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HOMOGENIZATION OF SINGULARLY PERTURBED OPERATORS

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To the memory of Serguei Kozlov

Abstract: The asymptotic behaviour of effective diffusion for a singularly perturbed parabolic operator

$$\frac{\partial}{\partial t} - \mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} - \mu b_i(x) \frac{\partial}{\partial x_i} - v(x)$$

is studied. We obtain logarithmic asymptotics of effective diffusion with respect to a small positive parameter μ in terms of auxiliary variational problems on the torus of periodicity.

1. Introduction.

It is well known in homogenization theory that the limiting behaviour of effective diffusion for singularly perturbed operators depends essentially on the structure of low order terms. For instance, in the presence of a vector field the effective diffusion is much more the initial diffusion in case of solenoidal vector field (see [1], [2]) and, conversely, decays exponentially in case of potential vector field (see [3], [4]) or vector field which in some sense is close to potential [5]. As was shown in [6], [7] the effective diffusion for the equations with a periodic potential is also exponentially small with respect to the initial diffusivity.

In the paper we study a singularly perturbed parabolic equation

$$\begin{aligned} \frac{\partial}{\partial t} u = \mu^2 \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} u \ + \ \mu b_i(x) \frac{\partial}{\partial x_i} u \ + \ v(x) u \\ u|_{t=0} \ = u_0(x) \end{aligned} \tag{1.1}$$

with 'potential' vector field

$$b_i(x) = a_{ij}(x) \frac{\partial}{\partial x_j} U(x)$$

We assume that both the coefficients of (1.1) and the potential $U(x)$ are smooth periodic functions and that the matrix $a_{ij}(x)$ is symmetric and uniformly elliptic. The goal of this work is to obtain the asymptotics of effective diffusion in (1.1) as the initial diffusion goes to zero. Let us recall the definition of effective diffusion. To this end consider on the torus of periodicity T^n the following eigenvalue problem

$$\left(\mu^2\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j} + \mu b_i(x)\frac{\partial}{\partial x_i} + v(x)\right)p = \lambda p \qquad (1.2)$$

and denote by λ_0 and $p_0(x)$ its first eigenvalue and first eigenfunction respectively. According to [8] for any positive μ the eigenvalue λ_0 is real and simple and the eigenfunction $p_0(x)$ is also real and positive. To fix the choice of $p_0(x)$ we assume the normalization condition $< p_0 >= 1$ to hold; here $< \cdot >$ means the average of a periodic function over the period. Then, as follows from [9], the solution $u(x,t)$ of (1.1) admits in the region $\{(x,t)|\ x^2 \le ct + c_1,\ t\to\infty\}$ the following asymptotics

$$u(x,t) = \exp(\lambda_0 t)p_0(x)\bar{u}(x,t)(1+o(1)) \qquad (1.3).$$

The function $\bar{u}(x,t)$ describing the diffusive properties of $u(x,t)$, satisfies the homogenized parabolic equation with constant coefficients

$$\frac{\partial}{\partial t}\bar{u} = \frac{\partial}{\partial x_i}\sigma_{ij}\frac{\partial}{\partial x_j}\bar{u}, \qquad \bar{u}|_{t=0} =< p_0^2 > u_0(x) \qquad (1.4)$$

The matrix $\{\sigma_{ij}\} = \{\sigma_{ij}(\mu)\}$ is called the effective diffusion.

The approach we use to find the asymptotics of $\{\sigma_{ij}(\mu)\}$ as $\mu\to 0$, relies on the results of [10] where the uniform logarithmic asymptotics of the ground state $p_0(x)$ all over the torus was obtained. This allows to transform equation (1.1) to an equation without potential. Then, taking into account the special form of the first order terms in the last equation and applying the rough estimates for the effective diffusion we replace this equation by another one of divergence form. Finally, we show that the effective diffusion $\sigma(\mu)$ satisfies the following limiting relation

$$\lim_{\mu\to 0}\mu\ln\sigma(\mu) = -\mathcal{A},$$

where $\mathcal{A}$ is a positive matrix whose coefficients can be found in terms of auxiliary variational problems.

2. The properties of the ground state of problem (1.2).

In this section the asymptotic behaviour of a ground state in (1.2) is studied. We start with the description of the first eigenvalue λ_0.

Theorem 2.1 *Let $U(x)$ be a 'potential' of the vector field $b(x)$: $b(x) = \{a_{ij}(x)\}\nabla U(x)$. Then,*

$$\lim_{\mu\to 0}\lambda_0 = -\min_{x\in T^n}\left(a_{ij}(x)\frac{\partial}{\partial x_i}U(x)\frac{\partial}{\partial x_j}U(x) - v(x)\right).$$

Proof: See [10], theorem 3. □

Denote the limit $(\lim_{\mu\to 0}\lambda_0)$ by $\bar\lambda$. To describe the asymptotic behaviour of $p_0(x)$ let us introduce the following functional: for absolutely continuous curves $x(t) = (x_1(t),...,x_n(t))$, $0 \le t \le T$, on T^n

$$I(x(\cdot),T) = \int_0^T \big(a^{ij}(x(t))[\dot x_i(t) - b_i(x(t))][\dot x_i(t) - b_i(x(t))] - v(x(t))\big)dt,$$

where $\{a^{ij}(x)\} = \{a_{ij}(x)\}^{-1}$. To extend this functional onto the whole space $C(0,T;T^n)$ of continuous functions we put $I(x(\cdot),T) = \infty$ for other $x(\cdot)$. It is clear that $I(x(\cdot),T)$ is bounded from below. Then, for any x^1, $x^2 \in T^n$ we define

$$S(x^1,x^2,T) = \inf_{x(\cdot),\, x(0)=x^1,\, x(T)=x^2} I(x(\cdot),T)$$

Now, we need the following

Definition 2.2 We say that the operator A^μ is recursive if the function

$$\Big(a_{ij}(x)\frac{\partial}{\partial x_i}U(x)\frac{\partial}{\partial x_j}U(x) - v(x)\Big)$$

has only one minimum point on T^n. This minimum point denoted by x^0, is called recursive point of A^μ.

For a recursive operator we define the functions $W_0(x)$ and $W(x)$ as follows

$$W_0(x) = \inf_{t>0} \inf_{\substack{x(\cdot),\, x(0)=x,\\ x(t)=x^0}} (I(x(\cdot),t) + \bar\lambda t) = \inf_{t>0}(S(x,x^0,t) + \bar\lambda t),$$

$$W(x) = W_0(x) - \min_{y\in T^n} W_0(y);$$

here x^0 is the recursive point of A^μ. Two assertions below are proved in [10].

Proposition 2.3 *$W(x)$ is well-defined Lipschitz function on T^n.*

Theorem 2.4 *Let operator A^μ be recursive. Then uniformly in $x \in T^n$*

$$\lim_{\mu\to 0} \mu \ln p_0(x) = -W(x).$$

3. The asymptotics of effective diffusion.

The aim of this section is to construct the logarithmic asymptotics of the effective diffusion $\sigma_{ij}(\mu)$. Later on we mainly follow the work [7]. First let us introduce new unknown function

$$z(x,t) = \exp(-\lambda_0 t)p_0^{-1}(x)u(x,t).$$

Substituting the expression $e^{\lambda_0 t}p_0(x)z(x,t)$ into (1.1) instead of $u(x,t)$ and considering (1.2) we find the equation for $z(x,t)$:

$$p_0(x)\frac{\partial}{\partial t}z(x,t) = p_0(x)\mu^2\frac{\partial}{\partial x_i}a_{ij}(x)\frac{\partial}{\partial x_j}z(x,t) + 2\mu^2 a_{ij}(x)\left(\frac{\partial}{\partial x_i}p_0(x)\right)\frac{\partial}{\partial x_j}z(x,t)+$$

$$+p_0(x)\mu b_i(x)\frac{\partial}{\partial x_i}z(x,t)$$

Multiplying the last equation by $p_0(x)\exp(U(x)/\mu)$ we obtain after simple transformation

$$p_0^2(x)e^{\frac{U(x)}{\mu}}\frac{\partial}{\partial t}z(x,t) = \mu^2\frac{\partial}{\partial x_i}p_0^2(x)e^{\frac{U(x)}{\mu}}a_{ij}(x)\frac{\partial}{\partial x_j}z(x,t)$$

$$z\Big|_{t=0} = u_0(x)p_0^{-1}(x)$$

This equation can be studied by the methods of classical homogenization theory. The homogenized equation takes a form (see[9])

$$\begin{aligned} < p_0^2 > \frac{\partial}{\partial t}\hat{z}(x,t) &= \mu^2\frac{\partial}{\partial x_i}\hat{a}_{ij}\frac{\partial}{\partial x_j}\hat{z}(x,t) \\ \hat{z}\Big|_{t=0} &= u_0(x)/ < p_0^2 > \end{aligned} \tag{3.1}$$

where $\hat{a}_{ij}$ is effective diffusion matrix for the elliptic part of the equation. This matrix differs from σ_{ij} only by the factor $< p_0^2(\cdot)\exp(U(\cdot)/\mu) > /\mu^2$ (see (1.4)).

The potential $U(x)$ is defined up to additive constant. It is convenient to fix the choice of this constant by the following condition

$$\min_{y\in T^n}(2W(y) - U(y)) = 0. \tag{3.2}$$

From (3.2), Proposition 2.3 and Theorem 2.4

$$\lim_{\mu\to 0}(\mu \ln < p_0^2(\cdot)\exp(U(x)/\mu) >) = 0. \tag{3.3}$$

Thus, the factor $< p_0^2(\cdot)\exp(U(\cdot)/\mu) > /\mu^2$ does not affect the logarithmic asymptotic of homogenized matrix and it is enough to study the asymptotic behaviour of the effective diffusion for the elliptic operator $\frac{\partial}{\partial x_i}p_0^2(x)e^{\frac{U(x)}{\mu}}a_{ij}(x)\frac{\partial}{\partial x_j}$.

We assume that the function $\mathcal{W}(x) = 2W(x) - U(x)$ has only one minimum point on T^n. Then, without loss of generality we can suppose that $\mathcal{W}(x)$ is defined on R^n/Z^n and that the set of minimum points of this function coincides with Z^n. By (3.2) $\mathcal{W}(x) = 0$ for $x \in Z^n$. Having the function $\mathcal{W}(x)$ we construct the matrix $A_{\mathcal{W}}$ as follows.
Denote by β_1 be the solution of the following minimax problem

$$\beta_1 = \inf_{i\in Z^n\backslash 0} F(i), \qquad F(i) = \inf_{\{x(t),\, x(0)=0,\, x(1)=i\}} \sup_t \mathcal{W}(x(t))$$

where inf is taken over all smooth paths connecting 0 with $i \in Z^n \setminus 0$. For arbitrary sequence $\{i_k\}_{k=1}^{\infty}$ of integer vectors $i_k \in Z^n \setminus 0$, denote by $\Lambda(\{i_k\})$ the set of the limiting points of the normalized sequence $\{i_k/|i_k|\}$. Next, we define Λ_{β_1} as a union of $\Lambda(\{i_k\})$ over all the sequences $\{i_k\}$ satisfying the relation $\lim\limits_{k\to\infty} F(i_k) = \beta_1$. According to the definition of β_1 and compactness of the unit sphere the set Λ_{β_1} is closed and not empty. We also set $\Gamma_{\beta_1} = \{x \in R^n : \ x/|x| \in \Lambda_{\beta_1}\} \cup \{0\}$.

Proposition 3.1 *Γ_{β_1} is a linear subspace of R^n. In Γ_{β_1} there exists a basis consisting of integer vectors. There exists $\delta_0 > 0$ such that the inequality $F(i) \geq \beta_1 + \delta_0$ holds for all $i \in Z^n \setminus \Gamma_{\beta_1}$.*

The proof of this statement is given in [7]. □

Now we set $\beta_2 = \inf\limits_{i \in Z^n \setminus \Gamma_{\beta_1}} F(i)$. According to Proposition 3.1 $\beta_2 > \beta_1$. Like Λ_{β_1} above Λ_{β_2} is defined as a union of $\Lambda(\{i_k\})$ over all the sequences $\{i_k\}$, $i_k \in Z^n \setminus 0$, such that $\limsup\limits_{k\to\infty} F(i_k) \leq \beta_2$. Let $\Gamma_{\leq\beta_2} = \{x \in R^n : \ \frac{x}{|x|} \in \Lambda_{\beta_2}\}$ and let Γ_{β_2} be the ortho-complement to Γ_{β_1} in $\Gamma_{\leq\beta_2}$. The following assertion is also proved in [7].

Proposition 3.2 *$\Gamma_{\leq\beta_2}$ is the linear subspace of R^n. There exists $\delta_0 > 0$ such that $F(i) > \beta_2 + \delta_0$ for any $i \in Z^n \setminus \Gamma_{\leq\beta_2}$. In Γ_{β_2} there exists the basis of integer vectors.*

The next step gives β_3, $\Gamma_{\leq\beta_3}$ and Γ_{β_3} and so on. Continuing the process we find $\beta_1, ..., \beta_s$ and $\Gamma_{\beta_1}, ..., \Gamma_{\beta_s}$, $\Gamma_{\beta_1} \oplus ... \oplus \Gamma_{\beta_s} = R^n$, where $1 \leq s \leq n$. Let $z_1, ..., z_n$ be the orthonormal basis in R^n consisting of the orthonormal basis of $\Gamma_{\beta_1}, ..., \Gamma_{\beta_s}$. We define the symmetric matrix $A_{\mathcal{W}}$ to be diagonal in the basis $z_1, ..., z_n$ with eigenvalues β_i in the corresponding subspaces Γ_{β_i}.
Let us now consider the matrix $A_{\mathcal{W}}$ as a function of $\mathcal{W}(\cdot)$.

Lemma 3.3 *$A_{\mathcal{W}}$ is a continuous monotone function of $\mathcal{W}$ in the functional space $C(T^n)$.*

Proof: see [7], Lemma 2. □

The next statement is the main result of the paper.

Theorem 3.4 *The effective diffusion matrix $\sigma(\mu)$ satisfies the following limiting relation*

$$\lim_{\mu\to 0} \mu \ln \sigma(\mu) = -2A_{\mathcal{W}}.$$

Proof: As was already mentioned it suffices to find the asymptotics of homogenized matrix for the operator $\frac{\partial}{\partial x_i} p_0^2(x) e^{\frac{U(x)}{\mu}} a_{ij}(x) \frac{\partial}{\partial x_j}$. For this purpose we approximate $\mathcal{W}(x)$ on T^n by the smooth functions $\mathcal{W}_\delta(x)$ with finite number of degenerate points and with the only global minimum point at the origin, in such a way that the estimate $|\mathcal{W} - \mathcal{W}_\delta|_{C(T^n)} < \delta/3$ holds. Under this choice of $\mathcal{W}_\delta$ for sufficiently small μ the following matrix inequality

$$e^{-\frac{\mathcal{W}_\delta(x)+2\delta}{\mu}} \leq p_0^2(x) e^{\frac{U(x)}{\mu}} a_{ij}(x) \leq e^{-\frac{\mathcal{W}_\delta(x)-2\delta}{\mu}} \tag{3.4}$$

holds for all $x \in T^n$. Further, the homogenized operator keeps the order relation of original operator. Hence, (3.4) implies the inequality

$$e^{-\frac{2\delta}{\mu}}\tilde{a}_{ij} \;\le\; \hat{a}_{ij}(\mu) \;\le\; e^{\frac{2\delta}{\mu}}\tilde{a}_{ij} \tag{3.5}$$

where $\tilde{a}_{ij}$ are the coefficients of homogenized operator for $\sum_{i=1}^{n} \frac{\partial}{\partial x_i} e^{-\frac{\mathcal{W}_\delta(x)}{\mu}} \frac{\partial}{\partial x_i}$ and $\hat{a}_{ij}$ are the coefficients of (3.1). The properties of $\mathcal{W}_\delta(x)$ allow us to apply the results of [4] to find the asymptotics of $\tilde{a}_{ij}$. This yields the following relation

$$\lim_{\mu\to 0} \mu \ln \tilde{a}_{ij} = -2A_{\mathcal{W}_\delta}. \tag{3.6}$$

Taking into account (3.5) we have

$$-2\delta I - 2A_{\mathcal{W}_\delta} \;\le\; \liminf_{\mu\to 0} \mu \ln \hat{a}_{ij} \;\le\; \limsup_{\mu\to 0} \mu \ln \hat{a}_{ij} \;\le\; 2\delta I - 2A_{\mathcal{W}_\delta}.$$

At last by Lemma 3.3 and the choice of $\mathcal{W}_\delta(x)$, the matrix $A_{\mathcal{W}_\delta}$ tends to $A_{\mathcal{W}}$ as $\delta \to 0$. Thus,

$$\lim_{\mu\to 0} \mu \ln \hat{a}_{ij} \;=\; \lim_{\mu\to 0} \mu \ln \sigma_{ij} \;=\; -2A_{\mathcal{W}}.$$

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References.

[1] Avellaneda M., 1991, *Enhanced diffusivity and intercell transition layers in 2-D models of passive advection*, J. Math. Phys., v.32, N11, p.3209-3212.
[2] Fannjiang A. and Papanicolaou G., 1992, *Convection enhanced diffusion for periodic flows*, SIAM J. Appl. Math., v.52, N2, p.412-429.
[3] Kozlov S.M. and Piatnitski A.L., 1991, *Averaging on a background of vanishing viscosity*, Math. USSR Sbornik, v.70, N1, p.241-261.
[4] Kozlov S.M., 1989, *Geometric aspects of homogenization*, Russ. Math. Surveys, v.44, N2, p.91-144.
[5] Piatnitski A.L. and Kozlov S.M., 1990, *Homogenization and vanishing viscosity*, Proceeding of the Fifth International Conference on Probability Theory and Mathematical Statistics. V2. MOKSLAS, 1990., p.330-339.
[6] Kozlov S.M. and Piatnitski A.L., 1993, *Effective diffusion for parabolic operator with periodic potential*, SIAM J. Appl. Math., v.53, N2, p.401-418.
[7] Kozlov S.M. and Piatnitski A.L., *Degeneration of effective diffusion in the presence of periodic potential*, Ann. Inst. Poincaré (to appear).
[8] Krasnoselskii M.A., Lifshitz E.A. and Sobolev A.V., 1989, *Positive linear systems*, Berlin, Helderman.

[9] Kozlov S.M., 1983, *Reducibility of quasiperiodic operators and homogenization*, Transactions of Moscow Math. Soc., v.46, p.99-123.
[10] Piatnitski A.L., 1994, *An asymptotic behaviour of a ground state of singularly perturbed equations*, Report 292, Institut für Mathematik, Augsburg.

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