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Homogenization of random Navier–Stokes-type system for electrorheological fluid

Andrey Piatnitski [a,b,*], Vasily Zhikov [c]

[a] *Faculty of Technology, Narvik University College, Norway*
[b] *Lebedev Physical Institute RAS, Moscow, Russia*
[c] *Vladimir State University, Vladimir, Russia*

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Abstract

The paper deals with homogenization of Navier–Stokes-type system describing electrorheological fluid with random characteristics. Under non-standard growth conditions we construct the homogenized model and prove the convergence result. The structure of the limit equations is also studied.

1. Introduction

Rheological properties of some fluids might change essentially in the presence of an electromagnetic field. For such fluids the viscous stress tensor is not only a nonlinear function of the deformation velocity tensor, it also depends on the spatial argument. A collection of interesting experimental data as well as a number of mathematical models of electrorheological fluids can be found in [10].

In this work we assume that the driving electromagnetic field has a random statistically homogeneous microstructure. Then the viscous stress tensor of the fluid is getting a random rapidly oscillating function of the spatial variables. The corresponding system of equations takes the form (the so-called generalized Navier–Stokes equations)

* Corresponding author at: Narvik University College, P.O. Box 385, Narvik 8505, Norway.
E-mail addresses: andrey@sci.lebedev.ru (A. Piatnitski), zhikov@vlsu.ru (V. Zhikov).

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$$\begin{cases} \dfrac{\partial u^\varepsilon}{\partial t} - \operatorname{div}\Big(A\big(\dfrac{x}{\varepsilon}, Du^\varepsilon\big)\Big) + \operatorname{div}(u^\varepsilon \otimes u^\varepsilon) + \nabla\pi = 0, \quad \text{in } G \times (0,T), \\ \operatorname{div} u^\varepsilon = 0, \qquad u^\varepsilon|_{\partial G} = 0, \qquad u|_{t=0} = u_0, \end{cases} \tag{1}$$

where the viscous stress tensor $A(y, \xi)$ satisfies non-standard $p(\cdot)$-growth conditions which are specified in detail in the next section. In (1) u^ε denotes the fluid velocity field and Du^ε stands for its symmetrized gradient, π is the pressure, $\operatorname{div}(u^\varepsilon \otimes u^\varepsilon)$ is the nonlinear convective term, and $A(x, Du^\varepsilon)$ is the viscosity stress tensor of the fluid; ε is a small positive parameter that characterizes the microscopic length scale.

The goal of this work is to study the limit behavior of u^ε as $\varepsilon \to 0$. We assume that $A(y, \xi)$ is a symmetric matrix being a random ergodic statistically homogeneous function of $y \in \mathbb{R}^d$. In particular, the exponent $p(y)$ that characterizes the growth conditions of $A(y, \xi)$ might be a random statistically homogeneous function. Under a monotonicity assumption and certain conditions on p, we construct the effective model and prove the homogenization result. We show in particular that the homogenized system is deterministic.

Similar results in the periodic framework have been obtained in [11]. Qualitative theory of a generalized Navier–Stokes system was developed in [3] and [12].

Our approach relies on a priori estimates, monotonicity arguments, generalized div–curl lemma and ergodic theorems.

2. Problem setup

Given a Lipschitz bounded domain G in $\mathbb{R}^d$ we study initial–boundary problem (1) in $Q_T = G \times [0, T]$ for a fixed $T > 0$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space with a measure preserving dynamical system τ_y, $y \in \mathbb{R}^d$. We recall that τ_y is a group of measurable mappings $\tau_y : \Omega \mapsto \Omega$ such that

- $\tau_{y_1+y_2} = \tau_{y_1} \circ \tau_{y_2}$, $\tau_0 = Id$.
- $\mathbf{P}(\tau_y(\mathcal{Q})) = \mathbf{P}(\mathcal{Q})$ for any $\mathcal{Q} \in \mathcal{F}$ and any $y \in \mathbb{R}^n$.
- $\tau : \Omega \times \mathbb{R}^n \mapsto \Omega$ is measurable; we assume here that $\mathbb{R}^d$ is equipped with the Borel σ-algebra.

In what follows we assume that the dynamical system $\tau.$ is ergodic that is any function which is invariant with respect to $\tau.$ is equal to a constant almost surely (a.s.).

We also assume that Ω is a compact metric space and that τ is continuous with respect to this topology.

Now we set

$$A(y, \xi) = \mathbf{A}(\tau_y \omega, \xi)$$

where $\mathbf{A} = \mathbf{A}(\omega, \xi)$ possesses the following properties:

h1. $\mathbf{A} : \Omega \times \mathbf{M} \mapsto \mathbf{M}$, where $\mathbf{M}$ is the space of symmetric $d \times d$-matrices which is identified with $\mathbb{R}^{\frac{d(d+1)}{2}}$. We assume that $\mathbf{A}$ is a Carathéodory function, that is $\mathbf{A}$ is continuous in ξ for almost all $\omega \in \Omega$ and measurable in ω for any ξ.

h2. For all $\omega \in \Omega$ and $\xi_1 \neq \xi_2$

$$\big(\mathbf{A}(\omega, \xi_1) - \mathbf{A}(\omega, \xi_2), \xi_1 - \xi_2\big) > 0.$$

h3. There exists $c_0 > 0$ such that

$$\big(\mathbf{A}(\omega, \xi), \xi\big) \ge c_0 |\xi|^{p(\omega)} - (c_0)^{-1}.$$

h4. There exists $c_1 > 0$ such that

$$\big|\mathbf{A}(\omega, \xi)\big|^{p'(\omega)} \le c_1 |\xi|^{p(\omega)} + c_1, \qquad p'(\omega) = \frac{p(\omega)}{p(\omega) - 1},$$

where the random variable $p(\omega)$ satisfies the following estimates:

$$1 < \alpha \le p(\omega) \le \beta < \infty. \tag{2}$$

2.1. Functional spaces

We introduce here several functional spaces. We denote

$$C^\infty_{0,\mathrm{sol}}(G) = \{\psi \in C_0^\infty(G; \mathbb{R}^d), \operatorname{div} \psi = 0\},$$

and H is the closure of $C^\infty_{0,\mathrm{sol}}(G)$ in $L^2(G; \mathbb{R}^d)$ norm. We also define X^ε as the closure of the space $C^\infty([0,T]; C^\infty_{0,\mathrm{sol}}(G))$ in the Luxemburg norm

$$\|D\psi\|_{L^{p_\varepsilon}(Q_T)} = \inf\Big\{\lambda > 0 : \int\limits_{Q_T} \big|\lambda^{-1} D\psi\big|^{p_\varepsilon(x)}\, dx dt \le 1\Big\};$$

here $Q_T = G \times (0, T)$ and $p_\varepsilon(x) = p(\tau_{x/\varepsilon}\omega)$. Observe that the space X^ε depends on ω.

We say that a vector function $u \in X^\varepsilon \cap L^\infty((0,T); H)$ is a *weak solution* of problem (1) if

(i) for any $\varphi \in C^\infty_{0,\mathrm{sol}}$ and for any $t', t'' \in [0, T]$ the relation holds

$$\int\limits_G [u(x,t'') - u(x,t')] \cdot \varphi(x)\, dx + \int\limits_{t'}^{t''} \int\limits_G \Big[A\big(\frac{x}{\varepsilon}, Du\big) - u \otimes u\Big] \cdot D\varphi\, dx dt = 0;$$

(ii)

$$\lim_{t \to +0} \int\limits_G u(x,t) \cdot \varphi(x)\, dx = \int\limits_G u_0(x) \cdot \varphi(x)\, dx;$$

(iii) the energy inequality

$$\frac{1}{2} \int\limits_G [u(x,t'') \cdot u(x,t'') - u(x,t') \cdot u(x,t')]\, dx + \int\limits_{t'}^{t''} \int\limits_G A\Big(\frac{x}{\varepsilon}, Du\Big) \cdot Du\, dx dt \le 0$$

holds for almost all $t', t'' \in [0, T]$.

Notice that from the definition of a solution it follows that $(u(\cdot, t), \varphi)$ is a continuous function of t for any $\varphi \in C^\infty_{0,\mathrm{sol}}$. In other words, $u(\cdot, t)$ is a weakly continuous function of t with values in H. However, it does not imply the energy equality. The theory admits the strict energy inequality, which means the violation of energy conservation law.

The following statement has been proved in [12].

Theorem 1. *Assume that*

$$\alpha \geq \alpha_0(d) = \max\left\{\frac{d + \sqrt{3d^2 + 4d}}{d+2}, \frac{3d}{d+2}\right\}$$

and

$$\alpha \leq p(x) \leq \beta < \infty.$$

Then generalized Navier–Stokes system (1) has a weak solution for any $u_0 \in H$.

Remark 1. In dimension $d = 3$ we have $\alpha_0(3) \in (1.84, 1.85)$.

The condition $\alpha \geq \alpha_0$ ensures that the convective term $u \otimes u$ can be estimated in terms of the viscous term. More precisely, the following statement holds.

Lemma 2.1. *If $u \in X \cap L^\infty(0, T, H)$, then*

$$|u|^2 \in L^1(0, T, L^{\alpha'}(G)).$$

Remark 2. In the classical case we have $p = \frac{3d+2}{d+2}$, see [7,8]. Notice that if $\alpha = \frac{3d+2}{d+2}$ then

$$|u|^2 \in L^{\alpha'}(0, T, L^{\alpha'}(G)) = L^{\alpha'}(Q_T);$$

here $\alpha' = \frac{\alpha}{\alpha-1}$. In this case the convective term is *completely* subjected to viscous one.

Due to Theorem 1, for each $\varepsilon > 0$ problem (1) has a solution. Our goal is to study the limit behavior of these solutions as $\varepsilon \to 0$.

The following sections deal with the homogenization procedure. This procedure relies on a number of auxiliary cell problems and the corresponding functional spaces. We introduce these spaces here.

We denote by $L^{p(\cdot)}(\Omega, \mathbb{R}^{d(d+1)/2})$ the space of functions defined on Ω with values in the space of $d \times d$ symmetric matrices and such that

$$\int\limits_{\Omega} |\phi(\omega)|^{p(\omega)}\, d\mathbf{P}(\omega) < \infty.$$

This space is equipped with the corresponding Luxemburg norm

$$\|\phi\|_{L^{p(\cdot)}(\Omega,\mathbb{R}^{d(d+1)/2})} = \inf\Big\{\lambda > 0 : \int\limits_\Omega |\lambda^{-1}\phi(\omega)|^{p(\omega)}\,d\mathbf{P}(\omega) \le 1\Big\}.$$

As an immediate consequence of the properties of dynamical system τ and the Fubini theorem we have

Lemma 2.2. *Let $\phi \in L^{p(\cdot)}(\Omega,\mathbb{R}^{d(d+1)/2})$. Then a.s. $\phi(\tau_x\omega) \in L^{p(\tau_x\omega)}_{\mathrm{loc}}(\mathbb{R}^d,\mathbb{R}^{d(d+1)/2})$. Moreover,*

$$\mathbf{E}\int\limits_S |\phi(\tau_x\omega)|^{p(\tau_x\omega)}dx = |S|\int\limits_\Omega |\phi(\omega)|^{p(\omega)}\,d\mathbf{P}(\omega)$$

for any bounded Borel set $S \subset \mathbb{R}^d$.

We now denote by ∂_i and $\mathcal{D}_i$ the generator of τ in the i-th coordinate direction and its domain in $L^2(\Omega)$, respectively. We also set $\mathcal{D} = \bigcap\limits_{i=1}^d \mathcal{D}_i$ and

$$\mathcal{D}^\infty = \{\varphi \in L^\infty(\Omega) : \partial_{i_1},\dots,\partial_{i_k}\varphi \in L^2(\Omega) \text{ for all } i_1,\dots,i_k\}.$$

The set $\mathcal{D}^\infty$ is dense in $L^p(\Omega)$ for any $p > 1$. The realizations of functions from $\mathcal{D}^\infty$ are a.s. smooth functions, see [5].

Denote $\mathcal{G}(\Omega)$ the closure of $\{D_\omega\phi : \phi \in (\mathcal{D}^\infty)^d, \operatorname{div}_\omega\phi = 0\}$ in $L^{p(\cdot)}(\Omega;\mathbb{R}^{d(d+1)/2})$, where $(D_\omega\phi)_{ij} = \frac{1}{2}(\partial_i\phi_j + \partial_j\phi_i)$, and $\operatorname{div}_\omega\phi = \partial_1\phi_1 + \ldots + \partial_d\phi_d$. We then define

$$\mathcal{G}^\perp(\Omega) = \Big\{\theta \in L^{p'(\cdot)}(\Omega;\mathbb{R}^{d(d+1)/2}) : \int\limits_\Omega \theta\cdot v\,d\mathbf{P}(\omega) = 0 \text{ for all } v \in \mathcal{G}(\Omega)\Big\}.$$

3. Homogenization

In this section we prove a number of auxiliary statements and formulate the homogenization result. From item (iii) of the definition of a solution to problem (1) it follows that for each $\varepsilon > 0$ and each $\omega \in \Omega$ we have

$$\sup_{0\le t\le T}\|u^\varepsilon(\cdot,t)\|^2_{L^2(G;\mathbb{R}^d)} + \int\limits_0^t\int\limits_G |Du^\varepsilon(x,s)|^{p_\varepsilon(x)}\,dxds \le C\|u_0\|^2_{L^2(G;\mathbb{R}^d)} \tag{3}$$

with a deterministic constant C. We recall that $p_\varepsilon(x) = p(\tau_{x/\varepsilon}\omega)$. Considering **h3.**, **h4.** and (2) we derive from (3)

Lemma 3.1. *For each $\omega \in \Omega$ the sequence Du^ε is bounded in $L^\alpha(Q_T;\mathbb{R}^{d(d+1)/2})$, and the sequence $A^\varepsilon = A(x/\varepsilon, Du^\varepsilon)$ is bounded in $L^{\beta'}(Q_T;\mathbb{R}^d)$.*

Using the standard arguments (see [12, Section 5]), one can show that $\{u^\varepsilon(\cdot,t)\}$ is a family of weakly equicontinuous functions $[0,T]\mapsto L^2(G;\mathbb{R}^{d(d+1)/2})$. Moreover, by the Aubin–Lions lemma, this family is compact in $L^2(Q_T;\mathbb{R}^d)$. This yields the following convergence result.

Lemma 3.2. *For* **P**-*almost all* ω, *for a subsequence, as* $\varepsilon\to 0$,

$$
\begin{aligned}
u^\varepsilon(\cdot,t) &\rightharpoonup u(\cdot,t) && \text{weakly in } L^2(G;\mathbb{R}^d) \quad \text{for all } t\in[0,T];\\
u^\varepsilon(\cdot,t) &\to u(\cdot,t) && \text{in } L^2(G;\mathbb{R}^d) \quad \text{for a.a. } t\in[0,T];\\
Du^\varepsilon &\rightharpoonup Du && \text{weakly in } L^\alpha(Q_T;\mathbb{R}^{d(d+1)/2});\\
A\Big(\frac{\cdot}{\varepsilon},Du^\varepsilon\Big) &\rightharpoonup z^0 && \text{weakly in } L^{\beta'}(Q_T;\mathbb{R}^{d(d+1)/2}).
\end{aligned}
$$

Notice that $u=u(x,t)$ and $z^0=z^0(x,t)$ might depend on ω.

Passing to the limit in the integral identity (i) we obtain

$$
\int_G [u(x,t'')-u(x,t')]\cdot\varphi(x)\,dx+\int_{t'}^{t''}\int_G \big[z^0-u\otimes u\big]\cdot D\varphi\,dxdt=0 \tag{4}
$$

for any $\varphi\in C^\infty_{0,\mathrm{sol}}(G)$ and for any $t',t''\in[0,T]$. The crucial step now is to determine a relation between z^0 and Du. To this end we consider the following auxiliary problem: given $\xi\in\mathbb{R}^{d(d+1)/2}$ find $v_\xi\in\mathcal{G}(\Omega)$ such that

$$
\int_\Omega \mathbf{A}(\omega,v_\xi(\omega)+\xi)\cdot\theta(\omega)\,d\mathbf{P}(\omega)=0 \quad \text{for any } \theta\in\mathcal{G}(\Omega). \tag{5}
$$

Lemma 3.3. *Under assumptions* **h1.–h4.** *problem (5) has a unique solution for each* $\xi\in\mathbb{R}^{d(d+1)/2}$.

Proof. The proof of Lemma 3.3 relies on classical result for monotone operators. Denote by $\mathcal{A}_\xi$ the operator mapping $\mathcal{G}(\Omega)$ to $\mathcal{G}^\perp(\Omega)$ and defined by $\mathcal{A}_\xi[\theta](\omega)=A(\omega,\xi+\theta(\omega))$. Due to assumption **h2.** this operator is monotone. From **h4.** it follows that $\mathcal{A}_\xi$ is bounded. Then, from **h1.** and **h4.** with the help of Lebesgue theorem one can derive that the function

$$
s\longrightarrow\int_\Omega \mathbf{A}(\omega,\xi+\theta_1(\omega)+s\theta_2(\omega))\cdot\theta_3(\omega)\,d\mathbf{P}(\omega)
$$

is continuous in $s\in\mathbb{R}$ for any $\theta_1,\theta_2,\theta_3\in\mathcal{G}(\Omega)$. Also, as an immediate consequence of **h3.**, we have $\|\theta\|^{-1}(\mathcal{A}_\xi(\theta),\theta)\to\infty$, as $\|\theta\|\to\infty$. Then, by [8, Theorem 2.2.1] problem (5) has a unique solution. $\square$

Remark 3. Notice that the proof of Lemma 3.3 relies on assumptions **h1.–h4.** only, it does not use ergodic properties of the dynamical system τ_x.

The homogenized diffusion tensor is now introduced by

$$A^{\mathrm{eff}}(\xi)=\int\limits_{\Omega}\mathbf{A}(\omega,\xi+v_\xi(\omega))\,d\mathbf{P}(\omega).$$

Consider an auxiliary variational problem

$$f(\xi)=\min_{v\in\mathcal{G}(\Omega)}\int\limits_{\Omega}\frac{|\xi+v(\omega)|^{p(\omega)}}{p(\omega)}\,d\mathbf{P}(\omega). \tag{6}$$

The conjugate (in the sense of Young) functional takes the form

$$f^*(\xi)=\Big\{\int\limits_{\Omega}\frac{|w|^{p'(\omega)}}{p'(\omega)}\,d\mathbf{P}(\omega)\,:\,w\in\mathcal{G}^{\perp}(\Omega),\ \int\limits_{\Omega}w\,d\mathbf{P}(\omega)=\xi\Big\}.$$

Both functionals f and f^* are convex, continuous and even. Moreover, $f(\xi)>0$ for $\xi\neq 0$, and $f^*(\xi)>0$ for $\xi\neq 0$.

Lemma 3.4. *Function $f(\xi)$ satisfies the following inequality*

$$f(\lambda\xi)\le\begin{cases}\lambda^{\alpha}f(\xi), & \text{if } \lambda\le 1,\\ \lambda^{\beta}f(\xi), & \text{if } \lambda\ge 1.\end{cases}$$

Proof. Denote w_ξ the function in $\mathcal{G}(\Omega)$ that provides the minimum in (6). We have

$$f(\lambda\xi)=\int\limits_{\Omega}\frac{|\lambda\xi+w_{\lambda\xi}(\omega)|^{p(\omega)}}{p(\omega)}\,d\mathbf{P}\le\int\limits_{\Omega}\frac{|\lambda\xi+\lambda w_{\xi}(\omega)|^{p(\omega)}}{p(\omega)}\,d\mathbf{P}$$
$$\le\int\limits_{\Omega}\lambda^{p(\omega)}\frac{|\xi+w_{\xi}(\omega)|^{p(\omega)}}{p(\omega)}\,d\mathbf{P}.$$

This implies the desired inequality. □

Let $L^f(Q_T)$ be the associated with f Orlicz space defined as

$$L^f(Q_T)=\Big\{\phi\in L^1(Q_T,\mathbb{R}^{d(d+1)/2})\,:\,\int\limits_{Q_T}f(\phi(x))\,dx<\infty\Big\}$$

with the norm

$$\|\phi\|_{L^f}=\inf\Big\{\lambda>0\,:\,\int\limits_{Q_T}f(\lambda^{-1}\phi)\,dx\le 1\Big\}.$$

We also need the following Sobolev–Orlicz spaces:

$$W_0^{1,f}(G) = \{\phi \in W_0^{1,1}(G) : \operatorname{div}\phi = 0,\ f(D\phi) \in L^1(G)\},$$
$$\|\phi\|_{W_0^{1,f}(G)} = \|D\phi\|_{L^f(G)},$$

and

$$X^f(Q_T) = \{\vartheta \in L^1((0,T), W_0^{1,1}(G;\mathbb{R}^d)) : \operatorname{div}_x \vartheta = 0,\ f(D\vartheta) \in L^1(Q_T)\},$$
$$\|\vartheta\|_{X^f(Q_T)} = \|D\vartheta\|_{L^f(Q_T)}.$$

The following statement has been proved in [4, Proposition X.2.6].

Lemma 3.5. *The space $C_{0,\mathrm{sol}}^\infty(G)$ is dense in $W_0^{1,f}(G)$, and the space $C^\infty([0,T], C_{0,\mathrm{sol}}^\infty(G))$ is dense in $X^f(Q_T)$.*

For star-shaped domains this result can be easily proved with the help of smoothing operators. For a generic Lipschitz domain the proof is more involved.

The properties of homogenized diffusion tensor A^{eff} are given in the following statement.

Lemma 3.6. *The homogenized tensor A^{eff} is strictly monotone and continuous. Moreover, the flux $\mathbf{A}(\xi + v_\xi(\cdot))$ is a weakly continuous function of ξ with values in $L^{p'(\cdot)}(\Omega, \mathbb{R}^{d(d+1)/2})$. There exist $c_0 > 0$ and $c_1 > 0$ such that*

$$\begin{aligned} &A^{\mathrm{eff}}(\xi)\cdot\xi \ge c_0 f(\xi) - c_0^{-1},\\ &f^*\big(A^{\mathrm{eff}}(\xi)\big) \le c_1 f(\xi) + c_1. \end{aligned} \tag{7}$$

Proof. Considering problem (5) and **h3.**, we have

$$A^{\mathrm{eff}}(\xi)\cdot\xi = \int_\Omega \mathbf{A}(\omega, \xi + v_\xi(\omega))\cdot\xi\, d\mathbf{P} = \int_\Omega \mathbf{A}(\omega, \xi + v_\xi(\omega))\cdot(\xi + v_\xi(\omega))\, d\mathbf{P}$$
$$\ge c_0 \int_\Omega |\xi + v_\xi(\omega)|^{p(\omega)}\, d\mathbf{P} - c_0^{-1} \ge c_0 f(\xi) - c_0^{-1}.$$

This gives the first inequality in (7). To justify the second one we notice that $\mathbf{A}(\xi + v_\xi) \in \mathcal{G}^\perp(\Omega)$, and $\int_\Omega \mathbf{A}(\xi + v_\xi)\, d\mathbf{P} = A^{\mathrm{eff}}(\xi)$. Therefore, by the definition of f^*,

$$\begin{aligned} f^*(A^{\mathrm{eff}}(\xi)) &\le \int_\Omega |\mathbf{A}(\omega, \xi + v_\xi(\omega))|^{p'(\omega)}\, d\mathbf{P}\\ &\le c_2 \int_\Omega |\mathbf{A}(\omega, \xi + v_\xi(\omega))\cdot(\omega, \xi + v_\xi(\omega))\, d\mathbf{P} + c_3\\ &= c_2 A^{\mathrm{eff}}(\xi)\cdot\xi + c_3 \le c_2\big(\gamma f^*(A^{\mathrm{eff}}(\xi)) + C(\gamma) f(\xi)\big) + c_3; \end{aligned}$$

here we have also used **h4.**, **h3.**, the Young inequality and Lemma 3.4. Choosing in the last expression $\gamma = (2c_2)^{-1}$, we obtain the second estimate in (7).

Strict monotonicity of $A^{\rm eff}(\xi)$ is an immediate consequence of the strict monotonicity of $\mathbf{A}(\omega,\xi)$ and the definition of $A^{\rm eff}$. Indeed,

$$\begin{aligned}
&(A^{\rm eff}(\xi_1) - A^{\rm eff}(\xi_2))\cdot(\xi_1-\xi_2)\\
&\quad = \int\limits_\Omega \big(\mathbf{A}(\omega,\xi_1+v_{\xi_1}(\omega)) - \mathbf{A}(\omega,\xi_2+v_{\xi_2}(\omega))\big)\cdot(\xi_1-\xi_2)\,d\mathbf{P}\\
&\quad = \int\limits_\Omega \big(\mathbf{A}(\omega,\xi_1+v_{\xi_1}(\omega)) - \mathbf{A}(\omega,\xi_2+v_{\xi_2}(\omega))\big)\cdot(\xi_1+v_{\xi_1}(\omega)-(\xi_2+v_{\xi_2}(\omega)))\,d\mathbf{P} > 0.
\end{aligned}$$

In order to prove weak continuity of $A(\xi+v_\xi(\cdot))$ we first show that $v_\xi(\cdot)$ is a weakly continuous in ξ function with values in $L^{p(\cdot)}(\Omega,\mathbb{R}^d)$. To this end we consider a sequence ξ_j that converges to ξ and notice that, due to condition **h3.**, we have $\|v_{\xi_j}\|_{L^{p(\cdot)}} \le C$. Then for a subsequence v_{ξ_j} converges to some $\eta\in\mathcal{G}(\Omega)$ weakly in $L^{p(\cdot)}(\Omega,\mathbb{R}^d)$. By monotonicity, for any $\zeta\in\mathcal{G}(\Omega)$ it holds

$$\int\limits_\Omega \mathbf{A}(\omega,\xi_j+\zeta)\cdot(v_{\xi_j}-\zeta)\,d\mathbf{P} = \int\limits_\Omega \big(\mathbf{A}(\omega,\xi_j+\zeta)-\mathbf{A}(\omega,\xi_j+v_{\xi_j})\big)\cdot(v_{\xi_j}-\zeta)\,d\mathbf{P} \le 0.$$

From **h1.** and **h4.** we deduce by the Lebesgue theorem that $\mathbf{A}(\omega,\xi_j+\zeta)\to\mathbf{A}(\omega,\xi+\zeta)$ strongly in $L^{p'(\cdot)}(\Omega,\mathbb{R}^{d(d+1)/2})$. Passing to the limit $j\to\infty$ in the last inequality yields

$$\int\limits_\Omega \mathbf{A}(\omega,\xi+\zeta(\omega))\cdot(\eta-\zeta(\omega))\,d\mathbf{P} \le 0.$$

This implies with the help of Minty's argument that η is a solution of problem (5). Since a solution of (5) is unique, $\eta = v_\xi$. Therefore, v_{ξ_j} converges to v_ξ.

Denote by z a weak limit (for a subsequence) of $\mathbf{A}(\cdot,\xi_j+v_{\xi_j}(\cdot))$, as $j\to\infty$. Since $z\in\mathcal{G}^\perp(\Omega)$,

$$\int\limits_\Omega z\cdot v_\xi\,d\mathbf{P} = 0, \qquad \int\limits_\Omega z\cdot\zeta\,d\mathbf{P} = 0 \quad \text{for all } \zeta\in\mathcal{G}(\Omega).$$

By monotonicity,

$$\int\limits_\Omega \big(\mathbf{A}(\omega,\xi_j+v_{\xi_j}(\omega))-\mathbf{A}(\omega,\xi_j+\zeta(\omega))\big)\cdot(v_{\xi_j}(\omega)-\zeta(\omega))\,d\mathbf{P} \ge 0.$$

Passing to the limit $j\to\infty$ we get

$$\int_{\Omega} \big(z - \mathbf{A}(\omega, \xi + \zeta(\omega))\big) \cdot (v_\xi(\omega) - \zeta(\omega))\, d\mathbf{P} \ge 0.$$

Using one more time Minty's technique we conclude that $z = \mathbf{A}(\omega, \xi + v_\xi(\omega))$. □

The homogenized problem reads

$$\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\big(A^{\mathrm{eff}}(Du)\big) + \operatorname{div}(u \otimes u) + \nabla \pi = 0, & (x,t) \in Q_T, \\ \operatorname{div} u = 0, \qquad u|_{\partial G} = 0, \qquad u|_{t=0} = u_0. \end{cases} \tag{8}$$

We say that a vector function $u \in X^f(Q_T) \cap L^\infty((0,T), H)$ is a solution of problem (8) if

(i) for any $\varphi \in C^\infty_{0,\mathrm{sol}}(G)$ and for any $t', t'' \in [0,T]$ it holds

$$\int_G [u(x,t'') - u(x,t')] \cdot \varphi(x)\, dx + \int_{t'}^{t''} \int_G \big[A^{\mathrm{eff}}(Du) - u \otimes u\big] \cdot D\varphi\, dx dt = 0;$$

(ii)

$$\lim_{t \to +0} \int_G u(x,t) \cdot \varphi(x)\, dx = \int_G u_0(x) \cdot \varphi(x)\, dx;$$

(iii) the inequality

$$\frac{1}{2} \int_G [u(x,t'') \cdot u(x,t'') - u(x,t') \cdot u(x,t')]\, dx + \int_{t'}^{t''} \int_G A^{\mathrm{eff}}(Du) \cdot Du\, dx dt \le 0$$

holds for almost all $t', t'' \in [0,T]$.

We proceed with the main homogenization result of this work.

Theorem 2. *Assume that*

$$\beta < \alpha^* = \begin{cases} \dfrac{\alpha d}{d - \alpha}, & \text{if } \alpha < d, \\ +\infty, & \text{if } \alpha \ge d. \end{cases}$$

Then almost surely, as $\varepsilon \to 0$, any limit point u of the family u^ε is a solution of the homogenized problem (8).

Remark 4. Notice that the previous theorem does not state that the limit function is deterministic. Although the limit problem is not random, a solution need not be unique. Then, the limit points of u^ε might be distinct for different realizations.

4. Stochastic two-scale convergence

We first recall the definition of stochastic two-scale convergence. Let $\{v^\varepsilon = v^\varepsilon(x,t,\widetilde{\omega}),\ 0 < \varepsilon \le \varepsilon_0\}$ be a family of functions such that for **P** almost all $\widetilde{\omega} \in \Omega$ we have $v^\varepsilon(\cdot,\cdot,\widetilde{\omega}) \in L^p(Q_T)$ for all $\varepsilon \in (0,\varepsilon_0]$.

Definition 4.1. We say that the family $v^\varepsilon \in L^p(Q_T)$ weakly stochastic two-scale converges, as $\varepsilon \to 0$, to a function $v = v(x,t,\omega)$, $v \in L^p(Q_T \times \Omega)$, if a.s.

$$\limsup_{\varepsilon\to 0} \|v_\varepsilon\|_{L^p(Q_T)} < \infty, \tag{9}$$

and for any $\varphi \in C_0^\infty(Q_T) \times \mathcal{D}^\infty(\Omega)$ it holds

$$\lim_{\varepsilon\to 0} \int\limits_{Q_T} v^\varepsilon(x,t)\varphi^\varepsilon(x,t)\,dxdt \longrightarrow \int\limits_{Q_T}\int\limits_{\Omega} v(x,t,\omega)\varphi(x,t,\omega)\,dxdtd\mathbf{P},$$

where $\varphi^\varepsilon(x,t) = \varphi(x,t,\tau_{x/\varepsilon}\omega)$.

We emphasize that in the above definition the functions v^ε need not be statistically homogeneous.

Notice that the two-scale limit function might also depend on the realization of the medium $\widetilde{\omega}$. Observe also that although the two-scale limit is defined separately for each typical realization of the medium, that is for a given $\widetilde{\omega}$, the limit function is defined on the whole Ω. We do not indicate the dependence on $\widetilde{\omega}$ explicitly.

We recall some of the main properties of stochastic two-scale convergence (see [13]) that are used in the further analysis. For the reader convenience we provide a proof of these statements. It should be noted that the proof of these statements relies on the ergodicity of τ_x.

Lemma 4.1. *Every family of functions $\{v^\varepsilon, \varepsilon > 0\}$ such that (9) holds, weakly two-scale converges for a subsequence to some $v = v(x,t,\omega)$, $v \in L^p(Q_T \times \Omega)$.*

Proof. With the help of the Birkhoff ergodic theorem we obtain that for any $\varphi \in C_0^\infty(Q_T)$, $\phi \in \mathcal{D}^\infty(\Omega)$ and for almost all $\widetilde{\omega} \in \Omega$

$$\limsup_{\varepsilon\to 0}\Bigg|\int\limits_{Q_T} v^\varepsilon(x)\varphi(x)\phi(\tau_{\frac{x}{\varepsilon}}\tilde{\omega})dx\Bigg| \le$$

$$\le \limsup_{\varepsilon\to 0}\|v^\varepsilon\|_{L^p(Q_T)}\Bigg(\int\limits_{Q_T}|\varphi(x)|^q\,|\phi(\tau_{\frac{x}{\varepsilon}}\tilde{\omega})|^q dx\Bigg)^{\frac{1}{q}} \le$$

$$\le C_{\tilde{\omega}}\lim_{\varepsilon\to 0}\Bigg(\int\limits_{Q_T}|\varphi(x)|^q\phi(\tau_{\frac{x}{\varepsilon}}\tilde{\omega})|^q dx\Bigg)^{\frac{1}{2}} = C_{\tilde{\omega}}\Bigg(\int\limits_{Q_T}\int\limits_{\Omega}|\varphi(x)|^q|\phi(\omega)|^q d\mathbf{P}(\omega)dx\Bigg)^{\frac{1}{q}}.$$

Using the diagonal procedure we can choose a subsequence $\varepsilon_j \to 0$ such that the limit $\lim\limits_{\varepsilon_j\to 0}\int\limits_{Q_T} v^\varepsilon(x)\varphi(x)\phi(\tau_{\frac{x}{\varepsilon}}\tilde\omega)dx$ exists for each φ and ϕ. It immediately follows from the last formula that this limit defines a linear bounded functional on $L^q(Q_T\times\Omega)$. Therefore, there exists a function $v\in L^p(Q_T\times\Omega)$ such that

$$\lim_{\varepsilon\to 0}\int\limits_{Q_T} v^\varepsilon(x)\varphi(x)\phi(\tau_{\frac{x}{\varepsilon}}\tilde\omega)dx = \int\limits_{Q_T}\int\limits_{\Omega} v(x,t,\omega)\varphi(x)\phi(\omega)dxd\mathbf{P}.$$

By the density arguments the last relation also holds for any test function $\varphi\in C_0^\infty(Q_T)\times\mathcal{D}^\infty(\Omega)$. This completes the proof. $\square$

Lemma 4.2. *Let a family v^ε be such that a.s.*

$$\|v^\varepsilon\|_{L^p(Q_T)}\le C,\qquad \lim_{\varepsilon\to 0}\varepsilon\|\nabla_x v^\varepsilon\|_{L^p(Q_T)}=0.$$

Then, for a subsequence,

$$v^\varepsilon \xrightharpoonup{2} v \qquad \textit{weakly two-scale in } L^p(Q_T),$$

with $v=v(x,t)$, $v\in L^p(Q_T)$.

Proof. Choosing a test function of the form $\varphi(x,t)\phi(\tau_{x/\varepsilon}\omega)$, we get for a subsequence

$$0=\lim_{\varepsilon\to 0}\int\limits_{Q_T}\varepsilon\nabla_x v^\varepsilon(x,t)\varphi(x,t)\phi(\tau_{x/\varepsilon}\omega)\,dx = -\lim_{\varepsilon\to 0}\int\limits_{Q_T} v^\varepsilon(x,t)\varphi(x,t)\,\mathrm{div}_\omega\,\phi(\tau_{x/\varepsilon}\omega)\,dx$$

$$=-\int\limits_{Q_T}\int\limits_{\Omega} v(x,t,\omega)\varphi(x,t)\,\mathrm{div}_\omega\,\phi(\omega)dxd\mathbf{P}.$$

Therefore, for almost all $(x,t)\in Q_T$ we have

$$\int\limits_{\Omega} v(x,t,\omega)\,\mathrm{div}_\omega\,\phi(\omega)d\mathbf{P}.$$

In the same way as in [13, Lemma 2.5] one can show that the set $\{\mathrm{div}_\omega\,\phi : \phi\in\mathcal{D}^\infty\}$ is dense in the space of $L^q(\Omega)$ functions with zero average. Therefore, v does not depend on ω. $\square$

Lemma 4.3. *Let a family v^ε satisfy a.s. the estimate*

$$\|v^\varepsilon\|_{L^p(Q_T)}+\|\nabla_x v^\varepsilon\|_{L^p(Q_T)}\le C$$

for all $\varepsilon\in(0,\varepsilon_0]$. Then, for a subsequence,

$$\nabla_x v^\varepsilon \xrightharpoonup{2} \nabla_x v(x,t)+v_1(x,t,\omega)\qquad \textit{weakly two-scale in } L^p(Q_T\times\Omega),$$

with $v = v(x,t)$, $v \in L^p((0,T); W^{1,p}(G))$ *and* $v_1 \in L^p(Q_T; L^p_{\mathrm{pot}}(\Omega))$, *where* $L^p_{\mathrm{pot}}(\Omega)$ *is the closure in* $L^p(\Omega)$ *of the set* $\{\partial_\omega u : u \in \mathcal{D}^\infty(\Omega)\}$.

Proof. According to the previous lemma a two-scale limit of v^ε does not depend on ω. Denote by $V = V(x,t,\omega)$ the two-scale limit of $\nabla_x v^\varepsilon$, and by $v = v(x,t)$ the two-scale limit of v^ε. Since the two-scale convergence in $L^p(Q_T \times \Omega)$ implies the weak convergence in $L^p(Q_T)$, we have $v \in L^p(0,T; W^{1,p}(Q))$. Taking a test function $\varphi(x,t)\phi(\tau_{x/\varepsilon}\omega)$ with $\mathrm{div}_\omega \phi = 0$, we arrive at the following relation

$$\begin{aligned}\int\limits_{Q_T}\int\limits_{\Omega} V(x,t,\omega)\varphi(x,t)\phi(\omega)dxd\mathbf{P} &= \lim_{\varepsilon\to 0}\int\limits_{Q_T} \nabla_x v^\varepsilon(x,t)\varphi(x,t)\phi(\tau_{x/\varepsilon}\omega)dx\\ &= -\int\limits_{Q_T}\int\limits_{\Omega} v(x,t)\nabla_x\varphi(x,t)\phi(\omega)dxd\mathbf{P}\\ &= \int\limits_{Q_T}\int\limits_{\Omega} \nabla_x v(x,t)\varphi(x,t)\phi(\omega)dxd\mathbf{P}.\end{aligned}$$

Denoting $v_1(x,t,\omega) = V(x,t,\omega) - \nabla_x v(x,t)$ we conclude that for almost all $(x,t) \in Q_T$ and for any $\phi \in \mathcal{D}^\infty$ such that $\mathrm{div}_\omega \phi = 0$ it holds

$$\int\limits_{\Omega} v_1(x,t,\omega)\phi(\omega)d\mathbf{P} = 0.$$

This implies the desired statement. □

Example *(Periodic case).* The periodic framework can be interpreted as a particular case of the random one. In this case $\Omega = [0,1)^d$, $\mathcal{F}$ is the Borel σ-algebra on Ω, and $\mathbf{P}$ is the Lebesgue measure. The dynamical system τ_y is the set of shifts on the torus, that is for any $\omega \in [0,1)^d$ we set $\tau_y\omega = \mathcal{I}(\omega + y)$, where $\mathcal{I}(\omega+y) \in [0,1)^d$, and $(\omega+y) - \mathcal{I}(\omega+y) \in \mathbb{Z}^d$. One can observe that in the periodic case for any ω_1 and ω_2 there exists $y \in \mathbb{Z}^d$ such that $\omega_2 = \tau_y\omega_1$. This property plays a crucial role in the analysis of periodic media.

In the periodic case Lemmas 4.1–4.3 are classical and can be found in [9,1].

Considering a priori estimate (3) and using the arguments from [13] and [11,12], one can justify the following statement:

Proposition 4.1. *For a subsequence,*

$$u^\varepsilon \overset{2}{\rightharpoonup} u(x,t) \qquad \textit{weakly two-scale in } L^\alpha(Q_T),$$

$$Du^\varepsilon \overset{2}{\rightharpoonup} Du(x,t) + u_1(x,t,\omega) \qquad \textit{weakly two-scale in } L^\alpha(Q_T \times \Omega),$$

where $u_1(x,t,\cdot) \in \mathcal{G}(\Omega)$ *a.a. in* Q_T *and*

$$\int\limits_{Q_T}\int\limits_{\Omega} |Du(x,t)+u_1(x,t,\omega)|^{p(\omega)}\,dxdtd\mathbf{P}(\omega)<\infty;$$

$$A(\cdot/\varepsilon, Du^\varepsilon) \xrightharpoonup{2} z(x,t,\omega) \qquad \textit{weakly two-scale in } L^{\beta'}(Q_T\times\Omega), \tag{10}$$

where

$$\int\limits_{Q_T}\int\limits_{\Omega} |z(x,t,\omega)|^{p'(\omega)}\,dxdtd\mathbf{P}(\omega)<\infty,$$

$z(x,t,\cdot)\in\mathcal{G}^\perp(\Omega)$ *a.a. in* Q_T. *Moreover,* $z_0(x,t)=\int_\Omega z(x,t,\omega)\,d\mathbf{P}(\omega)$ *with* z_0 *introduced in Lemma 3.2.*

Proof. The two-scale convergence follows from the previous lemmas. We should justify (10) and similar estimate for z. Denote for brevity $U(x,t,\omega)=Du(x,t)+u_1(x,t,\omega)$. For any $\gamma>0$ consider $U^\gamma\in C_0^\infty(Q_T)\times\mathcal{D}^\infty(\Omega)$ such that $\|U-U^\gamma\|_{L^\alpha(Q_T\times\Omega)}\le\gamma$. For any $\delta\in(0,1)$ by the convexity argument we have

$$\begin{aligned}&\int\limits_{Q_T}\big|(1-\delta)U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)+\delta Du^\varepsilon(t,x)\big|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}dxdt\\ &\quad\le(1-\delta)\int\limits_{Q_T}|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}dxdt+\delta\int\limits_{Q_T}|Du^\varepsilon(t,x)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}dxdt.\end{aligned}\tag{11}$$

Using the inequality $|a+\delta b|^p-|a|^p-\delta p|a|^{p-2}ab=o(\delta)\,(|a|^p+|b|^p)$, as $\delta\to0$, that holds uniformly in a and b, we obtain

$$\begin{aligned}&\big|(1-\delta)U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)+\delta Du^\varepsilon(t,x)\big|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}\\ &\quad=\big(1-\delta p(\tau_{\frac{x}{\varepsilon}}\omega)\big)|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}\\ &\qquad+\delta p(\tau_{\frac{x}{\varepsilon}}\omega)|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)-2}U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)Du^\varepsilon(t,x)\\ &\qquad+o(\delta)\big(|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}+|Du^\varepsilon(t,x)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}\big).\end{aligned}$$

Integrating the last equality over Q_T and combining the resulting relation with (11) after straightforward rearrangements we obtain

$$\begin{aligned}&\int\limits_{Q_T}|Du^\varepsilon(t,x)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}dxdt\\ &\quad\ge\int\limits_{Q_T}\big(1-p(\tau_{\frac{x}{\varepsilon}}\omega)\big)|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}dxdt\end{aligned}$$

$$+\int\limits_{Q_T} p(\tau_{\frac{x}{\varepsilon}}\omega)|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)-2}U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)Du^\varepsilon(t,x)\,dxdt$$
$$+o_\delta(1)\big(|U^\gamma(t,x,\tau_{\frac{x}{\varepsilon}}\omega)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}+|Du^\varepsilon(t,x)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}\big),$$

where $o_\delta(1)$ tends to zero as $\delta\to 0$. Due to the a priory estimates for Du^ε and by the Birkhoff theorem, the last term on the right-hand side does not exceed $o_\delta(1)$ for sufficiently small ε. Applying again the Birkhoff theorem we conclude that the first term on the right-hand side converges to the integral

$$\int\limits_{Q_T}\int\limits_{\Omega}(1-p(\omega))|U^\gamma(t,x,\omega)|^{p(\omega)}dxdtd\mathbf{P}.$$

Since $p(\cdot)|U^\gamma|^{p(\cdot)}U^\gamma$ can be used as a test function in the definition of two-scale convergence, the second term on the right-hand side converges to the integral

$$\int\limits_{Q_T}\int\limits_{\Omega} p(\omega)|U^\gamma(t,x,\omega)|^{p(\omega)-2}U^\gamma(t,x,\omega)U(t,x,\omega)\,dxdtd\mathbf{P}.$$

Summarizing the above relations yields

$$\liminf_{\varepsilon\to 0}\int\limits_{Q_T}|Du^\varepsilon(t,x)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}\,dxdt$$
$$\geq\int\limits_{Q_T}\int\limits_{\Omega}(1-p(\omega))|U^\gamma(t,x,\omega)|^{p(\omega)}dxdtd\mathbf{P}$$
$$+\int\limits_{Q_T}\int\limits_{\Omega} p(\omega)|U^\gamma(t,x,\omega)|^{p(\omega)-2}U^\gamma(t,x,\omega)U(t,x,\omega)\,dxdtd\mathbf{P}+o_\delta(1).$$

Sending first $\delta\to 0$ and choosing sufficiently small $\gamma>0$ we conclude that

$$\int\limits_{Q_T}\int\limits_{\Omega}|U^\gamma(t,x,\omega)|^{p(\omega)}dxdtd\mathbf{P}\leq C$$

with a constant C that does not depend on γ. By the Fatou lemma this yields the desired statement. Moreover, we have

$$\liminf_{\varepsilon\to 0}\int\limits_{Q_T}|Du^\varepsilon(t,x)|^{p(\tau_{\frac{x}{\varepsilon}}\omega)}\,dxdt\geq\int\limits_{Q_T}\int\limits_{\Omega}|U(t,x,\omega)|^{p(\omega)}dxdtd\mathbf{P}.\qquad\square$$

The last lemma implies that

$$Du\in L^f(Q_T),\quad u\in X^f(Q_T),\quad z_0\in L^{f^*}(Q_T). \tag{12}$$

Indeed, by Lemma 4.1,

$$\int\limits_{Q_T} f(Du)\,dxdt = \int\limits_{Q_T} \Big(\min_{w\in\mathcal{G}(\Omega)} \int\limits_{\Omega} |Du(x,t)+w(\omega)|^{p(\omega)}\,d\mathbf{P}(\omega)\Big)dxdt$$
$$\le \int\limits_{Q_T} \Big(\int\limits_{\Omega} |Du(x,t)+u_1(x,t,\omega)|^{p(\omega)}\,d\mathbf{P}(\omega)\Big)dxdt < \infty.$$

Similarly,

$$\int\limits_{Q_T} f^*(z_0)\,dxdt \le \int\limits_{Q_T}\int\limits_{\Omega} |z(x,t,\omega)|^{p'(\omega)}\,d\mathbf{P}(\omega)\Big)dxdt < \infty.$$

It also follows from Proposition 4.1 that

$$\int\limits_{t_1}^{t_2}\int\limits_{G} z_0\cdot Du\,dxdt = \int\limits_{t_1}^{t_2}\int\limits_{G}\int\limits_{\Omega} z(Du+u_1)\,dxdtd\mathbf{P}(\omega). \tag{13}$$

Our next goal is to pass to the limit in the viscous term in (1). To this end we take the difference between the relations of items (i) and (iii) of Section 2.1. The resulting relation reads

$$\int\limits_{t_0}^{t_1}\int\limits_{G} A\Big(\frac{x}{\varepsilon}, Du^\varepsilon\Big)\cdot Du^\varepsilon\,dxdt$$
$$\le \int\limits_{t_0}^{t_1}\int\limits_{G} \Big[A\Big(\frac{x}{\varepsilon}, Du^\varepsilon\Big) - u^\varepsilon\otimes u^\varepsilon\Big]\cdot\nabla\eta\,dxdt$$
$$-\int\limits_{G}\Big(\big[\frac{1}{2}|u^\varepsilon(x,t_1)|^2 - u^\varepsilon(x,t_1)\cdot\eta(x)\big] - \big[\frac{1}{2}|u^\varepsilon(x,t_0)|^2 - u^\varepsilon(x,t_0)\cdot\eta(x)\big]\Big)dx,$$

for any $\eta\in C^\infty_{0,\mathrm{sol}}(G)$. Considering the relation

$$\int\limits_{G}\Big(\frac{1}{2}|u^\varepsilon|^2 - u^\varepsilon\cdot\eta\Big)dx\Big|_{t=t_0}^{t_1} = \frac{1}{2}\int\limits_{G}(|u^\varepsilon-\eta|^2)\,dx\Big|_{t=t_0}^{t_1}$$

and the symmetry of matrices A and $u^\varepsilon\otimes u^\varepsilon$, we derive

$$\int\limits_{t_0}^{t_1}\int\limits_{G} A\Big(\frac{x}{\varepsilon}, Du^\varepsilon\Big)\cdot Du^\varepsilon\,dxdt \le \int\limits_{t_0}^{t_1}\int\limits_{G}\Big[A\Big(\frac{x}{\varepsilon}, Du^\varepsilon\Big) - u^\varepsilon\otimes u^\varepsilon\Big]\cdot D\eta\,dxdt$$
$$+\frac{1}{2}\int\limits_{G}|u^\varepsilon(x,t_0)-\eta(x)|^2\,dx.$$

Choosing t_0 in such a way that $u^\varepsilon(\cdot, t_0)$ converges to $u(\cdot, t_0)$ in $L^2(G)$ and $u(\cdot, t_0) \in W_0^{1,\alpha}(G)$, and passing to the limit $\varepsilon \to 0$ yields

$$\lim_{\varepsilon\to 0}\int_{t_0}^{t_1}\int_G A\Big(\frac{x}{\varepsilon}, Du^\varepsilon\Big)\cdot Du^\varepsilon\,dxdt \le \int_{t_0}^{t_1}\int_G [z_0 - u\otimes u]\cdot D\eta\,dxdt + \frac{1}{2}\int_G |u(x,t_0)-\eta(x)|^2\,dx. \tag{14}$$

We are going to show that $\eta = u(x, t_0)$ can be chosen as a test function in the last inequality. Let $\{\eta_N\}_{N=1}^\infty$ be a sequence of functions $\eta_N \in C_{0,\mathrm{sol}}^\infty(G)$ such that $\eta_N \to \eta$ in $W_0^{1,\alpha}(G)$. We substitute η_N for a test function in (14) and pass to the limit, as $N \to \infty$. It is clear that $\eta_N \to u(\cdot, t_0)$ in $L^2(G)$. Therefore, the last term on the right-hand side tends to zero.

Regarding the convection term by Lemma 2.1 we have

$$|u\otimes u| \in L^1((0,T), L^\alpha).$$

Then

$$\int_{t_0}^{t_1}\int_G (u\otimes u)\cdot D\eta_N\,dxdt = \int_G\int_{t_0}^{t_1}(u\otimes u)\,dt\cdot D\eta_N\,dx \to \int_G\int_{t_0}^{t_1}(u\otimes u)\cdot Du(x,t_0)\,dxdt.$$

By Lemma 3.5, the space $C_{0,\mathrm{sol}}^\infty(G)$ is dense in $W_0^{1,f}(G)$. Therefore, we can assume that η_N converges to $u(\cdot, t_0)$ in $W_0^{1,f}(G)$. This yields

$$\int_{t_0}^{t_1}\int_G z_0\cdot D\eta_N\,dxdt \longrightarrow \int_{t_0}^{t_1}\int_G z_0\cdot Du(x,t_0)\,dxdt;$$

here we used the fact that

$$\int_{t_0}^{t_1} z_0\,dt \in L^{f^*}(G).$$

Letting $t_1 = t_0 + h$ and combining (14) with the above limit relations, we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{1}{h}\int_{t_0}^{t_1}\int_G A\Big(\frac{x}{\varepsilon}, Du^\varepsilon\Big)\cdot Du^\varepsilon\,dxdt &\le \frac{1}{h}\int_{t_0}^{t_1}\int_G [z_0 - u\otimes u]\cdot Du(x,t_0)\,dxdt\\ &= \frac{1}{h}\int_{t_0}^{t_1}\int_G z_0\cdot Du\,dxdt - R(h)\end{aligned}$$

with

$$R(h)=\frac{1}{h}\int_{t_0}^{t_0+h}\int_G z_0\cdot(Du(x,t)-Du(x,t_0))\,dxdt-\int_G\frac{1}{h}\int_{t_0}^{t_0+h}u\otimes u\,dt\cdot Du(x,t_0)\,dx.$$

With the help of (13) we rearrange the last inequality as follows

$$\lim_{\varepsilon\to 0}\frac{1}{h}\int_{t_0}^{t_1}\int_G A\Big(\frac{x}{\varepsilon},Du^\varepsilon\Big)\cdot Du^\varepsilon\,dxdt\le\frac{1}{h}\int_{t_0}^{t_1}\int_G\int_\Omega z\cdot(Du+u_1)\,d\mathbf{P}(\omega)dxdt-R(h). \tag{15}$$

Due to monotonicity of $\mathbf{A}(\omega,\xi)$, for any $\Phi\in C_0^\infty(G,\mathcal{D}^\infty(\Omega,\mathbb{R}^{d(d+1)/2}))$ we have

$$\frac{1}{h}\int_{t_0}^{t_1}\int_G\Big[A\Big(\frac{x}{\varepsilon},Du^\varepsilon\Big)-A\Big(\frac{x}{\varepsilon},\Phi(x,\tau_{x/\varepsilon}\omega)\Big)\Big]\cdot[Du^\varepsilon-\Phi(x,\tau_{x/\varepsilon}\omega)]\,dxdt\ge 0.$$

We pass to the limit, as $\varepsilon\to 0$, in this relation. The term with the integrand $A(\varepsilon^{-1}x,Du^\varepsilon)\cdot Du^\varepsilon$ has been estimated in (15). In other three terms we pass to the two-scale limit. This yields

$$\frac{1}{h}\int_{t_0}^{t_1}\int_G\int_\Omega[z-\mathbf{A}(\omega,\Phi(x,\omega))]\cdot[Du+u_1-\Phi(x,\omega)]\,d\mathbf{P}(\omega)dxdt\ge R(h).$$

For an arbitrary Lebesgue point t_0 of functions $z(\cdot,t,\cdot)\cdot Du(\cdot,t)$ and $z(\cdot,t,\cdot)\cdot u_1(\cdot,t,\cdot)$ the left-hand side of the last inequality converges as $f\to 0$ to the following integral

$$\int_G\int_\Omega[z(x,t_0,\omega)-\mathbf{A}(\omega,\Phi(x,\omega))]\cdot[Du(x,t_0)+u_1(x,t_0,\omega)-\Phi(x,\omega)]\,d\mathbf{P}(\omega)dx.$$

It is also easy to check that both integrals in the definition of $R(h)$ tend to zero, as $h\to 0$. Therefore,

$$\int_G\int_\Omega[z(x,t_0,\omega)-\mathbf{A}(\omega,\Phi(x,\omega))]\cdot[Du(x,t_0)+u_1(x,t_0,\omega)-\Phi(x,\omega)]\,d\mathbf{P}(\omega)dx\ge 0$$

for any test function Φ. By the standard Minty's arguments

$$z(x,t_0,\omega)=\mathbf{A}\big(\omega,Du(x,t_0)+u_1(x,t_0,\omega)\big).$$

By Proposition 4.1 we have $z(x,t_0,\cdot)\in\mathcal{G}^\perp(\Omega)$. Therefore,

$$\int_\Omega\mathbf{A}\big(\omega,Du(x,t_0)+u_1(x,t_0,\omega)\big)\cdot v(\omega)\,d\mathbf{P}(\omega)=0$$

for any $v \in \mathcal{G}(\Omega)$, and thus $u_1(x, t_0, \omega)$ is a solution of problem (5) with $\xi = Du(x, t_0)$. We then conclude that

$$z_0(x, t_0) = \int_\Omega z(x, t_0, \omega)(\omega)\, d\mathbf{P}(\omega) = \int_\Omega \mathbf{A}\big(\omega, Du(x, t_0) + u_1(x, t_0, \omega)\big)\, d\mathbf{P}(\omega)$$
$$= A^{\mathrm{eff}}(Du(x, t_0)).$$

This completes the proof of Theorem 2.

5. Examples

In this section we consider examples of random diffusion tensors $A(x, \xi)$.

Example 1 *(Voronoi–Poisson tessellation model).* Consider a Poisson point process in $\mathbb{R}^d$ with intensity 1, and construct the Voronoi tessellation (diagram) for this point process. It is known (see [2]) that a.s. the said Voronoi tessellation consists of a countable number of convex polytopes, we denote them $H_1, H_2, \dots$. Moreover, the polytopes can be enumerated in such a way that the characteristic function $\mathbf{1}_{H_j}(y)$ is a $\mathcal{B} \times \mathcal{F}$-measurable function of y and ω for any $j = 1, 2, \dots$.

Let $\eta_1, \eta_2, \dots$ be a family of i.i.d. random variable taking on values in $[\alpha, \beta]$ with $\alpha_0(d) \le \alpha < \beta \le \alpha^*$. We then set

$$\mathrm{p}(y) = \sum_{j=1}^{\infty} \eta_j \mathbf{1}_{H_j}(y), \qquad A(y, \xi) = |\xi|^{\mathrm{p}(y)-1}, \quad \xi \in \mathbb{R}^{\frac{d(d+1)}{2}}.$$

This diffusion matrix $A = A(y, \xi)$ satisfies all the conditions of Theorem 2.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be the underlying probability space with an ergodic dynamical system τ_y such that $\mathrm{p}(y) = p(\tau_y \omega)$ with $p(\omega) = \mathrm{p}(0)$. Problem (5) then takes the form: find $v \in \mathcal{G}(\Omega)$ such that

$$\int_\Omega |\xi + v(\omega)|^{p(\omega)-1} \theta(\omega)\, d\mathbf{P}$$

for any $\theta \in \mathcal{G}(\Omega)$. In this case the effective tensor admits the following variational formula

$$A^{\mathrm{eff}}(\xi) = \min_{v \in \mathcal{G}(\Omega)} \int_\Omega \frac{1}{p(\omega)} |\xi + v(\omega)|^{p(\omega)}\, d\mathbf{P}.$$

One can easily modify the function p to make its realizations smooth. Taking the convolution of $\mathrm{p}(\tau_y \omega)$ with a $C_0^\infty(\mathbb{R}^d)$ even function $\varphi = \varphi(y)$ such that $\varphi \ge 0$ and $\int_{\mathbb{R}^d} \varphi\, dy = 1$, denoting the obtained function by $\widehat{\mathrm{p}}$ and letting $A(\omega, \xi) = |\xi|^{\widehat{p}(\omega)}$, $\widehat{p}(\omega) = \widehat{\mathrm{p}}(0)$, we define a diffusion matrix $A(\omega, \xi)$ with a.s. continuous in y realizations that also satisfies the assumptions of Theorem 2.

Example 2 *(Bernoulli percolation model).* Consider a checker board in $\mathbb{R}^d$ with the cell $[0, 1)^d$. We associate to each cell a random variable that takes on the value 1 with probability q and the value 0 with probability $1 - q$, and assume that these random variables are i.i.d. We denote these random variables by ζ_j, $j \in \mathbb{Z}^d$, and the corresponding cells by Q_j, so that $Q_j = [0,1)^d + j$. It is known (see [6]) that there is a $\mathbf{p}_{\rm cr}$, $0 < \mathbf{p}_{\rm cr} < 1$, such that for $q > \mathbf{p}_{\rm cr}$ the set $\{\bigcup_j Q_j \,:\, \zeta_j = 1\}$ a.s. has a unique unbounded connected component, the so-called infinite cluster. We denote it by $\mathcal{C}$, and introduce the following two random functions:

$$\mathrm{p}(y) = \alpha + (\beta - \alpha)\mathbf{1}_{\mathcal{C}}(y), \qquad a(y) = 1 + \sum_{j \in \mathbb{Z}^d} \zeta_j \mathbf{1}_{Q_j}(y)$$

with $\alpha_0(d) \le \alpha < \beta \le \alpha^*$. Then

$$A(y,\xi) = a(y)|\xi|^{\mathrm{p}(y)}, \quad \xi \in \mathbb{R}^{d(d+1)/2},$$

is an admissible diffusion matrix.

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