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On the asymptotic behaviour of eigenvalues of a boundary-value problem in a planar domain of Steklov sieve type

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Abstract. We consider a two-dimensional spectral problem of Steklov type for the Laplace operator in a domain divided into two parts by a perforated partition with a periodic microstructure. The Steklov boundary condition is imposed on the lateral sides of the perforation, the Neumann condition on the remaining part of the boundary, and the Dirichlet and Neumann conditions on the outer boundary of the domain. We construct and justify two-term asymptotic expressions for the eigenvalues of this problem. We also construct a two-term asymptotic formula for the corresponding eigenfunctions.

Keywords: asymptotic behaviour of eigenvalues, spectral problem, Steklov problem, homogenization of spectral problems.

Introduction

The homogenization of spectral problems in domains with microstructure is an important and, in many cases, difficult problem. The first results in this direction were obtained in [1] for perforated domains and in [2] for operators with rapidly oscillating coefficients. Later, many papers were published in which the homogenization of diverse elliptic spectral problems was considered. In particular, problems with a spectral condition of Steklov type were studied; see, for example, [3]–[6]. In these papers, homogenization problems of a spectral problem of Steklov type with a fast change of the type of the boundary condition were investigated (for the scalar equation, see [3], and for the system of elasticity theory, see [4]). In [5], the leading terms of the asymptotic expansion of the eigenvalue in a dense cascade connection were constructed. The behaviour of a solution of the problem in two domains connected by thin rods was studied in [6]. Problems of homogenization of operator pencils were considered in [7].

An asymptotic analysis of problems in domains with singular boundaries and of operators with singularities (for example, with high-contrast coefficients) was

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carried out, for example, in [8]–[10]. Problems concerning the homogenization and the asymptotic behaviour of multiple eigenvalues were discussed in the papers [11]–[16] and elsewhere.

Numerous papers are devoted to homogenization problems in domains with perforated partitions (see, for example, [17]–[23]). In Ch. I, § 3 of [18], the problem was considered in a domain perforated along a closed curve. It was proved that the solutions of the original problem converge uniformly to solutions of limit problems on compact subdomains that do not include the curve. Solutions of boundary-value problems in a domain divided into two parts by a perforated surface with different thicknesses were considered in [19] and [22]. In particular, the weak convergence in L_2 of solutions of the original problem to solutions of two independent problems in domains separated by this surface was proved. In [20], the asymptotic behaviour of solutions of the boundary-value problem in a domain perforated along a manifold was studied with different boundary conditions on the boundary of the cavities. The case when the perforation makes no contribution to the problem in the limit was considered. The paper [23] is devoted to the study of a problem of the type of a fine sieve with a spectral condition on a perforated partition. The Steklov problem on a periodic connection was treated in [24]. This domain can be regarded as half of a domain perforated along a hyperplane (but without the other fixed part of the domain). The asymptotic behaviour of the spectrum as the small parameter tends to zero was shown. The problem in a domain perforated along a segment was considered in the paper [21]. Here it was assumed that the width of the cavities is a small parameter, whereas the length is of finite size. The behaviour of eigenvalues as the small parameter tends to zero was investigated.

In this paper, we consider the problem in a domain divided into two parts by a thick partition with canals. The thickness of the partition and the period of the arrangement of the canals are of the same order and are equal to ε. The thickness of canals is $a\varepsilon$, where $a < 1$ is a constant. Here and below, the small parameter ε is determined by the equation $\varepsilon = 1/(2\mathcal{N}+1)$, where $\mathcal{N} \gg 1$ is a positive integer. We assume that the spectral condition of Steklov type is imposed on the boundaries of the canals, a homogeneous Dirichlet boundary condition on the outer boundary of the domain, and a homogeneous Neumann condition on the remaining part of the boundary of the partition. In [17], a limit (homogenized) problem was derived for this problem. However, the leading term of the asymptotic expansion of an eigenfunction gives no impression of the structure of this function in a small neighbourhood of the 'sieve' (the perforated partition). The behaviour of the eigenfunction in this neighbourhood is of particular interest since it is the very place at which a rearrangement of the initial spectral boundary conditions into effective ones occurs, and the eigenfunctions have rapidly oscillating structure. For this reason, to better understand the homogenization process and to improve the rate of convergence, it is required to construct the second terms of the asymptotic expansion of the eigenpairs, and that is the object of this paper.

We also discuss the meaning of the spectral conditions of Steklov type that are imposed on the lateral surface of the perforation. In problems of heat conduction and others related to diffusion equations, the Neumann–Dirichlet operator or its inverse, the Dirichlet–Neumann operator, plays an important role. For a harmonic function vanishing on a part of the boundary of the domain, this operator defines the profile of the solution for a given flow on the remaining part of the boundary. It is known that on the corresponding part of the boundary the Dirichlet–Neumann operator in the space L_2 is self-adjoint, positive and has compact resolvent. The Steklov-type spectral problem is equivalent to the spectral problem for the corresponding Dirichlet–Neumann operator. As usual, knowing the spectrum of the operator enables us to obtain information about the behaviour of solutions of evolutionary problems related to this operator.

In applied problems, the presence of thin partitions with a microstructure whose surface is equipped a flow of heat is quite natural. The reader can interpret such a partition as a thin heating pad which is permeated by warm canals or heaters.

§ 1. Statement of the problem and preliminaries

We denote by Ω a domain in $\mathbb{R}^2$ whose boundary Γ is smooth and, in a neighbourhood of the ends of the segment $\Gamma_1=[-1/2,1/2]$ on the abscissa axis, Γ coincides with the lines $x_1=-1/2$ and $x_1=1/2$, respectively. Consider a non-empty part of the boundary $\Gamma_2:=\{x\in\Gamma\colon x_2>c\}$ for some fixed $c>0$, and let $\Gamma_3=\Gamma\setminus\Gamma_2$.

Let Q be the rectangle $\{x\in\mathbb{R}^2\colon x_1\in(-1/2,1/2),\ x_2\in(-h\varepsilon/2,h\varepsilon/2)\}$ and let B be the rectangle $\{\xi\in\mathbb{R}^2\colon \xi_1\in(-a/2,a/2),\ \xi_2\in(-h/2,h/2)\}$, $0<a<1$, $h>0$. We recall that $\varepsilon=1/(2\mathcal{N}+1)$, where $\mathcal{N}\in\mathbb{N}$. We introduce the notation

$$B^j_\varepsilon=\{x\in\Omega\colon \varepsilon^{-1}(x_1-\varepsilon j,x_2)\in B\},\quad j\in\mathbb{Z},\qquad B_\varepsilon=\bigcup_j B^j_\varepsilon$$

and consider the perforated strip $Q_\varepsilon:=Q\setminus\overline{B_\varepsilon}$. We denote the vertical boundary of the canals by $\Gamma_\varepsilon=\partial B_\varepsilon\cap Q$. We define the domain Ω_ε as the set $\Omega\setminus\overline{Q_\varepsilon}$ (see Fig. 1).

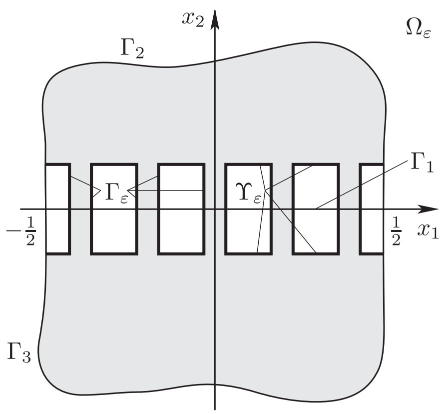

Figure 1. Structure of Ω_ε

We write

$$\Gamma^\varepsilon_3=\left\{x\in\Gamma_3\colon |x_2|>\frac{h\varepsilon}{2}\right\},\qquad \Upsilon_\varepsilon=\left\{x\in\partial Q_\varepsilon\colon |x_2|=\frac{h\varepsilon}{2}\right\}.$$

We define the space $H^1(\Omega_\varepsilon, \Gamma_2)$ as the closure, with respect to the norm of $H^1(\Omega_\varepsilon)$, of the set $C^\infty(\overline{\Omega_\varepsilon}\,)$ of functions vanishing on a neighbourhood of Γ_2.

Consider the following spectral problem of Steklov type:

$$\begin{cases} -\Delta u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Upsilon_\varepsilon \cup \Gamma_3^\varepsilon, \qquad \dfrac{\partial u_\varepsilon}{\partial \nu} = \lambda_\varepsilon u_\varepsilon \quad \text{on } \Gamma_\varepsilon. \end{cases} \tag{1.1}$$

Here and below, ν is the unit normal vector pointing outwards.

It is known (see [25]) that the spectral problem for the Laplace operator in a bounded domain with the Steklov condition on a part of the boundary is self-adjoint, and the resolvent of the corresponding operator is compact and positive. Therefore, the problem (1.1) has a discrete spectrum going to ∞. We denote the eigenvalues of the problem, renumbered taking their multiplicities into account, by $\lambda_{\varepsilon,1}, \lambda_{\varepsilon,2}, \dots, \lambda_{\varepsilon,j}, \dots \to \infty$, and the corresponding eigenfunctions by $u_{\varepsilon,1}, u_{\varepsilon,2}, \dots, u_{\varepsilon,j}, \dots$. The following normalisation condition is natural for the problem (1.1):

$$\int_{\Gamma_\varepsilon} u_{\varepsilon,i} u_{\varepsilon,j}\, dx_2 = \delta_i^j, \tag{1.2}$$

where $\delta_i^j = 1$ when $i = j$ and $\delta_i^j = 0$ when $i \neq j$.

In what follows, we also use the notation $[u]$ for the jump of a function u on Γ_1. We treat the problem (1.1) and the problems arising below with a jump on Γ_1 in the sense of integral identities with the corresponding solutions belonging to the space $H^1(\Omega_\varepsilon, \Gamma_2)$.

As was shown in [17], the homogenized problem for (1.1) acquires the form

$$\begin{cases} -\Delta u_0 = 0 \quad \text{in } \Omega, \\ u_0 = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial u_0}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \\ [u_0] = 0 \quad \text{on } \Gamma_1, \qquad \left[\dfrac{\partial u_0}{\partial x_2}\right] = -2h\lambda_0 u_0 \quad \text{on } \Gamma_1. \end{cases} \tag{1.3}$$

This problem is self-adjoint and has a discrete spectrum. The corresponding eigenvalues $\lambda_{0,k}$ indexed according to their multiplicities tend to $+\infty$ as $k \to \infty$. Moreover, as a special case of Theorem 3.1 in the paper [17], we obtain the following assertion.

Proposition 1.1. *Let the multiplicity of an eigenvalue $\lambda_0 = \lambda_{0,j}$ of the boundary-value problem (1.3) be equal to n, that is, $\lambda_{0,j} = \dots = \lambda_{0,j+n-1}$. Then the boundary-value problem (1.1) has precisely n eigenvalues $\lambda_\varepsilon^{(l)} = \lambda_{\varepsilon,j+l-1}$, $l = 1, \dots, n$, that tend to λ_0 as $\varepsilon \to 0$.*

Let $u_\varepsilon^{(l)}$ be the orthonormalised (in $L_2(\Omega_\varepsilon)$) eigenfunctions of the boundary-value problem (1.1) corresponding to $\lambda_\varepsilon^{(l)}$. Then from every sequence $\varepsilon_q \xrightarrow[q\to\infty]{} 0$ one can *single out a subsequence* ε_{q_i} *such that*

$$\|u^{(l)}_{\varepsilon_{q_i}} - u^{(l)}_*\|_{H^1(\Omega_{\varepsilon_{q_i}})} \xrightarrow[q\to\infty]{} 0, \tag{1.4}$$

where the $u^{(l)}_*$ *are orthonormalised* (*in* $L_2(\Omega)$) *eigenfunctions of the boundary-value problem* (1.3) *corresponding to* λ_0 (*and depending in general on the choice of both the sequence* $\varepsilon_q \xrightarrow[q\to\infty]{} 0$ *and its subsequence*).

Remark 1.1. Below, we omit the index j if possible, that is, we write λ_0 and $\lambda^{(l)}_\varepsilon$, $l = 1, \dots, n$.

We note an interesting specific feature of the limit problem (1.3). The coefficient in the spectral condition in this problem depends on the parameter h (the thickness of the partition), but not on the parameter a which characterizes the width of the holes (see the formula (11) in [17]). The dependence on a manifests itself only in the subsequent terms of the asymptotic expansions of the eigenpairs. This is related to the fact that the coefficient in the limit spectral condition is determined by the effective length of the vertical part of the boundary of the partition in the prelimit problem. This effective length does not depend on the parameter a.

In this paper, the cases of both a simple and a multiple eigenvalue λ_0 are considered (the multiplicity of λ_0 is $n \geqslant 1$). Two-term asymptotic expansions for the eigenvalues (of the boundary problem (1.1)) that converge to λ_0 are constructed and justified, together with the leading terms of the asymptotic expansions of the corresponding eigenfunctions.

§ 2. Auxiliary assertions

In this section, we prove some auxiliary assertions needed to construct asymptotic expansions of the solution of the problem (1.1).

Let $\Pi = \{\xi \colon -1/2 < \xi_1 < 1/2\}$ be a strip and let

$$\Pi_{ah} = \Pi \setminus \biggl\{\biggl\{\biggl[-\frac{1}{2}, -\frac{a}{2}\biggr] \cup \biggl[\frac{a}{2}, \frac{1}{2}\biggr]\biggr\} \times \biggl[-\frac{h}{2}, \frac{h}{2}\biggr]\biggr\}.$$

We write

$$\Upsilon = \Upsilon_+ \cup \Upsilon_-, \qquad \Upsilon_\pm = \biggl\{\xi \colon \xi_1 \in \biggl[-1, -\frac{a}{2}\biggr] \cup \biggl[\frac{a}{2}, 1\biggr],\ \xi_2 = \pm\frac{h}{2}\biggr\},$$

$$\Gamma_\pm = \biggl\{\xi \colon \xi_1 = \pm\frac{a}{2},\ \xi_2 \in \biggl[-\frac{h}{2}, \frac{h}{2}\biggr]\biggr\}.$$

We denote the remaining part of the boundary of the strip by Σ, that is,

$$\Sigma = \partial\Pi_{ah} \setminus (\Upsilon \cup \Gamma_+ \cup \Gamma_-)$$

(see Fig. 2).

For an arbitrary $s > 0$ we shall also write

$$\Pi^s_{ah} = \{\xi \in \Pi_{ah} \colon |\xi_2| < s\}.$$

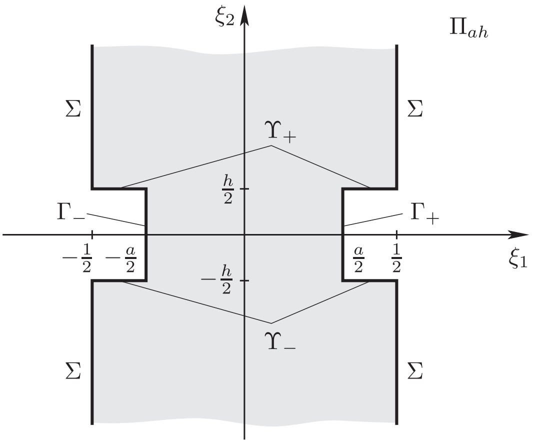

Figure 2. A cell of periodicity, Π_{ah}

In what follows, the index ξ of the operator Δ_ξ and of other operators means that this operator is taken with respect to the variables ξ.

All the auxiliary problems of this section are considered in the space of functions that are 1-periodic with respect to the variable ξ_1. In this connection, the solutions of the problems in Π_{ah} have periodic conditions on Σ. Moreover, by symmetry, they reduce to auxiliary problems with boundary conditions of Neumann or Dirichlet type on Σ.

We clarify this reduction in the case of the auxiliary problem

$$\begin{cases} \Delta_\xi X_0 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial X_0}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon, \qquad \dfrac{\partial X_0}{\partial \nu_\xi} = 1 \quad \text{on } \Gamma_\pm, \\ X_0 \quad \text{is 1-periodic with respect to } \xi_1, \end{cases} \tag{2.1}$$

whose solution is sought in the class of functions satisfying the condition

$$X_0(\xi) = -h|\xi_2| + o(1) \quad \text{as } \xi_2 \to \pm\infty. \tag{2.2}$$

It can readily be seen that the problem (2.1) is invariant with respect to replacing the variable ξ_1 by $-\xi_1$. By the maximum principle, the problem (2.1), (2.2) has at most one solution, and therefore, if a solution exists, then it is an even function with respect to ξ_1 and, as a consequence, it satisfies the problem

$$\begin{cases} \Delta_\xi X_0 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial X_0}{\partial \nu_\xi} = 0 \quad \text{on } \Sigma \cup \Upsilon, \qquad \dfrac{\partial X_0}{\partial \nu_\xi} = 1 \quad \text{on } \Gamma_\pm. \end{cases} \tag{2.3}$$

The converse assertion can also readily be proved: if a function X_0 is a solution of the problem (2.3), (2.2), then it is also a solution of the problem (2.1). Therefore, in what follows, instead of (2.1), (2.2), we study the problem (2.3), (2.2).

We also consider the auxiliary problems

$$\begin{cases} \Delta_\xi \widetilde{X}_0 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial \widetilde{X}_0}{\partial \nu_\xi} = 0 \quad \text{on } \Sigma \cup \Upsilon \cup \Gamma_\pm \end{cases} \tag{2.4}$$

and

$$\begin{cases} \Delta_\xi Y_0 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial Y_0}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon, \qquad Y_0 = 0 \quad \text{on } \Sigma, \qquad \dfrac{\partial Y_0}{\partial \nu_\xi} = \pm 1 \quad \text{on } \Gamma_\pm. \end{cases} \tag{2.5}$$

The following assertion holds.

Lemma 2.1. *There are solutions of the problems* (2.3) *and* (2.4) *having the following asymptotic behaviour, respectively*:

$$\begin{aligned} X_0(\xi) &= -h|\xi_2| + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } \xi_2 \to \pm\infty, \\ \widetilde{X}_0(\xi) &= \xi_2 \pm C_{ah} + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } \xi_2 \to \pm\infty, \end{aligned} \tag{2.6}$$

where $C_{ah} > 0$. These solutions are unique and are even functions with respect to ξ_1.

There is a unique solution of (2.5) *decaying exponentially by the rule $\mathcal{O}(e^{-\pi|\xi_2|})$ as $\xi_2 \to \pm\infty$. This solution is an odd function of ξ_1.*

Proof. We shall prove the existence of a solution of (2.4) satisfying the second condition in (2.6). To this end, we write $\widetilde{Z}_0 = \widetilde{X}_0 - \xi_2$. The problem for $\widetilde{Z}_0$ becomes

$$\begin{cases} \Delta_\xi \widetilde{Z}_0 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial \widetilde{Z}_0}{\partial \nu_\xi} = 0 \quad \text{on } \Sigma \cup \Gamma_\pm, \qquad \dfrac{\partial \widetilde{Z}_0}{\partial \nu_\xi} = \mp 1 \quad \text{on } \Upsilon_\pm. \end{cases} \tag{2.7}$$

We write $\Pi_{ah}^N = \{\xi \in \Pi_{ah} : -N < \xi_2 < N\}$ and consider the sequence of problems

$$\begin{cases} \Delta_\xi \widetilde{Z}_0^N = 0 \quad \text{in } \Pi_{ah}^N, \\ \dfrac{\partial \widetilde{Z}_0^N}{\partial \nu_\xi} = 0 \quad \text{on } \partial\Pi_{ah}^N \setminus \Upsilon, \qquad \dfrac{\partial \widetilde{Z}_0^N}{\partial \nu_\xi} = \mp 1 \quad \text{on } \Upsilon_\pm. \end{cases} \tag{2.8}$$

The solubility condition for these problems is satisfied. We choose an additive constant in such a way that the solution is odd with respect to ξ_2. Then $\widetilde{Z}_0^N(\xi_1, 0) = 0$. Therefore, by the theorem on traces and by the Friedrichs inequality (see, for example, Ch. III, § 5 in [26]), the following inequalities hold:

$$\int_\Upsilon (\widetilde{Z}_0^N)^2 \, d\xi_1 \leqslant \widetilde{C} \|\widetilde{Z}_0^N\|^2_{H^1(\Pi_{ah}^{h+1})} \leqslant C \|\nabla \widetilde{Z}_0^N\|^2_{L_2(\Pi_{ah}^{h+1})} \leqslant C \|\nabla \widetilde{Z}_0^N\|^2_{L_2(\Pi_{ah}^N)}, \tag{2.9}$$

and the constant C does not depend on N. Multiplying the equation (2.8) by $\widetilde{Z}_0^N$ and integrating by parts, we arrive at the relation

$$\int_{\Pi_{ah}^N} |\nabla \widetilde{Z}_0^N|^2 \, d\xi \leqslant \int_\Upsilon |\widetilde{Z}_0^N| \, d\xi_1 \leqslant C_1 \biggl(\int_\Upsilon |\widetilde{Z}_0^N|^2 \, d\xi_1\biggr)^{1/2} \leqslant C_2 \biggl(\int_{\Pi_{ah}^{h+1}} |\nabla \widetilde{Z}_0^N|^2 \, d\xi\biggr)^{1/2}.$$

Hence, $\|\nabla \widetilde{Z}_0^N\|_{L_2(\Pi_{ah}^N)} \leqslant C_3$. Then (2.9) implies the inequality

$$\|\widetilde{Z}_0^N\|_{L_2(\Pi_{ah}^{h+2})} \leqslant C_3.$$

Since $\widetilde{Z}_0^N$ is a harmonic function, it follows that, using the Schauder estimates (see, for example, Ch. 6, § 6.1 in [27]), we obtain $\|\widetilde{Z}_0^N(\,\cdot\,, \pm(h+1))\|_{L_\infty(-1/2,1/2)} \leqslant C_4$, and, by the maximum principle,

$$\|\widetilde{Z}_0^N\|_{L_\infty(\Pi_{ah}^N \setminus \Pi_{ah}^{h+1})} \leqslant C_4. \tag{2.10}$$

Passing to the limit as $N \to \infty$, we obtain a bounded solution $\widetilde{Z}_0$ of the problem (2.7). Indeed, by the elliptic estimates for a harmonic function, for any $k \geqslant 2$ and $N > k+1$ the following estimate holds:

$$\|\widetilde{Z}_0^N(\,\cdot\,, h+k)\|_{C^2[-1/2,1/2]} \leqslant C_5,$$

and C_5 depends neither on N nor on k. Therefore, for the specified values of N the functions $\widetilde{Z}_0^N$ admit the bound $\|\widetilde{Z}_0^N\|_{H^1(\Pi_{ah}^{h+k})} \leqslant C(k)$. Passing to the weak limit along a subsequence, we obtain a harmonic function on Π_{ah}^{h+k} satisfying the boundary conditions required in (2.7). Further, using the diagonal procedure, we construct a harmonic function on Π_{ah} which is a solution of the problem (2.7). Its boundedness on the set $\Pi_{ah} \setminus \Pi_{ah}^{h+1}$ follows from (2.10), and the boundedness on the set Π_{ah}^{h+1} is a consequence of the standard elliptic estimates.

Using the method of separation of variables, we can readily prove that a harmonic function which is bounded in the infinite half-strip $\{\xi \in \mathbb{R}^2\colon -1/2<\xi_1<1/2,\ \xi_2 > h\}$ and satisfies the condition of periodicity or the homogeneous Neumann condition on the lateral surface of this half-strip converges with an exponential rate to a constant. For more general operators, this result can be found. for example, in [28] and [29]. Therefore, the function $\widetilde{Z}_0$ converges with an exponential rate to a constant as $\xi_2 \to +\infty$. We denote this constant by C_{ah}. By construction, the solution $\widetilde{Z}_0$ is odd with respect to ξ_2, and therefore Z_0 converges with exponential rate to $\pm C_{ah}$ as $\xi_2 \to \pm\infty$. As a result, we arrive at the bound $|\widetilde{Z}_0(\xi) - C_{ah}| \leqslant Ce^{-\pi\xi_2}$, which can be obtained by the method of separation of variables, taking into account that the width of the strip is equal to 1. Therefore, $\widetilde{X}_0(\xi) = \xi_2 \pm C_{ah} + \mathcal{O}(e^{-\pi|\xi_2|})$ as $\xi_2 \to \pm\infty$.

The uniqueness of the solution $\widetilde{X}_0$ satisfying the condition (2.6) is a simple consequence of the maximum principle. The evenness with respect to ξ_1 is obvious.

It remains to prove that C_{ah} is positive. We claim that the function $\widetilde{Z}_0$ is positive. Since it is odd with respect to ξ_2, we have $\widetilde{Z}_0(x_1, 0) = 0$. Since $\widetilde{Z}_0$ is harmonic, periodic with respect to ξ_1 and bounded, it follows that its mean value with respect to the period is a constant as a function of ξ_2, which is equal to C_{ah}. If $C_{ah} \leqslant 0$, then $\widetilde{Z}_0$ takes negative values in $\{\xi \in \Pi_{ah}\colon \xi_2 > 0\}$. Taking into account the boundary conditions in (2.7), we can readily see that this assumption contradicts the maximum principle. Thus, $\widetilde{Z}_0$ is positive, and $C_{ah} > 0$.

We now claim that the problem (2.3) has a solution satisfying the first relation in (2.6) for some constant $C > 0$. To this end, we choose a smooth even function θ

on $\mathbb{R}$ such that $\theta(s) = h|s|$ for $|s| \geqslant h/2$. Then the function $Z_0(\xi) = X_0(\xi) + \theta(\xi_2)$ must be a solution of the problem

$$\begin{cases} \Delta_\xi Z_0 = \theta''(\xi_2) \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial Z_0}{\partial \nu_\xi} = 0 \quad \text{on } \Sigma, \\ \dfrac{\partial Z_0}{\partial \nu_\xi} = 1 \quad \text{on } \Gamma_\pm, \qquad \dfrac{\partial Z_0}{\partial \nu_\xi} = -h \quad \text{on } \Upsilon \end{cases} \tag{2.11}$$

and satisfy the condition $|Z_0(\xi)| \leqslant Ce^{-\pi|\xi_2|}$. It can readily be seen that

$$\int_{\Gamma_\pm} d\xi_2 + \int_\Upsilon (-h)\, d\xi_1 - \int_{\Pi_{ah}} \theta''(\xi_2)\, d\xi = 0. \tag{2.12}$$

Consider the family of problems

$$\begin{cases} \Delta_\xi Z_0^N = \theta''(\xi_2) \quad \text{in } \Pi_{ah}^N, \\ \dfrac{\partial Z_0^N}{\partial \nu_\xi} = 0 \quad \text{on } \partial\Pi_{ah}^N \setminus (\Gamma_\pm \cup \Upsilon), \\ \dfrac{\partial Z_0^N}{\partial \nu_\xi} = 1 \quad \text{on } \Gamma_\pm, \qquad \dfrac{\partial Z_0^N}{\partial \nu_\xi} = -h \quad \text{on } \Upsilon. \end{cases} \tag{2.13}$$

The equation (2.12) ensures the solubility of this problem. We choose a corresponding additive constant in such a way that the following equation holds:

$$\int_{\Pi_{ah}^{h+2}} Z_0^N\, d\xi = 0.$$

Then, by Poincaré's inequality (see, for example, Ch. I, § 1.4 in [30]), we have the bound

$$\int_{\Pi_{ah}^{h+2}} |Z_0^N|^2\, d\xi \leqslant C \int_{\Pi_{ah}^{h+2}} |\nabla Z_0^N|^2\, d\xi \tag{2.14}$$

with a constant C independent of N. Multiplying the equation in (2.13) by Z_0^N and integrating the equation thus obtained over Π_{ah}^N, we arrive (after an integration by parts) at the relation

$$\int_{\Pi_{ah}^N} |\nabla Z_0^N|^2\, d\xi + \int_{\Pi_{ah}^h} Z_0^N \theta''(\xi_2)\, d\xi - \int_{\Gamma_\pm} Z_0^N\, d\xi_2 + h \int_\Upsilon Z_0^N\, d\xi_1 = 0.$$

Thus, using (2.14) and the theorem on traces for H^1-functions, we derive the bound

$$\int_{\Pi_{ah}^N} |\nabla Z_0^N|^2\, d\xi \leqslant C.$$

Using the Schauder estimates (see, for example, Ch. 6, § 6.1 in [27]), we obtain $\|Z_0^N(\,\cdot\,, \pm(h+1))\|_{L_\infty(-1/2,1/2)} \leqslant C_5$. This implies that (a subsequence of) Z_0^N converges as $N \to \infty$ to a bounded solution of the problem (2.11). By construction, this solution is an even function of the variables ξ_2 and ξ_1. According to [29], the solution Z_0^N converges as $\xi_2 \to \pm\infty$ with exponential rate to $C_\pm$. Because of the evenness, we have the equation $C_- = C_+$. By subtracting the constant $C_\pm$ from the solution thus constructed, we obtain a solution of the problem (2.11) with the desired properties.

The uniqueness follows readily from the maximum principle.

The last assertion of the lemma can be proved similarly, with substantial simplifications. This completes the proof of Lemma 2.1. □

The last lemma implies immediately the following assertion.

Corollary 2.1. *The periodic continuation of the function X_0 with respect to ξ_1 is a solution of the problem* (2.1)*. The function $\widetilde{X}_0$, periodically continued with respect to ξ_1, satisfies the problem* (2.4) *in which the Neumann condition on Σ is replaced by the periodicity condition. Similarly, the periodic continuation of the function Y_0 is a solution of the problem* (2.5) *with periodic boundary conditions on Σ.*

Remark 2.1. We now give a rather explicit formula for the constant C_{ah}. To this end, we use the following considerations. We note that the function $\widetilde{X}_0$ is odd with respect to ξ_2, and therefore it is possible to study the problem in the half-strip $\Pi_{ah} \cap \{\xi_2 > 0\}$, introducing the homogeneous Dirichlet condition on the segment $\Theta = \{(\xi_1, \xi_2)\colon\ -a/2 \geqslant \xi_1 \geqslant a/2,\, \xi_2 = 0\}$. Further, by conformally mapping the domain thus obtained onto the upper half-plane in such a way that the points $(-a/2, 0)$ and $(a/2, 0)$ pass to -1 and 1, respectively, on the real axis, the point at infinity passes to the point at infinity on the half-plane, and the points $a/2 + \mathrm{i}h/2$ and $1/2 + \mathrm{i}h/2$ pass to $b_1 > 1$ and $b_2 > b_1$ on the real axis, respectively, we can implicitly evaluate the constant C_{ah}.

Indeed, using the Christoffel–Schwartz theorem to construct such a mapping, we have the formula

$$w = F(z) = -\frac{\mathrm{i}}{\pi} \int_0^z \sqrt{\frac{\zeta^2 - b_1^2}{(\zeta^2 - 1)(\zeta^2 - b_2^2)}}\, d\zeta, \tag{2.15}$$

where w are the coordinates on the plane on which the half-strip is given, and z are the coordinates on the given half-plane onto which the half-strip is mapped. Under this mapping, the original boundary-value problem passes to a new one in a half-plane on whose border the solution has the homogeneous Neumann condition on the axis, except for the segment $[-1, 1]$, on which the homogeneous Dirichlet condition is given. The solution of this problem can be expressed explicitly. It has the form $\operatorname{Re} \ln \sqrt{z^2 - 1}$ with the asymptotic behaviour $\ln|z| - \ln 2$ at infinity.

Now, finding the constants b_1 and b_2 from the equations

$$\begin{cases} -\dfrac{\mathrm{i}}{\pi} \displaystyle\int_0^1 \sqrt{\frac{\zeta^2 - b_1^2}{(\zeta^2 - 1)(\zeta^2 - b_2^2)}}\, d\zeta = \dfrac{a}{2}, \\[2ex] -\dfrac{1}{\pi} \displaystyle\int_1^{b_1} \sqrt{\frac{\zeta^2 - b_1^2}{(\zeta^2 - 1)(\zeta^2 - b_2^2)}}\, d\zeta = \dfrac{h}{2} \end{cases}$$

and using the formula for the inverse mapping $z = F^{-1}(w)$ with the constants b_1 and b_2 substituted, we can find the constant C_{ah}:

$$C_{ah} = 2\int_{b_1}^{b_2} \operatorname{Re}\ln\sqrt{(F^{-1})^2 - 1}\left|\frac{\partial F^{-1}}{\partial w}\right| d\eta_1, \qquad w = \eta_1 + \mathrm{i}\eta_2$$

(here $|\partial F^{-1}/\partial w|$ stands for the Jacobian).

Consider the following auxiliary problems:

$$\begin{cases} \Delta_\xi X_1 = \dfrac{\partial Y_0(\xi)}{\partial \xi_1} & \text{in } \Pi_{ah}, \\ \dfrac{\partial X_1}{\partial \nu_\xi} = 0 & \text{on } \Sigma \cup \Upsilon \cup \Gamma_\pm, \end{cases} \qquad \begin{cases} \Delta_\xi X_2 = 1 & \text{in } \Pi_{ah}, \\ \dfrac{\partial X_2}{\partial \nu_\xi} = 0 & \text{on } \Sigma \cup \Upsilon \cup \Gamma_\pm, \end{cases} \tag{2.16}$$

$$\begin{cases} \Delta_\xi X_3 = 0 & \text{in } \Pi_{ah}, \\ \dfrac{\partial X_3}{\partial \nu_\xi} = 0 & \text{on } \Sigma \cup \Upsilon, \\ \dfrac{\partial X_3}{\partial \nu_\xi} = X_0 & \text{on } \Gamma_\pm, \end{cases} \qquad \begin{cases} \Delta_\xi X_4 = 0 & \text{in } \Pi_{ah}, \\ \dfrac{\partial X_4}{\partial \nu_\xi} = 0 & \text{on } \Sigma \cup \Upsilon, \\ \dfrac{\partial X_4}{\partial \nu_\xi} = \widetilde{X}_0 & \text{on } \Gamma_\pm, \end{cases} \tag{2.17}$$

$$\begin{cases} \Delta_\xi X_5 = 0 & \text{in } \Pi_{ah}, \\ \dfrac{\partial X_5}{\partial \nu_\xi} = 0 & \text{on } \Sigma \cup \Upsilon, \\ \dfrac{\partial X_5}{\partial \nu_\xi} = \pm Y_0 & \text{on } \Gamma_\pm, \end{cases} \tag{2.18}$$

$$\begin{cases} \Delta_\xi Y_1 = \dfrac{\partial X_0(\xi)}{\partial \xi_1} & \text{in } \Pi_{ah}, \\ Y_1 = 0 & \text{on } \Sigma, \\ \dfrac{\partial Y_1}{\partial \nu_\xi} = 0 & \text{on } \Gamma_\pm \cup \Upsilon, \end{cases} \qquad \begin{cases} \Delta_\xi Y_2 = \dfrac{\partial \widetilde{X}_0(\xi)}{\partial \xi_1} & \text{in } \Pi_{ah}, \\ Y_2 = 0 & \text{on } \Sigma, \\ \dfrac{\partial Y_2}{\partial \nu_\xi} = 0 & \text{on } \Gamma_\pm \cup \Upsilon, \end{cases} \tag{2.19}$$

$$\begin{cases} \Delta_\xi Y_3 = 0 & \text{in } \Pi_{ah}, \\ \dfrac{\partial Y_3}{\partial \nu_\xi} = 0 & \text{on } \Upsilon, \\ Y_3 = 0 & \text{on } \Sigma, \\ \dfrac{\partial Y_3}{\partial \nu_\xi} = \pm X_0 & \text{on } \Gamma_\pm, \end{cases} \qquad \begin{cases} \Delta_\xi Y_4 = 0 & \text{in } \Pi_{ah}, \\ \dfrac{\partial Y_4}{\partial \nu_\xi} = 0 & \text{on } \Upsilon, \\ Y_4 = 0 & \text{on } \Sigma, \\ \dfrac{\partial Y_4}{\partial \nu_\xi} = \pm \widetilde{X}_0 & \text{on } \Gamma_\pm, \end{cases} \tag{2.20}$$

$$\begin{cases} \Delta_\xi Y_5 = 0 & \text{in } \Pi_{ah}, \\ \dfrac{\partial Y_5}{\partial \nu_\xi} = 0 & \text{on } \Upsilon, \\ Y_5 = 0 & \text{on } \Sigma, \\ \dfrac{\partial Y_5}{\partial \nu_\xi} = Y_0 & \text{on } \Gamma_\pm. \end{cases} \tag{2.21}$$

To describe the solutions of the above problems, we need the following constants:

$$\begin{aligned}
&\frac{h}{2}(1-a)\widetilde{A}_{ah} = A_{ah} = \frac{1}{2}\int_{-a/2}^{a/2}\int_{-h/2}^{h/2}\frac{\partial Y_0(\xi)}{\partial \xi_1}\,d\xi,\\
&\frac{h}{2}(1-a)\widetilde{B}_{ah} = B_{ah} = \int_{-h/2}^{h/2} X_0\left(\frac{a}{2},\xi_2\right)d\xi_2.
\end{aligned} \tag{2.22}$$

We note that, by the Newton–Leibniz formula and the oddness of the function Y_0, we have

$$\int_{-a/2}^{a/2}\int_{-h/2}^{h/2}\frac{\partial Y_0(\xi)}{\partial \xi_1}\,d\xi = \int_{-h/2}^{h/2} Y_0(\xi_1,\xi_2)\,d\xi_2\Big|_{-a/2}^{a/2} = 2\int_{-h/2}^{h/2} Y_0\left(\frac{a}{2},\xi_2\right)d\xi_2. \tag{2.23}$$

The following lemma holds.

Lemma 2.2. *There are unique solutions of the problems* (2.16)–(2.18) *having the following asymptotic behaviour*:

$$\begin{aligned}
X_1(\xi) &= \frac{h}{2}(1-a)\widetilde{A}_{ah}|\xi_2| + \mathcal{O}(e^{-\pi|\xi_2|}) &&\text{as } \xi_2\to\pm\infty,\\
X_2(\xi) &= \frac{1}{2}\xi_2^2 + \frac{h}{2}(1-a)|\xi_2| + \mathcal{O}(e^{-\pi|\xi_2|}) &&\text{as } \xi_2\to\pm\infty,\\
X_3(\xi) &= \frac{h}{2}(1-a)\widetilde{B}_{ah}|\xi_2| + \mathcal{O}(e^{-\pi|\xi_2|}) &&\text{as } \xi_2\to\pm\infty,\\
X_4(\xi) &= \pm C_{ah} + \mathcal{O}(e^{-\pi|\xi_2|}) &&\text{as } \xi_2\to\pm\infty,\\
X_5(\xi) &= \frac{h}{2}(1-a)\widetilde{A}_{ah}|\xi_2| + \mathcal{O}(e^{-\pi|\xi_2|}) &&\text{as } \xi_2\to\pm\infty.
\end{aligned} \tag{2.24}$$

These solutions are even functions of ξ_1. *Moreover, the constant* $\widetilde{A}_{ah}$ *is positive.*

There are unique solutions of the problems (2.19)–(2.21) *that decay exponentially by the rule* $\mathcal{O}(e^{-\pi|\xi_2|})$ *as* $\xi_2\to\pm\infty$. *These solutions are odd functions of* ξ_1.

Proof. Taking into account the equations (2.22) and (2.23), we see that the proofs of all the assertions of the lemma, except for the positivity of the constant $\widetilde{A}_{ah}$, are simple modifications of the proof of the previous lemma.

We claim that $\widetilde{A}_{ah} > 0$. It suffices to prove that $A_{ah} > 0$. By (2.23), this will follow from the pointwise-positivity of the function $Y_0(\xi)$ on the set Γ_+. We claim that $Y_0(\xi)$ is non-negative on the closure of the set $\Pi_{ah}\cap\{\xi_1>0\}$. Since $Y_0(\xi)$ is an odd function of ξ_1, it follows that $Y_0(0,\xi_2)=0$. Suppose that $Y_0(\xi)$ takes negative values at some points of the set $\Pi_{ah}\cap\{\xi_1>0\}$. Since $Y_0(\xi)$ tends to zero as $\xi_2\to\pm\infty$, it follows that a negative minimum is attained at some point of the closure of the set $\Pi_{ah}\cap\{\xi_1>0\}$. For our choice of the boundary conditions, this contradicts the maximum principle. Moreover, $Y_0(\xi)$ is positive on Γ_+, since the corresponding normal derivative is positive. □

Lemma 2.3. *Let* $f\in H^{1/2}(\Gamma_1)$ *and* $g\in L_2(\Gamma_1)$ *be arbitrary functions and let* $u_0^{(1)},\dots,u_0^{(n)}$ *be orthonormalised (in* $L_2(\Gamma_1)$*) eigenfunctions of the homogenised problem* (1.3) *corresponding to the eigenvalue* λ_0. *Then the necessary and sufficient* *conditions for the solubility of the boundary-value problem*

$$\begin{cases} \Delta U = 0 \quad \textit{in} \ \Omega, \\ U = 0 \quad \textit{on} \ \Gamma_2, \\ \dfrac{\partial U}{\partial \nu} = 0 \quad \textit{on} \ \Gamma_3, \\ [U] = f(x_1) \quad \textit{on} \ \Gamma_1, \\ \left[\dfrac{\partial U}{\partial x_2}\right] = -2h\lambda_0 \dfrac{U(x_1,+0)+U(x_1,-0)}{2} + g(x_1) \quad \textit{on} \ \Gamma_1 \end{cases} \tag{2.25}$$

have the following form: *for all* $l = 1, \dots, n$,

$$\int_{\Gamma_1} g(x_1) u_0^{(l)}\, dx_1 + \frac{1}{2}\int_{\Gamma_1} \left(\frac{\partial u_0^{(l)}}{\partial x_2}(x_1,+0) + \frac{\partial u_0^{(l)}}{\partial x_2}(x_1,-0) \right) f(x_1)\, dx_1 = 0. \tag{2.26}$$

Proof. We first consider two problems with zero jump of the function on Γ_1:

$$\begin{cases} -\Delta W = 0 \quad \text{on } \Omega \setminus \Gamma_1, \\ W = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial W}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \\ [W] = 0 \quad \text{on } \Gamma_1, \\ \left[\dfrac{\partial W}{\partial x_2}\right] = -2h\lambda_0 W + G(x_1) \quad \text{on } \Gamma_1; \end{cases} \tag{2.27}$$

$$\begin{cases} -\Delta w = 0 \quad \text{on } \Omega \setminus \Gamma_1, \\ w = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial w}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \\ [w] = 0 \quad \text{on } \Gamma_1, \\ \left[\dfrac{\partial w}{\partial x_2}\right] = \widetilde{g}(x_1) \quad \text{on } \Gamma_1, \end{cases} \tag{2.28}$$

where $\widetilde{g} \in H^{-1/2}(\Gamma_1)$. The integral identity of the problem (2.28) has the form

$$\int_{\Omega} \nabla w \nabla v\, dx = \langle \widetilde{g}, v \rangle_{\Gamma_1} \quad \forall\, v \in H^1(\Omega, \Gamma_2), \tag{2.29}$$

where $\langle\,\cdot\,,\cdot\,\rangle_{\Gamma_1}$ stands for the action of a functional in $H^{-1/2}(\Gamma_1)$ on a function in $H^{1/2}(\Gamma_1)$. The operator of the problem(2.28) assigns to a function $\widetilde{g}(x_1)$ in the space $H^{-1/2}(\Gamma_1)$ a function w in the space $H^1(\Omega)$, as follows immediately from the integral identity by Riesz' theorem. In particular, this also implies the unique solubility of the problem (2.28) in $H^1(\Omega, \Gamma_2)$. Further, the trace of a function in the

Sobolev space $H^1(\Omega)$ is an element of the space $H^{1/2}(\Gamma_1)$, and therefore the linear operator $\mathcal{A}$ taking $\widetilde{g}$ to $w|_{\Gamma_1}$ is a bounded operator from $H^{-1/2}(\Gamma_1)$ to $H^{1/2}(\Gamma_1)$. As an operator from $L_2(\Gamma_1)$ to $L_2(\Gamma_1)$, this operator is compact. Using the integral identity (2.29), we can readily prove that it is positive definite and symmetric.

Thus, the eigenvalue problem (1.3) and the boundary-value problem (2.27), treated in $H^1(\Omega,\Gamma_2)$, are equivalent, respectively, to the equations

$$\begin{gathered} u_0|_{\Gamma_1} + 2h\lambda_0\mathcal{A}u_0|_{\Gamma_1} = 0, \\ W|_{\Gamma_1} + 2h\lambda_0\mathcal{A}W|_{\Gamma_1} = G \end{gathered} \tag{2.30}$$

in $L_2(\Gamma_1)$. The Fredholm alternative can be applied to the equation (2.30). Therefore, necessary and sufficient conditions for the solubility of the problem (2.27) are given by the orthogonality conditions

$$\int_{\Gamma_1} G(x_1)u_0^{(l)}(x_1,0)\,dx_1 = 0, \qquad l=1,\dots,n. \tag{2.31}$$

We note that for $G \in H^{-1/2}(\Gamma_1)$ the solubility condition for the problem (2.27) acquires the form $\langle G, u_0^{(l)}|_{\Gamma_1}\rangle_{\Gamma_1} = 0$.

We now reduce the problem (2.25) to a problem with zero jump of the function on Γ_1. To this end, we consider the auxiliary problem

$$\begin{cases} \Delta\mathcal{U}^\pm = 0 \quad \text{on } \Omega^\pm, \\ \mathcal{U}^\pm = 0 \quad \text{on } \Gamma_2\cap\Omega^\pm, \qquad \dfrac{\partial\mathcal{U}^\pm}{\partial\nu} = 0 \quad \text{on } \Gamma_3\cap\Omega^\pm, \\ \mathcal{U}^\pm = \mp\dfrac{f(x_1)}{2} \quad \text{on } \Gamma_1, \end{cases} \tag{2.32}$$

where $\Omega^\pm \equiv \Omega\cap\{x_2 \gtrless 0\}$, and write $\mathcal{U} = \mathcal{U}^\pm$ in $\Omega^\pm$. As in the above case of the integral identity (2.29), it can readily be derived from the integral identity corresponding to the problem (2.32) that $[\partial\mathcal{U}/\partial x_2] \in H^{-1/2}(\Gamma_1)$. We multiply the equation of the problem by $u_0^{(l)}$ and integrate over the domain, using Green's formula twice. We obtain

$$\int_{\Gamma_1}\biggl[\frac{\partial\mathcal{U}}{\partial x_2}\biggr]u_0^{(l)}\,dx_1 + \frac12\int_{\Gamma_1}\biggl(\frac{\partial u_0^{(l)}}{\partial x_2}(x_1,+0) + \frac{\partial u_0^{(l)}}{\partial x_2}(x_1,-0)\biggr)f(x_1)\,dx_1 = 0. \tag{2.33}$$

We will seek the function U in the form $U = W - \mathcal{U}$. Then we obtain the problem (2.27) for W, where

$$G(x_1) = g(x_1) - \biggl[\frac{\partial\mathcal{U}}{\partial x_2}\biggr].$$

It follows from this equation, from the necessary and sufficient conditions (2.31) for the solubility of the boundary-value problem (2.27), and from the equation (2.33) that the conditions (2.26) are necessary and sufficient for the solubility of the problem (2.25). This completes the proof of Lemma 2.3. □

We also need the following lemma (see [31]).

Lemma 2.4. *Let $\mathcal{K}\colon \mathcal{H}\to\mathcal{H}$ be a self-adjoint operator with discrete spectrum on a Hilbert space $\mathcal{H}$. Suppose that the following relations hold for $v\in\mathcal{H}$ and $\varpi\in\mathbb{R}$:*

$$\|v\|_{\mathcal{H}}=1,\qquad \varkappa:=\|\mathcal{K}v-\varpi v\|_{\mathcal{H}}<|\varpi|.$$

Then there is an eigenvalue ϖ_j of $\mathcal{K}$ such that

$$|\varpi_j-\varpi|\leqslant\varkappa.$$

Moreover, for every $\varkappa_1\in(\varkappa,|\varpi|)$ there are $\{a_j\}\in\mathbb{R}$ such that

$$\Bigl\|v-\sum a_j u_j\Bigr\|_{\mathcal{H}}\leqslant 2\frac{\varkappa}{\varkappa_1},$$

where the sum is taken over all the eigenvalues of $\mathcal{K}$ in the interval $[\varpi-\varkappa_1,\varpi+\varkappa_1]$ and $\{u_j\}$ are the corresponding eigenfunctions. The coefficients a_j satisfy the relation $\sum|a_j|^2=1$.

§ 3. Statement and proof of the main assertion

We denote by $(u,v)_{L_2(\Gamma_1)}$ and $\|u\|_{L_2(\Gamma_1)}$ the inner product and the norm in $L_2(\Gamma_1)$ and retain this notation for the traces of functions on Γ_1. We write

$$\langle u\rangle(x_1):=\frac{u(x_1+0)+u(x_1-0)}{2}.$$

This notation is used also for vector functions.

Let $u_0^{(l)}$ be the eigenfunctions of the boundary-value problem (1.3) corresponding to an eigenvalue λ_0 of multiplicity n and satisfying the following normalisation conditions:

$$\|u_0^{(l)}\|_{L_2(\Gamma_1)}=1;\qquad \bigl(u_0^{(l)},u_0^{(k)}\bigr)_{L_2(\Gamma_1)}=0\quad\text{for } l\neq k;$$
$$\biggl(\Bigl\langle\frac{\partial u_0^{(l)}}{\partial x_2}\Bigr\rangle,\Bigl\langle\frac{\partial u_0^{(k)}}{\partial x_2}\Bigr\rangle\biggr)_{L_2(\Gamma_1)}+(1-3\widetilde{A}_{ah})\biggl(\frac{\partial u_0^{(l)}}{\partial x_1},\frac{\partial u_0^{(k)}}{\partial x_1}\biggr)_{L_2(\Gamma_1)}=0\quad\text{for } l\neq k. \tag{3.1}$$

To prove the existence of these functions $u_0^{(l)}$ we first choose an arbitrary basis of the eigensubspace orthonormalised in $L_2(\Gamma_1)$. For the chosen matrix, the matrix whose components are defined by the left-hand side of the inequality in the second line of (3.1) (including $l=k$) is symmetric. This matrix can be reduced to diagonal form by an orthogonal transformation. Applying this transformation to the basis chosen originally, we obtain a basis satisfying all the conditions in (3.1).

We write

$$\lambda_1^{(l)}=\frac{1-a}{2}\biggl(\widetilde{B}_{ah}\lambda_0^2+\biggl\|\Bigl\langle\frac{\partial u_0^{(l)}}{\partial x_2}\Bigr\rangle\biggr\|^2_{L_2(\Gamma_1)}+(1-3\widetilde{A}_{ah})\biggl\|\frac{\partial u_0^{(l)}}{\partial x_1}\biggr\|^2_{L_2(\Gamma_1)}\biggr), \tag{3.2}$$

where the constants $\widetilde{A}_{ah}$ and $\widetilde{B}_{ah}$ are defined by the equations (2.22). If $\lambda_1^{(l)}=\lambda_1^{(l+1)}=\dots=\lambda_1^{(l+n_l-1)}$ and the other $\lambda_1^{(k)}$ have values different from $\lambda_1^{(l)}$, then we say that the multiplicity of $\lambda_1^{(l)}$ is equal to n_l. We refer to the linear subspace spanned by the corresponding eigenfunctions $u_0^{(l)}, u_0^{(l+1)}, \dots, u_0^{(l+n_l-1)}$ as the eigensubspace of $\lambda_1^{(l)}$.

By Proposition 1.1, there are n eigenvalues $\lambda_\varepsilon^{(l)}$ of the problem (1.1) (listed according to multiplicity) that converge to λ_0. We denote the corresponding orthonormalised (in $L_2(\Gamma_\varepsilon)$) eigenfunctions by $u_\varepsilon^{(l)}$. We extend the functions $u_\varepsilon^{(j)}$ to Ω in such a way that the following inequalities hold:

$$\|u_\varepsilon^{(j)}\|_{H^1(\Omega)} \leqslant C\|u_\varepsilon^{(j)}\|_{H^1(\Omega_\varepsilon)}$$

with a constant C independent of ε. This extension is possible by [32] and by our assumptions about the structure of Ω_ε. We retain the same notation for the extended functions.

The main content of this paper is the proof of the following theorem.

Theorem 3.1. *Let λ_0 be an eigenvalue of the problem (1.3) of multiplicity n and let $u_0^{(l)}$ be the corresponding eigenfunctions normalised by the conditions (3.1). Then the asymptotic formulae for the eigenvalues $\lambda_\varepsilon^{(l)}$ of the problem (1.1) that converge to λ_0 as $\varepsilon \to 0$ have the form*

$$\lambda_\varepsilon^{(l)} = \lambda_0 + \varepsilon\lambda_1^{(l)} + o(\varepsilon^{3/2-\mu}), \tag{3.3}$$

where $\lambda_1^{(l)}$ is defined by the equation (3.2) and μ is an arbitrarily small positive number.

If the multiplicity of $\lambda_1^{(l)}$ is equal to n_l, then the joint multiplicity of the eigenvalues $\lambda_\varepsilon^{(l)}$ (of the problem (1.1)) having the asymptotic behaviour (3.3) is also equal to n_l, and the subspace formed by the corresponding eigenfunctions $u_\varepsilon^{(l)}$ converges to the eigensubspace of $\lambda_1^{(l)}$ in $L_2(\Gamma_1)$.

Proof. To construct the asymptotic expansions, we use the method of matching different asymptotic expansions (see [33], [34], and [12]–[16] for a multiple eigenvalue). For the convenience of the reader, let us explain the main ideas of this method in connection with the problem considered in this paper. We are looking for the eigenfunctions and eigenvalues of the problem (1.1) in the form of asymptotic series in powers of the small parameter ε, and for the leading terms we choose the eigenfunction and the eigenvalue of the limit problem. In a neighbourhood of the perforated partition, we introduce stretched (internal) variables $\xi = x/\varepsilon$, after which we rewrite the functions entering the asymptotic expansion, and also the operator Δ and the boundary conditions on the partition in the coordinates ξ and x, and we assume the periodicity with respect to ξ_1. The periodicity condition is chosen in accordance with the fact that the perforation and boundary conditions on the partition have periodic microstructure. We say that the resulting expansion is internal. In the remaining part of the domain, we preserve the variables x in the asymptotic expansion. This expansion is said to be external. On the part of the domain on which $|x_2| \ll 1$ and $|\xi_2| \gg 1$, the coefficients of the Taylor series in x_2 for the functions of the external expansion must be consistent with the asymptotic expansion of the functions of the internal expansion as $|\xi_2| \to \infty$, which leads to matching conditions for the internal and external expansions.

Before passing to the proof of the theorem, we note that the convergence in (1.4) is equivalent to the convergence

$$\|u_\varepsilon^{(l)} - u_*^{(l)}\|_{L_2(\Gamma_1)} \to 0,$$

where $u_*^{(l)}$ is an element of the eigensubspace in $L_2(\Gamma_1)$ generated by the eigenfunctions $u_0^{(1)}, \dots, u_0^{(n)}$.

We denote by $\mathbf{u}_0$ the vector function with the components $u_0^{(1)}, \dots, u_0^{(n)}$.

Lemma 3.1. *The function $\mathbf{u}_0$ is infinitely smooth in the closed domains*

$$[-1/2, 1/2] \times [0, \pm s], \quad 0 < s < c.$$

Proof. Denoting the sets $[-1/2, 1/2] \times [0, \pm s]$ by the symbols $\Omega_{\pm s}$, we can readily see that the function $\mathbf{u}_0$ satisfies the following problem on the set $\Omega_{-s} \cup \Omega_{+s}$:

$$\begin{cases} -\Delta \widetilde{\mathbf{u}}_0 = 0 \quad \text{in } \Omega_{-s} \cup \Omega_{+s}, \\ \dfrac{\partial \widetilde{\mathbf{u}}_0}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \qquad [\widetilde{\mathbf{u}}_0] = 0 \quad \text{on } \Gamma_1, \\ \widetilde{\mathbf{u}}_0 = \mathbf{u}_0 \quad \text{on } \{x \in \Omega \colon x_2 = \pm s\}, \qquad \left[\dfrac{\partial \widetilde{\mathbf{u}}_0}{\partial x_2}\right] = -2h\lambda_0 \widetilde{\mathbf{u}}_0 \quad \text{on } \Gamma_1. \end{cases}$$

Using the symmetric reflection of the coefficients and the right-hand sides of this problem with respect to the vertical boundary, we reduce this problem to a problem with periodic boundary conditions on Γ_3. The corresponding periodic continuation is assumed in problems arising in the proof of the lemma.

It can readily be seen that this problem is coercive for sufficiently small $s = s(\lambda_0) > 0$, and therefore the solution $\widetilde{\mathbf{u}}_0$ exists and is unique.

Consider the auxiliary problems

$$\begin{cases} -\Delta \mathbf{u}_0^\pm = 0 \quad \text{in } \Omega_{\pm s}, \\ \dfrac{\partial \mathbf{u}_0^\pm}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \\ \mathbf{u}_0^\pm = \mathbf{h}_\pm \quad \text{on } \{x \in \Omega \colon x_2 = \pm s\}, \\ \dfrac{\partial \mathbf{u}_0^\pm}{\partial x_2} = \pm \mathbf{r}_\pm \quad \text{on } \Gamma_1 \end{cases}$$

and denote by $\mathfrak{A}_\pm$ the operators assigning to the functions $\mathbf{h}_\pm$ and $\mathbf{r}_\pm$ the restrictions to Γ_1 of the solutions $\mathbf{u}_0^\pm$. By Proposition 1.2 in [35], these Neumann–Dirichlet operators taking $\mathbf{r}_\pm$ to $\mathbf{u}_0^\pm\big|_{\Gamma_1}$ are elliptic pseudodifferential operators of order -1 with smooth symbols. We also consider the problem

$$\begin{cases} -\Delta \widehat{\mathbf{u}}_0 = 0 \quad \text{in } \Omega_{-s} \cup \Omega_{+s}, \\ \dfrac{\partial \widehat{\mathbf{u}}_0}{\partial \nu} = 0 \quad \text{on } \Gamma_3, \qquad [\widehat{\mathbf{u}}_0] = 0 \quad \text{on } \Gamma_1, \\ \widehat{\mathbf{u}}_0 = \mathbf{h}_\pm \quad \text{on } \{x \in \Omega \colon x_2 = \pm s\}, \qquad \left[\dfrac{\partial \widehat{\mathbf{u}}_0}{\partial x_2}\right] = \mathbf{r} \quad \text{on } \Gamma_1. \end{cases} \tag{3.4}$$

We note that the operator $\mathfrak{A} = \mathfrak{A}_+ + \mathfrak{A}_-$ taking the functions $\mathbf{h}_\pm$ and $\mathbf{r}$ to $\widehat{\mathbf{u}}_0|_{\Gamma_1}$ is also an elliptic pseudodifferential operator of order -1 with smooth symbol.

Substituting the functions $\mathbf{u}_0$ and $-2h\lambda_0\widetilde{\mathbf{u}}_0$ for $\mathbf{h}_\pm$ and $\mathbf{r}$, respectively, into (3.4) and considering the smoothing properties of the operator $\mathfrak{A}$, we obtain the desired assertion on the smoothness of $\mathfrak{A}$. □

We introduce the functions

$$\boldsymbol{\alpha}_0(x_1) := \mathbf{u}_0(x_1, 0), \quad \boldsymbol{\alpha}_{1,\pm}(x_1) := \frac{\partial \mathbf{u}_0}{\partial x_2}(x_1, \pm 0).$$

By Lemma 3.1 and properties of solutions of the boundary-value problem (1.3), for the function $\mathbf{u}_0(x)$ we have

$$\frac{\partial^2 \mathbf{u}_0}{\partial x_2^2}(x_1, \pm 0) = -\boldsymbol{\alpha}_0''(x_1) \in C^\infty\left[-\frac{1}{2}, \frac{1}{2}\right], \tag{3.5}$$

$$\boldsymbol{\alpha}_0'\left(\pm\frac{1}{2}\right) = \frac{\partial \mathbf{u}_0}{\partial x_1}\left(\pm\frac{1}{2}, 0\right) = 0, \tag{3.6}$$

$$\begin{aligned}\boldsymbol{\alpha}_{1,+}(x_1) - \boldsymbol{\alpha}_{1,-}(x_1) &= \frac{\partial \mathbf{u}_0}{\partial x_2}(x_1, +0) - \frac{\partial \mathbf{u}_0}{\partial x_2}(x_1, -0) \\ &= -2h\lambda_0 \mathbf{u}_0(x_1, 0) = -2h\lambda_0 \boldsymbol{\alpha}_0(x_1), \end{aligned}\tag{3.7}$$

$$\mathbf{u}_0(x) = \boldsymbol{\alpha}_0(x_1) + \boldsymbol{\alpha}_{1,\pm}(x_1)x_2 - \boldsymbol{\alpha}_0''(x_1)\frac{x_2^2}{2} + \mathcal{O}(x_2^3) \quad \text{as } x_2 \to \pm 0.$$

By making the change $x_2 = \varepsilon\xi_2$ in the last equation, we obtain

$$\mathbf{u}_0(x) = \boldsymbol{\alpha}_0(x_1) + \varepsilon\boldsymbol{\alpha}_{1,\pm}(x_1)\xi_2 - \varepsilon^2\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} + \mathcal{O}(\varepsilon^3\xi_2^3) \quad \text{as } x_2 = \varepsilon\xi_2 \to \pm 0.$$

By the method of matching asymptotic expansions, we conclude that the leading terms of the *internal* expansion in a neighbourhood of Γ_1 must have the form

$$\widehat{\mathbf{v}}_\varepsilon(x) = \mathbf{v}_0(\xi; x_1) + \varepsilon\mathbf{v}_1(\xi; x_1) + \varepsilon^2\mathbf{v}_2(\xi; x_1), \qquad \text{where } \xi = \frac{x}{\varepsilon}, \tag{3.8}$$

$$\mathbf{v}_0(\xi; x_1) \sim \boldsymbol{\alpha}_0(x_1) \qquad \text{as } \xi_2 \to \pm\infty, \tag{3.9}$$

$$\mathbf{v}_1(\xi; x_1) \sim \boldsymbol{\alpha}_{1,\pm}(x_1)\xi_2 \quad \text{as } \xi_2 \to \pm\infty, \tag{3.10}$$

$$\mathbf{v}_2(\xi; x_1) \sim -\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} \quad \text{as } \xi_2 \to \pm\infty. \tag{3.11}$$

Here x_1 is a 'slow' variable and $\xi = (\xi_1, \xi_2)$ is a 'fast' variable.

In the variables $(\xi; x_1)$, the Laplace operator and the boundary operator become

$$\Delta = -\varepsilon^{-2}\Delta_\xi - 2\varepsilon^{-1}\frac{\partial^2}{\partial x_1\, \partial \xi_1} - \frac{\partial^2}{\partial x_1^2}, \qquad \frac{\partial}{\partial \nu} = (\nu, \varepsilon^{-1}\nabla_\xi) + (\nu, \nabla_x). \tag{3.12}$$

It is sometimes convenient to use the notation

$$\frac{\partial}{\partial \nu_\xi} = (\nu, \varepsilon^{-1}\nabla_\xi), \qquad \frac{\partial}{\partial \nu_x} = (\nu, \varepsilon^{-1}\nabla_x).$$

Remark 3.1. By the ε-periodicity of the boundary of Ω_ε in a neighbourhood of the partition Γ_1, we seek the functions $\mathbf{v}_j(\xi; x_1)$ that are 1-periodic with respect to ξ_1.

We write

$$\widehat{\Lambda}_\varepsilon = \lambda_0 E + \varepsilon \Lambda_1, \tag{3.13}$$

where Λ_1 is an $n \times n$ diagonal matrix with, for now, arbitrary entries $\lambda_1^{(1)}, \dots, \lambda_1^{(n)}$, and E stands for the identity matrix. We denote the diagonal entries of the matrix $\widehat{\Lambda}_\varepsilon$ by $\widehat{\lambda}_\varepsilon^{(l)}$, $l = 1, \dots, n_1$. Then, taking into account Remark 3.1, substituting (3.8), (3.13) and (3.12) into (1.1), and equating coefficients of like powers of ε in the equations and boundary conditions thus obtained, we derive the following boundary value problems for $\mathbf{v}_j$:

$$\begin{cases} \Delta_\xi \mathbf{v}_0 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial \mathbf{v}_0}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon \cup \Gamma_\pm, \\ \mathbf{v}_0 \quad \text{is 1-periodic with respect to } \xi_1, \end{cases} \tag{3.14}$$

$$\begin{cases} \Delta_\xi \mathbf{v}_1 = -2\, \dfrac{\partial^2 \mathbf{v}_0}{\partial \xi_1\, \partial x_1} \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial \mathbf{v}_1}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon, \qquad \dfrac{\partial \mathbf{v}_1}{\partial \nu_\xi} = -\dfrac{\partial \mathbf{v}_0}{\partial \nu_x} + \lambda_0 \mathbf{v}_0 \quad \text{on } \Gamma_\pm, \\ \mathbf{v}_1 \quad \text{is 1-periodic with respect to } \xi_1, \end{cases} \tag{3.15}$$

$$\begin{cases} \Delta_\xi \mathbf{v}_2 = -2\, \dfrac{\partial^2 \mathbf{v}_1}{\partial \xi_1\, \partial x_1} - \dfrac{\partial^2 \mathbf{v}_0}{\partial x_1^2} \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial \mathbf{v}_2}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon, \qquad \dfrac{\partial \mathbf{v}_2}{\partial \nu_\xi} = -\dfrac{\partial \mathbf{v}_1}{\partial \nu_x} + \lambda_0 \mathbf{v}_1 + \Lambda_1 \mathbf{v}_0 \quad \text{on } \Gamma_\pm, \\ \mathbf{v}_2 \quad \text{is 1-periodic with respect to } \xi_1, \end{cases} \tag{3.16}$$

where the functions $\mathbf{v}_j$, $j = 0, 1, 2$, must satisfy the asymptotic conditions (3.9)–(3.11).

Obviously, the function

$$\mathbf{v}_0(\xi; x_1) \equiv \boldsymbol{\alpha}_0(x_1) \tag{3.17}$$

is a solution of the boundary-value problem (3.14) and has the required behaviour (3.9) as $\xi_2 \to \pm\infty$. Taking this identity into account, we can represent the boundary-value problem (3.15) in the form

$$\begin{cases} \Delta_\xi \mathbf{v}_1 = 0 \quad \text{in } \Pi_{ah}, \\ \dfrac{\partial \mathbf{v}_1}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon \cup \Sigma, \\ \dfrac{\partial \mathbf{v}_1}{\partial \nu_\xi} = \mp \boldsymbol{\alpha}_0'(x_1) + \lambda_0 \boldsymbol{\alpha}_0(x_1) \quad \text{on } \Gamma_\pm, \end{cases} \tag{3.18}$$

with the condition of periodicity on Σ replaced by the homogeneous Neumann condition. These problems are equivalent because of their symmetry with respect to the vertical coordinate axis ξ_2 and the uniqueness of the solution of each of them.

Taking (3.7) into account, we can readily see that for every function $\boldsymbol{\beta}(x_1)$, the function

$$\mathbf{v}_1(\xi;x_1) = \left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle(x_1)\, \widetilde{X}_0(\xi) + \lambda_0 \boldsymbol{\alpha}_0(x_1) X_0(\xi) - \boldsymbol{\alpha}_0'(x_1) Y_0(\xi) + \boldsymbol{\beta}(x_1) \tag{3.19}$$

is a solution of the boundary-value problem (3.18) and has the desired asymptotic behaviour (3.10). We note that a more precise asymptotic expansion for the function $\mathbf{v}_1$ is of the form

$$\mathbf{v}_1(\xi;x_1) = \boldsymbol{\alpha}_{1,\pm}(x_1)\xi_2 \pm C_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle(x_1) + \boldsymbol{\beta}(x_1) + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } \xi_2 \to \pm\infty. \tag{3.20}$$

Recalculating the asymptotic expansion of the sum (3.8) as $\xi_2 \to \pm\infty$ in the variables x_2 and taking (3.20) into account, we see that the external expansion of the eigenfunctions should be sought in the form

$$\widehat{\mathbf{u}}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x), \tag{3.21}$$

where

$$\mathbf{u}_1(x) \sim \pm C_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle(x_1) + \boldsymbol{\beta}(x_1) \quad \text{as } x_2 \to \pm 0.$$

Obviously, the last conditions are equivalent to the boundary conditions

$$\mathbf{u}_1(x_1, \pm 0) = \pm C_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle(x_1) + \boldsymbol{\beta}(x_1),$$

which, in turn, in the notation (3.5), are equivalent to the conditions

$$[\mathbf{u}_1](x_1) = 2C_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle(x_1), \tag{3.22}$$

$$\boldsymbol{\beta}(x_1) = \langle \mathbf{u}_1 \rangle(x_1). \tag{3.23}$$

Rewriting the asymptotic expansion of the function (3.21) as $x_2 \to 0$ in the internal variables ξ, we can refine the asymptotic behaviour of the function $\mathbf{v}_2(\xi;x_1)$ at infinity (3.11):

$$\mathbf{v}_2(\xi;x_1) \sim -\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} + \frac{\partial \mathbf{u}_1}{\partial x_2}(x_1, \pm 0)\xi_2 \quad \text{as } xi_2 \to \pm\infty. \tag{3.24}$$

On the other hand, it follows from Lemmas 2.1 and 2.2 that for every function $\boldsymbol{\varphi}(x_1)$ the function

$$\begin{aligned}
\mathbf{v}_2(\xi;x_1) = {} & -2\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle'(x_1)\, Y_2(\xi) - 2\lambda_0 \boldsymbol{\alpha}_0'(x_1) Y_1(\xi) + 2\boldsymbol{\alpha}_0''(x_1) X_1(\xi) \\
& - \boldsymbol{\alpha}_0''(x_1) X_2(\xi) - \left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle'(x_1)\, Y_4(\xi) - \lambda_0 \boldsymbol{\alpha}_0'(x_1) Y_3(\xi) \\
& + \boldsymbol{\alpha}_0''(x_1) X_5(\xi) + \lambda_0 \left\langle \frac{\partial \mathbf{u}_0}{\partial x_2} \right\rangle(x_1)\, X_4(\xi) + \lambda_0^2 \boldsymbol{\alpha}_0(x_1) X_3(\xi) \\
& - \lambda_0 \boldsymbol{\alpha}_0'(x_1) Y_5(\xi) + \Lambda_1 \boldsymbol{\alpha}_0(x_1) X_0(\xi) + \lambda_0 \boldsymbol{\beta}(x_1) X_0(\xi) \\
& - \boldsymbol{\beta}'(x_1) Y_0(\xi) + \boldsymbol{\varphi}(x_1)\widetilde{X}_0(\xi)
\end{aligned} \tag{3.25}$$

is a solution of the boundary-value problem (3.16) and has the asymptotic behaviour

$$\begin{aligned}\mathbf{v}_2(\xi;x_1) = -\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} + \biggl(\boldsymbol{\alpha}_0''(x_1)\Bigl(3A_{ah} - \frac{h}{2}(1-a)\Bigr) + \lambda_0^2\boldsymbol{\alpha}_0(x_1)B_{ah} \\ - h\Lambda_1\boldsymbol{\alpha}_0(x_1) - h\lambda_0\boldsymbol{\beta}(x_1)\biggr)|\xi_2| + \boldsymbol{\varphi}(x_1)\xi_2 \pm C_{ah}\boldsymbol{\varphi}(x_1) \\ \pm \lambda_0\Bigl\langle\frac{\partial\mathbf{u}_0}{\partial x_2}\Bigr\rangle(x_1)\,C_{ah} + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } |\xi_2|\to\infty.\end{aligned}$$

Comparing this equation with (3.24) and taking (2.22) into account, we obtain

$$\begin{aligned}\Bigl[\frac{\partial\mathbf{u}_1}{\partial x_2}\Bigr](x_1) &= h(1-a)\bigl(\boldsymbol{\alpha}_0''(x_1)(3\widetilde{A}_{ah}-1) + \lambda_0^2\boldsymbol{\alpha}_0(x_1)\widetilde{B}_{ah}\bigr) \\ &\qquad - 2h\bigl(\Lambda_1\boldsymbol{\alpha}_0(x_1) + \lambda_0\boldsymbol{\beta}(x_1)\bigr), \\ \Bigl\langle\frac{\partial\mathbf{u}_1}{\partial x_2}\Bigr\rangle(x_1) &= \boldsymbol{\varphi}(x_1).\end{aligned} \tag{3.26}$$

In turn, this, together with (3.5) and (3.23), implies that

$$\begin{aligned}\Bigl[\frac{\partial\mathbf{u}_1}{\partial x_2}\Bigr](x_1) = -2h\lambda_0\langle\mathbf{u}_1\rangle(x_1) + h(1-a)\biggl(\frac{\partial^2\mathbf{u}_0}{\partial x_1^2}(x_1,0)(3\widetilde{A}_{ah}-1) \\ + \lambda_0^2\mathbf{u}_0(x_1,0)\widetilde{B}_{ah}\biggr) - 2h\Lambda_1\mathbf{u}_0(x_1,0).\end{aligned} \tag{3.27}$$

Substituting (3.21) into (1.1), we obtain the following equation and boundary conditions for $\mathbf{u}_1$:

$$\begin{cases}-\Delta\mathbf{u}_1 = 0 \quad \text{in } \Omega\setminus\Gamma_1, \\ \mathbf{u}_1 = 0 \quad \text{on } \Gamma_2, \qquad \dfrac{\partial\mathbf{u}_1}{\partial\nu} = 0 \quad \text{on } \Gamma_3.\end{cases} \tag{3.28}$$

Since

$$\int_{\Gamma_1}\frac{\partial^2 u_0^{(l)}}{\partial x_1^2}u_0^{(k)}\,dx_1 = -\int_{\Gamma_1}\frac{\partial u_0^{(l)}}{\partial x_1}\,\frac{\partial u_0^{(k)}}{\partial x_1}\,dx_1$$

by (3.6), it follows from Lemma 2.3 and the equations (3.1) that the boundary-value problem (3.28), (3.22), (3.27) is soluble when the entries of the diagonal matrix Λ_1 are defined by the equations (3.2). Thus, the formulae (3.2) are obtained (at the formal level).

We note that the functions $\boldsymbol{\beta}(x_1)$ and $\boldsymbol{\varphi}(x_1)$ are defined by (3.23) and (3.26), respectively. Thus, we finally determine $\mathbf{v}_1(\xi)$ and $\mathbf{v}_2(\xi)$ and achieve the validity of the relations

$$\begin{aligned}\mathbf{v}_1(\xi) &= \frac{\partial\mathbf{u}_0}{\partial x_2}(x_1,\pm0)\xi_2 + \mathbf{u}_1(x_1,\pm0) + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } \xi_2\to\pm\infty, \\ \mathbf{v}_2(\xi) &= -\frac{\partial^2\mathbf{u}_0}{\partial x_1^2}(x_1,\pm0)\frac{\xi_2^2}{2} + \frac{\partial\mathbf{u}_1}{\partial x_2}(x_1,\pm0)\xi_2 \\ &\qquad + C_{ah}\boldsymbol{\varphi}(x_1) \pm \lambda_0\Bigl\langle\frac{\partial\mathbf{u}_0}{\partial x_2}\Bigr\rangle(x_1)\,C_{ah} + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } \xi_2\to\pm\infty.\end{aligned} \tag{3.29}$$

We proceed with the rigorous justification of the formulae thus obtained (that is, with the completion of the proof of Theorem 3.1).

Let $\chi(t)$ be an infinitely differentiable truncating function vanishing for $|t| < 1$ and equal to 1 for $|t| > 2$ and let γ be an arbitrary number in the interval $(0, 1)$. Write

$$\widetilde{\mathbf{u}}_\varepsilon(x) = \chi(\varepsilon^{-\gamma}x_2)\widehat{\mathbf{u}}_\varepsilon(x) + \big(1 - \chi(\varepsilon^{-\gamma}x_2)\big)\widehat{\mathbf{v}}_\varepsilon(x). \tag{3.30}$$

Then it follows from (3.21), (1.3), (3.28) and (3.8), (3.13)–(3.16), (3.18) that this function is a solution of the following problem:

$$\begin{cases} -\Delta\widetilde{\mathbf{u}}_\varepsilon = \mathbf{F}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \widetilde{\mathbf{u}}_\varepsilon = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial\widetilde{\mathbf{u}}_\varepsilon}{\partial\nu} = 0 \quad \text{on } \Upsilon_\varepsilon \cup \Gamma_3^\varepsilon, \qquad \dfrac{\partial\widetilde{\mathbf{u}}_\varepsilon}{\partial\nu} = \widehat{\Lambda}_\varepsilon\widetilde{\mathbf{u}}_\varepsilon + \mathbf{g}_\varepsilon \quad \text{on } \Gamma_\varepsilon, \end{cases} \tag{3.31}$$

where

$$\mathbf{F}_\varepsilon(x) = \varepsilon\mathbf{F}_{1,\varepsilon}(x) + \mathbf{F}_{2,\varepsilon}(x) + \mathbf{F}_{3,\varepsilon}(x), \tag{3.32}$$

$$\begin{aligned} \mathbf{F}_{1,\varepsilon}(x) = -\left(1 - \chi\left(\frac{x_2}{\varepsilon^\gamma}\right)\right)\Bigg(2\,\frac{\partial^2\mathbf{v}_2}{\partial\xi_1\,\partial x_1}\left(x_1, \frac{x}{\varepsilon}\right) \\ + \frac{\partial^2\mathbf{v}_1}{\partial x_1^2}\left(x_1, \frac{x}{\varepsilon}\right) + \varepsilon\,\frac{\partial^2\mathbf{v}_2}{\partial x_1^2}\left(x_1, \frac{x}{\varepsilon}\right)\Bigg), \end{aligned} \tag{3.33}$$

$$\mathbf{F}_{2,\varepsilon}(x) = -\varepsilon^{-2\gamma}\chi''\left(\frac{x_2}{\varepsilon^\gamma}\right)\big(\widehat{\mathbf{u}}_\varepsilon(x) - \widehat{\mathbf{v}}_\varepsilon(x)\big), \tag{3.34}$$

$$\mathbf{F}_{3,\varepsilon}(x) = -\varepsilon^{-\gamma}\chi'\left(\frac{x_2}{\varepsilon^\gamma}\right)\frac{\partial}{\partial x_2}\big(\widehat{\mathbf{u}}_\varepsilon(x) - \widehat{\mathbf{v}}_\varepsilon(x)\big), \tag{3.35}$$

$$\mathbf{g}_\varepsilon(x) = \frac{\partial\mathbf{v}_\varepsilon}{\partial\nu} - \widehat{\Lambda}_\varepsilon\mathbf{v}_\varepsilon = \varepsilon^2\left(\frac{\partial\mathbf{v}_2}{\partial\nu_x} - \lambda_0\mathbf{v}_2 - \Lambda_1(\mathbf{v}_1 + \varepsilon\mathbf{v}_2)\right). \tag{3.36}$$

By (3.33) and (3.29), we see that $\operatorname{supp}\mathbf{F}_{1,\varepsilon} \subset \{x\colon |x_2| \leqslant 2\varepsilon^\gamma\}$, and hence

$$\|\varepsilon\mathbf{F}_{1,\varepsilon}\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{3\gamma/2}). \tag{3.37}$$

It follows from (3.21), (3.8), (3.17) and (3.29) that

$$\widehat{\mathbf{u}}_\varepsilon(x) - \widehat{\mathbf{v}}_\varepsilon(x) = \mathcal{O}(\varepsilon^{3\gamma} + \varepsilon^2) \quad \text{for } \varepsilon^\gamma \leqslant |\xi_2| \leqslant 2\varepsilon^\gamma, \tag{3.38}$$

$$\frac{\partial\widehat{\mathbf{u}}_\varepsilon}{\partial x_2}(x) - \frac{\partial\widehat{\mathbf{v}}_\varepsilon}{\partial x_2}(x) = \mathcal{O}(\varepsilon^{2\gamma}) \quad \text{for } \varepsilon^\gamma \leqslant |\xi_2| \leqslant 2\varepsilon^\gamma. \tag{3.39}$$

By (3.34) and (3.38), we see that $\operatorname{supp}\mathbf{F}_{2,\varepsilon} \subset \{x\colon \varepsilon^\gamma \leqslant |x_2| \leqslant 2\varepsilon^\gamma\}$, and hence

$$\|\mathbf{F}_{2,\varepsilon}\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{3\gamma/2}). \tag{3.40}$$

Similarly, it follows from (3.35) and (3.39) that $\|\mathbf{F}_{3,\varepsilon}\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{3\gamma/2})$. This, together with (3.32), (3.37) and (3.40), implies that

$$\|\mathbf{F}_\varepsilon\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}(\varepsilon^{3\gamma/2}). \tag{3.41}$$

Finally, it follows from (3.36), (3.17), (3.19), (3.25) and (3.23), (3.26) that

$$\|\mathbf{g}_\varepsilon\|_{L_2(\Gamma_\varepsilon)} = \mathcal{O}(\varepsilon^2), \tag{3.42}$$

$$\|\widetilde{\mathbf{u}}_\varepsilon - \mathbf{u}_0\|_{L_2(\Gamma_\varepsilon)} = \mathcal{O}(\varepsilon). \tag{3.43}$$

Consider the boundary-value problem

$$\begin{cases} -\Delta \mathbf{w}_\varepsilon = \mathbf{F}_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ \mathbf{w}_\varepsilon = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial \mathbf{w}_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Upsilon_\varepsilon \cup \Gamma_3^\varepsilon, \qquad \dfrac{\partial \mathbf{w}_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Gamma_\varepsilon. \end{cases} \tag{3.44}$$

It can readily be seen that this problem is uniquely soluble in $H^1(\Omega, \Gamma_2)$, and, by (3.41), the solution satisfies the bound

$$\|\mathbf{w}_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \leqslant C\varepsilon^{3\gamma/2}. \tag{3.45}$$

Consider the boundary-value problem

$$\begin{cases} -\Delta z_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ z_\varepsilon = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial z_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Upsilon_\varepsilon \cup \Gamma_3^\varepsilon, \qquad \dfrac{\partial z_\varepsilon}{\partial \nu} = g \quad \text{on } \Gamma_\varepsilon, \end{cases} \tag{3.46}$$

where $g \in H^{-1/2}(\Gamma_\varepsilon)$. We denote by $\mathcal{A}_\varepsilon$ the linear operator $\mathcal{A}_\varepsilon\colon H^{-1/2}(\Gamma_\varepsilon) \to H^{1/2}(\Gamma_\varepsilon)$ assigning to a function g the restriction of the solution z_ε to Γ_ε:

$$\mathcal{A}_\varepsilon g = z_\varepsilon|_{\Gamma_\varepsilon}.$$

Being an operator from $L_2(\Gamma_\varepsilon)$ to $L_2(\Gamma_\varepsilon)$, this is a compact self-adjoint operator for any fixed ε, and its characteristic numbers coincide with the eigenvalues of the problem (1.1). By the integral identity for the problem (3.46), we can readily prove that

$$\|\mathcal{A}_\varepsilon\|_{\mathcal{L}(H^{-1/2}(\Gamma_\varepsilon), H^{1/2}(\Gamma_\varepsilon))} \leqslant C \tag{3.47}$$

with a constant C independent of ε.

By (3.31) and (3.44), the function

$$\check{\mathbf{u}}_\varepsilon = \widetilde{\mathbf{u}}_\varepsilon - \mathbf{w}_\varepsilon \tag{3.48}$$

is a solution of the problem

$$\begin{cases} -\Delta \check{\mathbf{u}}_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon, \\ \check{\mathbf{u}}_\varepsilon = 0 \quad \text{on } \Gamma_2, \\ \dfrac{\partial \widetilde{\mathbf{u}}_\varepsilon}{\partial \nu} = 0 \quad \text{on } \Upsilon_\varepsilon \cup \Gamma_3^\varepsilon, \\ \dfrac{\partial \check{\mathbf{u}}_\varepsilon}{\partial \nu} = \widehat{\Lambda}_\varepsilon \check{\mathbf{u}}_\varepsilon - \widehat{\Lambda}_\varepsilon \mathbf{w}_\varepsilon + \mathbf{g}_\varepsilon \quad \text{on } \Gamma_\varepsilon. \end{cases}$$

Hence, $\widehat{\Lambda}_\varepsilon \mathcal{A}_\varepsilon(\check{\mathbf{u}}_\varepsilon - \mathbf{w}_\varepsilon)|_{\Gamma_\varepsilon} = \check{\mathbf{u}}_\varepsilon|_{\Gamma_\varepsilon} - \mathcal{A}_\varepsilon \mathbf{g}_\varepsilon$. Therefore, by (3.45), (3.47) and (3.42), we see that

$$\begin{aligned}\|\mathcal{A}_\varepsilon \check{\mathbf{u}}_\varepsilon - (\widehat{\Lambda}_\varepsilon)^{-1}\check{\mathbf{u}}_\varepsilon\|_{L_2(\Gamma_\varepsilon)} &= \|\mathbf{w}_\varepsilon - (\widehat{\Lambda}_\varepsilon)^{-1}\mathcal{A}_\varepsilon \mathbf{g}_\varepsilon\|_{L_2(\Gamma_\varepsilon)} \\ &= \mathcal{O}(\varepsilon^{3\gamma/2}) = o(\varepsilon^{3/2-\mu}) \quad \forall\, \mu > 0. \qquad (3.49)\end{aligned}$$

We now show how to derive the assertions of Theorem 3.1 from this bound using Lemma 2.4 and Proposition 1.1. As is shown below (see the end of the proof of the theorem),

$$0 < c_1 \leqslant \|\check{u}_\varepsilon^{(l)}\|_{L_2(\Gamma_\varepsilon)} \leqslant c_2 < \infty, \qquad (3.50)$$

$$(\check{u}_\varepsilon^{(l)}, \check{u}_\varepsilon^{(k)})_{L_2(\Gamma_\varepsilon)} = o(1), \qquad l \neq k, \qquad (3.51)$$

where $\check{u}_\varepsilon^{(l)}$ is the lth component of the vector $\check{\mathbf{u}}_\varepsilon$. It follows immediately from (3.50), (3.49) and Lemma 2.4 that, for every $\lambda_1^{(l)}$ defined by the equation (3.2), there is an eigenvalue of the problem (1.1) that has the asymptotic behaviour (3.3). Here, if $\lambda_1^{(l)} \neq \lambda_1^{(k)}$ when $l \neq k$, then by Proposition 1.1, all the eigenvalues of the problem (1.1) are simple and have the asymptotic behaviour (3.3).

Let a number $\lambda_1^{(l)} = \cdots = \lambda_1^{(l+n_l-1)}$ defined by the equation (3.2) have multiplicity n_l. We set $\varkappa_1 = \varepsilon^{3/2-2\mu}$ in Lemma 2.4. Then we may assume without loss of generality that

$$\left\| \check{u}_\varepsilon^{(l+i-1)} - \sum_{k=1}^{m_i(\varepsilon)} a_k^i u_\varepsilon^{(l+k-1)} \right\|_{L_2(\Gamma_\varepsilon)} \leqslant C\varepsilon^\mu, \qquad i = 1, \ldots, n_l, \qquad (3.52)$$

where the $u_\varepsilon^{(k)}$ are the orthonormalised (in $L_2(\Gamma_\varepsilon)$) eigenfunctions of the problem (1.1) that correspond to all the eigenvalues $\lambda_\varepsilon^{(k)}$ in the interval $I_l(\varepsilon) := (\lambda_0 + \varepsilon\lambda_1^{(l)} - \varepsilon^{3/2-2\mu}, \lambda_0 + \varepsilon\lambda_1^{(l)} + \varepsilon^{3/2-2\mu})$. We stress that

$$\lambda_0 + \varepsilon\lambda_1^{(j)} \notin I_l(\varepsilon), \qquad j \neq l, \ldots, l + n_l - 1.$$

It follows from (3.52), (3.50) and (3.51) that $m_i(\varepsilon) \geqslant n_l$. On the other hand, since the total number of eigenvalues of the problem (1.1) that converge to $\lambda_{0,j}$ is equal to n, it follows that $m_i(\varepsilon) = n_l$. Thus, we have proved that the number of eigenvalues of (1.1) that have the asymptotic behaviour (3.3) is equal to n_l.

We now pass to the proof of the convergence of eigenfunctions on Γ_1. We note that, by the integral identities of the problems (1.1) and (1.3) and the normalisation conditions for the eigenfunctions $u_\varepsilon^{(j)}$ and $u_0^{(j)}$, the following bounds hold:

$$\|u_\varepsilon^{(j)}\|_{H^1(\Omega_\varepsilon)} \leqslant C, \qquad \|u_0^{(j)}\|_{H^1(\Omega)} \leqslant C.$$

We extend the functions $\check{u}_\varepsilon^{(j)}$ to Ω in such a way that the following inequalities hold:

$$\|\check{u}_\varepsilon^{(j)}\|_{H^1(\Omega)} \leqslant C\|\check{u}_\varepsilon^{(j)}\|_{H^1(\Omega_\varepsilon)}$$

with a constant C independent of ε, retaining the same notation for the extended functions.

By Lemma 3.5 in [17], for arbitrary $v_1 \in H^1(\Omega_\varepsilon)$ and $v_2 \in H^1(\Omega_\varepsilon)$ we have

$$\left| \int_{\Gamma_\varepsilon} v_1 v_2 \, dx_2 - h \int_{-1/2}^{1/2} v_1(x_1, \varepsilon h) v_2(x_1, \varepsilon h) \, dx_1 \right| \leqslant C\varepsilon^{1/2} \|v_1\|_{H^1(\Omega_\varepsilon)} \|v_2\|_{H^1(\Omega_\varepsilon)}.$$

Further, by the theorem on the continuity of traces for $v_1, v_2 \in H^1(\Omega)$, we obtain

$$\left| \int_{\Gamma_1} v_1 v_2 \, dx_1 - \int_{-1/2}^{1/2} v_1(x_1, \varepsilon h) v_2(x_1, \varepsilon h) \, dx_1 \right| \leqslant C\varepsilon^{1/2} \|v_1\|_{H^1(\Omega_\varepsilon)} \|v_2\|_{H^1(\Omega_\varepsilon)}.$$

The last two inequalities imply the bound

$$\left| \int_{\Gamma_\varepsilon} v_1 v_2 \, dx_2 - h \int_{\Gamma_1} v_1 v_2 \, dx_1 \right| \leqslant C\varepsilon^{1/2} \|v_1\|_{H^1(\Omega_\varepsilon)} \|v_2\|_{H^1(\Omega_\varepsilon)}. \tag{3.53}$$

The definitions (3.48) and (3.30) of $\breve{u}_\varepsilon^{(j)}$ and $\widetilde{u}_\varepsilon^{(j)}$, the integral identity for the problem (3.44), and the boundedness of the operator of extension from $H^1(\Omega_\varepsilon)$ to $H^1(\Omega)$, the bounds (3.43) and (3.45) yield that

$$\|\breve{u}_\varepsilon^{(j)}\|_{H^1(\Omega)} \leqslant C, \quad \|\mathbf{w}_\varepsilon\|_{H^1(\Omega)} = o(1), \quad \|\breve{u}_\varepsilon^{(j)} - u_0^{(j)}\|_{H^1(\Omega)} = o(1). \tag{3.54}$$

Taking into account (3.53), (1.2) and (3.52), we derive from (3.54) the bound

$$\left\| \breve{u}_\varepsilon^{(j)} - \sum_{k=1}^{n_l} a_k^j u_\varepsilon^{(k)} \right\|_{L_2(\Gamma_1)} \leqslant C\varepsilon^{\mu}, \qquad j = 1, \dots, n_l.$$

Therefore,

$$\left\| u_0^{(j)} - \sum_{k=1}^{n_1} a_k^j u_\varepsilon^{(k)} \right\|_{L_2(\Gamma_1)} = o(1), \qquad j = 1, \dots, n_1.$$

To complete the proof of the theorem, it remains to justify the relations (3.50) and (3.51). It follows from the third bound in (3.54) and from (3.53) that

$$\|\breve{u}_\varepsilon^{(j)} - u_0^{(j)}\|_{L_2(\Gamma_\varepsilon)} = o(1).$$

Again using the bound (3.53) and recalling the normalisation conditions (3.1), we obtain $\|\breve{u}_\varepsilon^{(j)}\|_{L_2(\Gamma_\varepsilon)} = h^{-1} + o(1)$, which gives the bound (3.50). The relation (3.51) can be proved analogously.

This completes the proof of Theorem 3.1. □

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