla u_0(x)\Big)dz\Big| \qquad (114)\\
&\le \frac{\alpha_2^2}{2}\max|\nabla\nabla\nabla u_0| \int\limits_{\mathbb{R}^d} \varepsilon\big|\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\big|\,\chi_{<3L}(x-\varepsilon z)\,a(z)z^2\,dz.
\end{aligned}
$$

Taking into account the relation

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d} \varepsilon\big|\theta\big(\frac{x}{\varepsilon}-z,\omega\big)\big|\,\chi_{<3L}(x-\varepsilon z)\,a(z)z^2\,dz\Big)^2 dx \\
&= \int\limits_{\mathbb{R}^d} a(z_1)z_1^2dz_1 \int\limits_{\mathbb{R}^d} a(z_2)z_2^2dz_2 \qquad (115)\\
&\quad \times \int\limits_{\mathbb{R}^d} \varepsilon^2\big|\theta\big(\frac{x}{\varepsilon}-z_1,\omega\big)\big|\big|\theta\big(\frac{x}{\varepsilon}-z_2,\omega\big)\big|\chi_{<3L}(x-\varepsilon z_1)\chi_{<3L}(x-\varepsilon z_2)dx
\end{aligned}
$$

and applying the Cauchy-Schwartz inequality to the last integral on its right hand side we conclude with the help of Proposition 4.4 that $\|\gamma_\varepsilon^<(x,\omega)\|_{L^2(\mathbb{R}^d)} \to 0$ as $\varepsilon \to 0$.

If $|x| < L$ and $|\varepsilon z| > 2L$, then $|x-\varepsilon z| > L$, and $u_0(x-\varepsilon z) = 0$. The right-hand side of the second formula in (113) can be rearranged as follows:

$$
\begin{aligned}
\gamma_\varepsilon^>(x,\omega) &= \mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x) \int\limits_{|z|>\frac{2L}{\varepsilon}} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon}-z,\omega\big) \\
&\qquad\qquad \times \Big(-\frac{1}{\varepsilon}\nabla u_0(x)+z\nabla\nabla u_0(x)\Big)dz \\
&= \mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x) \int\limits_{|z|>\frac{2L}{\varepsilon}} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\big(\theta\big(\frac{x}{\varepsilon}-z,\omega\big)-\theta\big(\frac{x}{\varepsilon},\omega\big)\big) \qquad (116)\\
&\qquad\qquad \times \Big(-\frac{1}{\varepsilon}\nabla u_0(x)+z\nabla\nabla u_0(x)\Big)dz \\
&+ \mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x) \int\limits_{|z|>\frac{2L}{\varepsilon}} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\theta\big(\frac{x}{\varepsilon},\omega\big) \\
&\qquad\qquad \times \Big(-\frac{1}{\varepsilon}\nabla u_0(x)+z\,\nabla\nabla u_0(x)\Big)dz
\end{aligned}
$$

The second term on the right-hand side in (116) is estimated in the same way as the function $\phi_\varepsilon^{(2)}\,\chi_{|x|>L}$ in (111). Thus the $L^2(\mathbb{R}^d)$ norm of this term tends to 0 as $\varepsilon \to 0$.

The first term on the right-hand side of (116) admits the following upper bound:

$$
\begin{aligned}
&\Big|\mu\big(\frac{x}{\varepsilon},\omega\big)\chi_{<L}(x)\int\limits_{|z|>\frac{2L}{\varepsilon}} a(z)\mu\big(\frac{x}{\varepsilon}-z,\omega\big)\zeta_{-z}\big(T_{\frac{x}{\varepsilon}}\omega\big)\\
&\qquad\qquad\times\Big(-\frac{1}{\varepsilon}\nabla u_0(x)+z\,\nabla\nabla u_0(x)\Big)\,dz\Big|\\
&\le \alpha_2^2\int\limits_{|z|>\frac{2L}{\varepsilon}} a(z)\Big|\zeta_{-z}\big(T_{\frac{x}{\varepsilon}}\omega\big)\Big|\,\Big|-\frac{1}{\varepsilon}\nabla u_0(x)+z\,\nabla\nabla u_0(x)\Big|\,dz\\
&\le \alpha_2^2 C(L)\int\limits_{|z|>\frac{2L}{\varepsilon}} |z|a(z)\Big|\zeta_{-z}\big(T_{\frac{x}{\varepsilon}}\omega\big)\Big|dz\big(\big|\nabla u_0(x)\big|+\big|\nabla\nabla u_0(x)\big|\big).\\
&\le \alpha_2^2 C(L)\Big(\int\limits_{|z|>\frac{2L}{\varepsilon}} |z|^2a(z)dz\Big)^{\frac12}\Big(\int\limits_{\mathbb{R}^d} a(z)\big|\zeta_{-z}\big(T_{\frac{x}{\varepsilon}}\omega\big)\big|^2\,dz\Big)^{\frac12}\\
&\qquad\qquad\times\big(\big|\nabla u_0(x)\big|+\big|\nabla\nabla u_0(x)\big|\big).
\end{aligned}
\tag{117}
$$

Since $\zeta_{-z}(\omega)\in L^2_M$, we have

$$
\mathbf{E}\int\limits_{\mathbb{R}^d} a(z)|\zeta_{-z}(\omega)|^2\,dz<\infty.
$$

Taking into account the convergence

$$
\int\limits_{|z|>\frac{2L}{\varepsilon}} |z|^2a(z)dz\to 0,\quad \text{as } \varepsilon\to 0,
$$

by the Birkhoff ergodic theorem we obtain that the $L^2(\mathbb{R}^d)$ norm of the first term on the right-hand side of (116) tends to zero a.s., as $\varepsilon\to 0$. Therefore, $\|\gamma_\varepsilon^{>}(x,\omega)\|_{L^2(\mathbb{R}^d)}\to 0$ as $\varepsilon\to 0$.

From (112) it follows that $\|\phi_\varepsilon^{(2)}(x,\omega)\chi_{<L}(x)\|_{L^2(\mathbb{R}^d)}\to 0$ as $\varepsilon\to 0$, and together with (111) this implies that

$$
\|\phi_\varepsilon^{(2)}(x,\omega)\|_{L^2(\mathbb{R}^d)}\to 0 \quad \text{as}\quad \varepsilon\to 0. \tag{118}
$$

Finally, (107) follows from (108) and (118). Lemma is proved. □

7. Proof of the main results

We begin this section by proving relation (14) for $f\in\mathcal{S}_0(\mathbb{R}^d)$. For such f we have $u_0\in C_0^\infty(\mathbb{R}^d)$. It follows from (15), Proposition 4.4 and Lemmas 5.1, 5.5 that

$$
\|w^\varepsilon-u_0\|_{L^2(\mathbb{R}^d)}\to 0,\quad \text{as } \varepsilon\to 0. \tag{119}
$$

By the definition of v^ε, u_2^ε and u_3^ε,

$$
\begin{aligned}
(L^\varepsilon-m)w^\varepsilon&=(\hat L-m)u_0-m\varepsilon\theta\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla u_0+\phi_\varepsilon=f-m\varepsilon\theta\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla u_0+\phi_\varepsilon\\
&=(L^\varepsilon-m)u^\varepsilon-m\varepsilon\theta\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla u_0+\phi_\varepsilon.
\end{aligned}
$$

Therefore,

$$(L^\varepsilon - m)(w^\varepsilon - u^\varepsilon) = -m\varepsilon\theta\Big(\frac{x}{\varepsilon}\Big)\cdot\nabla u_0 + \phi_\varepsilon.$$

According to Proposition 4.4 and Lemma 6.1 the L^2 norm of the functions on the right-hand side of the last formula tends to zero as $\varepsilon\to 0$. Consequently,

$$\|w^\varepsilon - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \to 0, \quad \text{as } \varepsilon\to 0.$$

Combining this relation with (119) yields the desired relation (14) for $f\in\mathcal{S}_0(\mathbb{R}^d)$.

To complete the proof of Theorem 2.1 we should show that the last convergence holds for any $f\in L^2(\mathbb{R}^d)$.

For any $f\in L^2(\mathbb{R}^d)$ there exists $f_\delta\in\mathcal{S}_0$ such that $\|f-f_\delta\|_{L^2(\mathbb{R}^d)}<\delta$. Since the operators $(L^\varepsilon-m)^{-1}$ and $(\hat{L}-m)^{-1}$ are bounded uniformly in ε, then

$$\|u^\varepsilon_\delta - u^\varepsilon\|_{L^2(\mathbb{R}^d)} \le C_1\delta, \qquad \|u_{0,\delta}-u_0\|_{L^2(\mathbb{R}^d)} \le C_1\delta, \tag{120}$$

where

$$\begin{aligned} u^\varepsilon &= (L^\varepsilon - m)^{-1} f, \;\; u_0 = (\hat{L}-m)^{-1} f,\\ u^\varepsilon_\delta &= (L^\varepsilon - m)^{-1} f_\delta, \;\; u_{0,\delta} = (\hat{L}-m)^{-1} f_\delta. \end{aligned}$$

Recalling that $f_\delta\in\mathcal{S}_0$, we obtain $\|u^\varepsilon_\delta - u_{0,\delta}\|_{L^2(\mathbb{R}^d)}\to 0$. Therefore, by (120)

$$\overline{\lim_{\varepsilon\to 0}}\|u^\varepsilon - u_0\|_{L^2(\mathbb{R}^d)} \le 2C_1\delta$$

with an arbitrary $\delta>0$. This implies the desired convergence in (11) for an arbitrary $f\in L^2(\mathbb{R}^d)$ and completes the proof of the main theorem.

7.1. Proof of Corollary 2.1

Here we assume that the operator $L^{\varepsilon,\mathrm{ns}}$ is defined by (12). Multiplying equation (13) by $\rho^\varepsilon(x,\omega) = \rho\big(\frac{x}{\varepsilon},\omega\big) = \mu\big(\frac{x}{\varepsilon},\omega\big)\big(\lambda\big(\frac{x}{\varepsilon},\omega\big)\big)^{-1}$ we obtain

$$L^\varepsilon u_\varepsilon - m\rho^\varepsilon u_\varepsilon = \rho_\varepsilon f, \tag{121}$$

where the symmetrized operator L^ε is given by (7). Letting $\langle\rho\rangle = \mathbf{E}\boldsymbol{\rho} = \mathbf{E}\big(\frac{\boldsymbol{\mu}}{\boldsymbol{\lambda}}\big)$ we consider an auxiliary equation

$$L^\varepsilon g_\varepsilon - m\langle\rho\rangle g_\varepsilon = \langle\rho\rangle f. \tag{122}$$

By Theorem 2.1 the functions g_ε converge a.s. in $L^2(\mathbb{R}^d)$, as $\varepsilon\to 0$, to a solution of the equation $\hat{L}g - m\langle\rho\rangle g = \langle\rho\rangle f$. Our goal is to show that $\|g_\varepsilon - u_\varepsilon\|_{L^2(\mathbb{R}^d)}\to 0$ as $\varepsilon\to 0$. To this end we subtract equation (121) from (122). After simple rearrangements this yields

$$L^\varepsilon\alpha_\varepsilon - m\rho_\varepsilon\alpha_\varepsilon = \big(\langle\rho\rangle - \rho_\varepsilon\big)g_\varepsilon + \big(\langle\rho\rangle-\rho_\varepsilon\big)f, \tag{123}$$

with $\alpha_\varepsilon(x) = g_\varepsilon(x) - u_\varepsilon(x)$. In a standard way one can derive the following estimate

$$m\int\limits_{\mathbb{R}^d}(\alpha_\varepsilon(x))^2dx + \frac{1}{\varepsilon^2}\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} a(z)(\alpha_\varepsilon(x-\varepsilon z) - \alpha_\varepsilon(x))^2dzdx < C. \tag{124}$$

As was shown in the proof of Lemma 5.3, this estimate implies compactness of the family $\{\alpha_\varepsilon\}$ in $L^2(B)$ for any cube B. Multiplying (123) by α_ε and integrating the resulting relation over $\mathbb{R}^d$ we obtain

$$\|\alpha_\varepsilon\|^2_{L^2(\mathbb{R}^d)} \le C_1\big|\big((\langle\rho\rangle-\rho_\varepsilon)g_\varepsilon,\alpha_\varepsilon\big)_{L^2(\mathbb{R}^d)}\big| + \big|\big((\langle\rho\rangle-\rho_\varepsilon)f,\alpha_\varepsilon\big)_{L^2(\mathbb{R}^d)}\big| \tag{125}$$

By the Birkhoff ergodic theorem $(\langle\rho\rangle-\rho_\varepsilon)$ converges to zero weakly in $L^2_{\rm loc}(\mathbb{R}^d)$. Considering the boundedness of $(\langle\rho\rangle-\rho_\varepsilon)$ and the properties of α_ε and g_ε, and approximating f and g_ε by functions with a compact support, we conclude that both terms on the right-hand side in (125) tend to zero, as $\varepsilon\to 0$. So does $\|\alpha_\varepsilon\|^2_{L^2(\mathbb{R}^d)}$. Therefore, u_ε converges to the solution of equation $\hat{L}u - m\langle\rho\rangle u = \langle\rho\rangle f$. Dividing this equation by $\langle\rho\rangle$, we rewrite the limit equation as follows

$$\Big(\mathbf{E}\big\{\frac{\boldsymbol{\mu}}{\boldsymbol{\lambda}}\big\}\Big)^{-1}\Theta_{ij}\frac{\partial^2 u}{\partial x_i\partial x_j} - mu = f$$

with Θ defined in (103). This completes the proof of Corollary.

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