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HOMOGENIZATION OF DIFFUSION IN HIGH CONTRAST RANDOM MEDIA AND RELATED MARKOV SEMIGROUPS

Brahim Amaziane[1], Andrey Piatnitski[*2,3] and Elena Zhizhina[3,2]

[1]Universite de Pau et des Pays de l'Adour, E2S UPPA, CNRS, LMAP, Pau, France

[2]The Arctic University of Norway, UiT, campus Narvik

[3]Kharkevich Institute for Information Transmission Problems, RAS , Moscow, Russia

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Abstract. The goal of the paper is to describe the large time behavior of a Markov process associated with a symmetric diffusion in a high contrast random stationary environment and to characterize the limit process under the diffusive scaling. The advantage of the proposed approach is that we explicitly present the limit operator on the extended space. That gives us a possibility to use this operator for approximations of diffusion processes in porous media with random structure of inclusions. We also describe the spectrum of the limit operator. Our approach uses auxiliary Markov processes defined on extended state spaces, as well as tools from homogenization theory in random media.

1. **Introduction.** Elliptic and parabolic operators with high contrast rapidly oscillating periodic coefficients have been widely studied in the homogenization theory. The first rigorous results for parabolic operators of this type were obtained in [15] and [5]. In particular, it was shown that, under proper choice of the scaling coefficient, the homogenized problem contains a non-local in time operator which reflects the so-called memory effect. Later on in [2], with the help of the two-scale convergence technique, the limit problem was written as a coupled system of parabolic PDEs in the space with higher number of variables. Homogenization problems for elliptic and hyperbolic operators in strongly inhomogeneous periodic media were originally studied in [18], where the asymptotic expansion of solutions was constructed. In [24, 25] high contrast problems in domains with singular or asymptotically singular periodic geometry were considered. A number of interesting results on homogenization of high contrast operators has been obtained in the recent work [14]. At present, there are many works in the existing mathematical literature that describe the effective behavior of high contrast periodic media. Under proper scaling, in parabolic problems this usually results in the memory effect while homogenization of spectral problems leads to a non-linear dependence on the spectral parameter. Rigorous homogenization results for high contrast random stationary media have been obtained in [7, 8] and some other works.

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*Corresponding author: Andrey Piatnitski.

This work focuses on a problem that is situated within double-porosity models which are typically used to simulate flow in fractured formations. During the last decade, there appeared a significant body of literature devoted to the modeling of such problems. There is an extensive literature on this subject. We will not attempt a literature review here but will merely mention a few references. A recent review of the mathematical homogenization methods developed for flow in double porosity media can be viewed in [4, 19].

The paper focuses on the large time behavior of diffusion in high contrast random statistically homogeneous media. We also study the limit behavior of the corresponding semigroups. Equivalently, we consider the limit behavior of diffusion defined in a high contrast environment with a random microstructure on a finite time interval.

In this paper we deal with second order divergence form operators in $\mathbb{R}^d$. Each such an operator is a generator of a Markov semigroup. The corresponding Markov process (generalized diffusion) has continuous trajectories. However, the presence of a non-local in temporal variable term in the effective operator means that the limit dynamics of the coordinate process is not Markov.

The goal of this work is to equip the coordinate process with additional components in such a way that the dynamics of the enlarged process remains Markovian in the limit. We show that it is sufficient to combine the coordinate process in $\mathbb{R}^d$ with a position of the diffusion inside the rescaled inclusions for the time intervals when the diffusion is trapped by one of the inclusions.

It is interesting to observe that, although in the original processes the additional components are functions of the coordinate process, in the limit process these components are getting independent while the coordinate process becomes coupled with them.

The explicit form of the limit operator on the extended space gives us a possibility to use this operator as a generator of an approximation dynamics for the processes in high contrast random stationary disperse porous media. The discrete version of such approximation process was constructed in [22], where we considered a discrete diffusion in a high-contrast random environment given by a jump random walk on the lattice $\mathbb{Z}^d$. The crucial step in this construction is to describe the "clock process" governing transitions from the observable "real" space to the supplementary "astral" spaces and back. The "clock process" is a continuous time finite Markov chain with transition rates depending on parameters of the limit operator.

To our best knowledge, the questions considered in this paper have not been studied in the existing literature. In the discrete framework the results on scaling limit of symmetric random walk in high contrast periodic environments were obtained in [22]. The construction of related Markov semigroups for symmetric diffusions in high contrast periodic media on large time scales has been discussed in [21].

The limit behaviour of the spectrum of high contrast operators in a random disperse type environment is the subject of recent works [8, 9]. In [8] the authors consider a random double porosity model in a regular bounded domain and show that, for some particular examples of random media, the spectrum of the original operator converges in the sense of Hausdorff to the spectrum of the homogenized operator. In [9] it is explained that in the case of the whole space $\mathbb{R}^d$ the Hausdorff convergence need not take place. The approach used in the present work also allows us to study the asymptotic behaviour of the spectrum of the original operators. This is illustrated in Section 5.

The rest of the paper is organized as follows. Section 2 deals with problem setup. We describe the model equation and provide the assumptions on the data. In Section 3 we introduce proper functional spaces, and construct the limit semigroup and the homogenized operator in the extended space, the semigroup convergence is then proved in Section 4. Finally, in Section 5 we study the spectrum of the limit operator. Then the semigroup convergence in L^2 spaces allows us to provide some information about the limit behavior of the spectrum of the original operators. Our approach essentially relies on the approximation technique developed in [10] and the technique of correctors in random media. In contrast with the periodic framework, the auxiliary operators used to introduce correctors need not be of Fredholm type in the case of random inclusions. The construction of the first corrector can be found in the existing literature, see for instance [13]. However, when defining the higher order correctors we face additional difficulties. Lastly, in Section 6 some concluding remarks are forwarded.

2. **Problem setup.** Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space. Consider a symmetric diffusion operator in divergence form

$$\tilde{A}_\varepsilon^\omega f(y) = \operatorname{div}\left(\tilde{a}_\varepsilon^\omega(y)\nabla f(y)\right), \tag{1}$$

where

$$\tilde{a}_\varepsilon^\omega(y) = \begin{cases} \mathbb{I}, & y \in \mathbb{R}^d \setminus G^\omega = \left(G^\omega\right)^c, \\ \varepsilon^2\mathbb{I}, & y \in G^\omega, \end{cases} \tag{2}$$

$\mathbb{I}$ being the unit matrix. Here $G^\omega \subset \mathbb{R}^d$, $\omega \in \Omega$, is a classical disperse set, i.e. a random statistically homogeneous ergodic set $G^\omega \subset \mathbb{R}^d$ that consists of a countable number of uniformly bounded simply connected domains with uniformly Lipschitz boundary. Moreover, the distance between any two such domains admits a uniform deterministic lower bound. The complement $\mathbb{R}^d \setminus G^\omega$ is almost surely (a.s.) connected and unbounded. The set G^ω corresponds to the matrix blocks (inclusions), and $\mathbb{R}^d \setminus G^\omega \subset \mathbb{R}^d$ to the fractures system, see [7].

To be more specific, we consider in this work disperse media that satisfy the following additional condition: there is a **finite** collection of bounded C^2 regular domains (patterns) in $\mathbb{R}^d$ such that a.s. any connected component of G^ω can be obtained by a proper translation and rotation of one of these domains. For the elements of this collection we use the notation $\mathcal{D}_j$, $j = 1, \dots, N$, $N \in \mathbb{N}$, and the whole collection is denoted $\boldsymbol{\mathcal{D}}$, $\boldsymbol{\mathcal{D}} = \{\mathcal{D}_j\}_{j=1}^N$. Thus N is the number of elements in $\boldsymbol{\mathcal{D}}$.

We assume without loss of generality that each pattern $\mathcal{D}_j$, $j \geq 1$, contains the origin. The union of all subsets of G^ω that have the same geometry as $\mathcal{D}_j$ is denoted $\mathcal{G}_j^\omega$, the copies $\mathcal{G}_j^{\omega,i}$ of $\mathcal{D}_j$ in $\mathcal{G}_j^\omega$ are enumerated by index $i \in \mathbb{N}$. So we have

$$G^\omega = \bigcup_{j=1}^N \mathcal{G}_j^\omega, \quad \mathcal{G}_j^\omega = \bigcup_{i\in\mathbb{N}} \mathcal{G}_j^{\omega,i},$$

and for the complement $\mathbb{R}^d \setminus G^\omega$ is used the notation $\mathcal{G}_0^\omega$.

Under these assumptions for each $\mathcal{G}_j^{\omega,i}$ there exist a shift $x_j^{\omega,i} \in \mathbb{R}^d$ and a rotation $\mathcal{S}_j^{\omega,i} \in SO(d)$ such that $\mathcal{G}_j^{\omega,i}$ is the image of $\mathcal{D}_j$ under the mapping $\xi \mapsto (\mathcal{S}_j^{\omega,i})^{-1}\xi + x_j^{\omega,i}$. The inverse mapping reads $\xi \mapsto \mathcal{S}_j^{\omega,i}(\xi - x_j^{\omega,i})$.

Letting

$$\alpha_0 = \mathbf{P}\{0 \in \mathcal{G}_0^\omega\}, \quad \alpha_j^o = \mathbf{P}\{0 \in \mathcal{G}_j^\omega\} \tag{3}$$

we have $\alpha_0 + \sum_{j\ge 1} \alpha_j^o = 1$, and assume without loss of generality that $\alpha_j^o > 0$ for all j. Define

$$\alpha_j = |\mathcal{D}_j|^{-1}\alpha_j^o = |\mathcal{D}_j|^{-1}\mathbf{P}\{0 \in \mathcal{G}_j^\omega\}. \tag{4}$$

An example of such a disperse medium is associated with a Bernoulli site percolation model on the lattice $\mathbb{Z}^d$ embedded in $\mathbb{R}^d$. Let $\{\xi_j,\ j \in \mathbb{Z}^d\},\ \xi_j \in \{0,1\}$, be a sequence of i.i.d. random variables having the Bernoulli law:

$$\mathbb{P}(\xi_j = 1) = p, \quad \mathbb{P}(\xi_j = 0) = 1 - p \quad \text{with } 0 < p < 1.$$

We then define $\mathtt{B}_j = j + [-\frac{1}{2}, \frac{1}{2}]^d$, $j \in \mathbb{Z}^d$, and consider the set $\mathtt{G}^{1,\omega} = \bigcup\limits_{\{j\,:\,\xi_j=1\}} \mathtt{B}_j$. This set is a.s. a union of countable number of bounded connected sets (components) and not more than one unbounded connected component, see [12]. We consider the generic bounded connected component of $\mathtt{G}^{1,\omega}$, call it $\mathcal{E}$ and denote by $\mathcal{E}_\delta$ its open δ-neighbourhood with $0 < \delta < \frac{1}{4}$. Let $\tilde{\mathcal{E}}_\delta$ be the minimal open simply connected set that contains $\mathcal{E}_\delta$. Smoothing the boundary of $\tilde{\mathcal{E}}_\delta$ we denote the obtained set by $\widehat{\mathcal{E}}_\delta$. Then we fix an integer number $M \ge 1$ and keep only those sets that contain not more than M points of $\mathbb{Z}^d$. By construction, there exists a finite collection of open bounded simply connected sets with a smooth boundary such that each $\widehat{\mathcal{E}}_\delta$ can be obtained by a proper shift of one of the sets of this collection. This is the desired collection $\boldsymbol{\mathcal{D}}$. By the standard arguments of percolation theory, G^ω is statistically homogeneous and ergodic.

After the scaling $x = \varepsilon y,\ t = \varepsilon^2 s$ we get the following diffusion operator

$$A_\varepsilon^\omega f(x) = \operatorname{div}\big(a_\varepsilon^\omega(x)\nabla f(x)\big), \tag{5}$$

where

$$a_\varepsilon^\omega(x) = \tilde{a}_\varepsilon^\omega(\frac{x}{\varepsilon}) = \begin{cases} \mathbb{I}, & x \in \mathbb{R}^d \setminus \varepsilon G^\omega = \big(G_\varepsilon^\omega\big)^c, \\ \varepsilon^2\mathbb{I}, & x \in \varepsilon G^\omega, \end{cases} \tag{6}$$

and $G_\varepsilon^\omega = \varepsilon G^\omega$. The corresponding Cauchy problem reads

$$\frac{\partial}{\partial t}u_\varepsilon^\omega(x,t) \ = \ \operatorname{div}\big(a_\varepsilon^\omega(x)\nabla u_\varepsilon^\omega(x,t)\big), \quad u_\varepsilon^\omega(x,0) = \varphi(x). \tag{7}$$

For each $\varepsilon > 0$ the operator A_ε^ω has random statistically homogeneous coefficients in $\mathbb{R}^d$, where the randomness is defined through the random geometry of G_ε^ω. These operators are also called metrically transitive with respect to the unitary group of the space translations in $\mathbb{R}^d$. In $L^2(\mathbb{R}^d)$ we introduce a domain of A_ε^ω by

$$\begin{aligned} D(A_\varepsilon^\omega) = \Big\{ f \in H^1(\mathbb{R}^d)\ :\ & f \in H^2(G_\varepsilon^\omega) \cap H^2(\mathbb{R}^d \setminus G_\varepsilon^\omega),\ \varepsilon^2 \nabla f(x)\big|_{\partial G_\varepsilon^\omega} \cdot n^+ \\ &= -\nabla f(x)\big|_{\partial G_\varepsilon^\omega} \cdot n^- \Big\} \end{aligned} \tag{8}$$

The last relation in (8) represents the continuity of the normal flux $a_\varepsilon \nabla f$ through the boundary $\partial G_\varepsilon^\omega$. Here n^-, n^+ are respectively the internal and external normals on $\partial G_\varepsilon^\omega$.

Remark 2.1. Notice that, according to the classical trace theorem, see for instance [16], for any function $v \in D(A_\varepsilon^\omega)$ its trace and the trace of its flux on the interface $\partial G_\varepsilon^\omega$ is a well-defined $L^2(\partial G_\varepsilon^\omega)$ function.

Then $(A_\varepsilon^\omega, D(A_\varepsilon^\omega))$ is almost surely a self-adjoint operator in $L^2(\mathbb{R}^d)$, and for any $\lambda > 0$ the operator $(\lambda\mathbf{I} - A_\varepsilon^\omega)$ is coercive, here and in what follows the notation $\mathbf{I}$ is used for the identity operator. By the Hille-Yosida theorem, A_ε^ω is a generator of a strongly continuous, positive, contraction semigroup $T_\varepsilon^\omega(t)$ on $L^2(\mathbb{R}^d)$ for a.e. $\omega \in \Omega$, and the solution to (7) can be written as

$$u_\varepsilon^\omega(x,t) = T_\varepsilon^\omega(t)\varphi(x). \tag{9}$$

In the next sections we construct a Markov semigroup $T(t)$ acting on an extended space and prove ω-a.s. convergence $T_\varepsilon^\omega(t) \to T(t)$ as $\varepsilon \to 0$ on any finite time intervals. Thus the Markov semigroup $T(t)$ describes the limit behaviour of the process corresponding to T_ε^ω, and the projection on the first component of this limit process gives an approximation for the solutions (9) of Cauchy problem (7).

We omit in this paper discussions on the choice of the extended space and the generator, and refer the reader to [21], where the analogous idea was realised in the case of periodic high contrast media.

3. **The limit semigroup $T(t)$.** In this section we describe the generator A of the limit Markov semigroup $T(t)$. Denote

$$E = \mathbb{R}^d \times \boldsymbol{\mathcal{D}}^\star, \qquad \text{where } \boldsymbol{\mathcal{D}}^\star = \{\star\} \cup \boldsymbol{\mathcal{D}};$$

here the symbol $\star$ stands for a single point set that corresponds to the unbounded connected (fast) component in the effective dynamics. We consider functions F defined on E of the following vector form

$$F(x,\xi) = \begin{cases} f_0(x), & \text{if } x \in \mathbb{R}^d,\ \xi = \star, \\ f_j(x,\xi), & \text{if } x \in \mathbb{R}^d,\ \xi \in \mathcal{D}_j,\ j = 1, \dots, N, \end{cases} \tag{10}$$

with $f_0 \in L^2(\mathbb{R}^d)$, $\quad f_j \in L^2(\mathbb{R}^d \times \mathcal{D}_j)$. Equipped with the norm

$$\|F\|^2 = \alpha_0 \int\limits_{\mathbb{R}^d} f_0^2(x)\,dx + \sum_{j=1}^N \alpha_j \int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{D}_j} (f_j)^2(x,\xi)\,d\xi dx \tag{11}$$

where $\alpha_0,\ \alpha_j$ was defined in (4), $\alpha_0 > 0,\ \alpha_j > 0$, this set of functions is a Hilbert space. We call this Hilbert space $L^2(E,\alpha)$.

Let us consider in $L^2(E,\alpha)$ an operator of the following form

$$(AF)(x,\xi) = \begin{pmatrix} \Theta \cdot \nabla\nabla f_0(x) + \frac{1}{\alpha_0} \sum\limits_{j=1}^N \alpha_j \int\limits_{\partial\mathcal{D}_j} \frac{\partial f_j(x,\xi)}{\partial n_\xi^-} d\sigma(\xi) \\ \triangle_\xi f_1(x,\xi) \\ \dots \\ \triangle_\xi f_N(x,\xi) \end{pmatrix} \tag{12}$$

where a positive definite constant matrix Θ will be defined later on, see (43), $\sigma(\xi)$ is the element of the surface volume on the Lipshitz boundary $\partial\mathcal{D}_j^k$, n_ξ^- is the (inner) normal to $\partial\mathcal{D}_j^k$. The domain of definition of this operator is specified below in (15). Observe that, for each $j = 1, \dots, N$, the argument ξ of the function $f_j(x,\xi)$ belongs to $\mathcal{D}_j$, and that the function in the first line on the right-hand side depends only on x. Using the relation $n^+ = -n^-$ and the Stokes formula one can rewrite the operator (12) as follows:

$$(AF)(x,\xi) = \begin{pmatrix} \Theta \cdot \nabla\nabla f_0(x) - \frac{1}{\alpha_0} \sum\limits_{j=1}^N \alpha_j \int\limits_{\mathcal{D}_j} \triangle_\xi f_j(x,\xi) d\xi \\ \triangle_\xi f_1(x,\xi) \\ \dots \\ \triangle_\xi f_N(x,\xi) \end{pmatrix}. \tag{13}$$

We denote

$$\Upsilon(x) = -\frac{1}{\alpha_0}\sum_{j=1}^{N}\alpha_j \int\limits_{\mathcal{D}_j} \triangle_\xi f_j(x,\xi) d\xi. \tag{14}$$

For each set $\mathcal{D}_j$, $j \geq 1$, denote by $\mathbb{D}_j(\Delta)$ the domain of a self-adjoint operator in $L^2(\mathcal{D}_j)$ that corresponds to the Laplacian in $\mathcal{D}_j$ with homogeneous Dirichlet boundary conditions. Since the boundary of $\mathcal{D}_j$ is C^2 regular, we have $\mathbb{D}_j(\Delta) = H^2(\mathcal{D}_j) \cap H^1_0(\mathcal{D}_j)$. Notice that this operator is positive. The space $\mathbb{D}_j(\Delta)$ is equipped with the norm $\|g\|_{\mathbb{D}_j(\Delta)} = \|\Delta g\|_{L^2(\mathcal{D}_j)}$.

Defining an operator A in $L^2(E,\alpha)$ by formulas (12), (13), one can easily check that, with a domain

$$\begin{aligned}\hat{D}(A) = \Big\{ F = (f_0,\dots,f_N) \in L^2(E,\alpha) \, : \, f_0 \in H^2(\mathbb{R}^d),\\ f_j - f_0 \in L^2(\mathbb{R}^d; \mathbb{D}_j(\Delta)),\ f_j(x,\xi)\Big|_{\xi\in\partial\mathcal{D}_j} = f_0(x),\ \Big(\Theta \cdot \nabla\nabla f_0(x)\\ -\tfrac{1}{\alpha_0}\sum_{j=1}^{N}\alpha_j \int\limits_{\mathcal{D}_j} \triangle_\xi f_j(x,\xi)d\xi,\ \triangle_\xi f_1(x,\xi),\dots,\triangle_\xi f_N(x,\xi)\Big) \in L^2(E,\alpha)\Big\},\end{aligned} \tag{15}$$

the operator $(A, \hat{D}(A))$ is a closed symmetric operator in $L^2(E,\alpha)$, and $\hat{D}(A)$ is dense in $L^2(E,\alpha)$.

We introduce the following two spaces:

$$H^1_D(E,\alpha) = \Big\{ f_0 \in H^1(\mathbb{R}^d),\ \ f_j - f_0 \in L^2(\mathbb{R}^d; H^1_0(\mathcal{D}_j)) \Big\} \tag{16}$$

and

$$H^2_D(E,\alpha) = \Big\{ f_0 \in H^2(\mathbb{R}^d),\ \ f_j - f_0 \in L^2(\mathbb{R}^d; H^2(\mathcal{D}_j) \cap H^1_0(\mathcal{D}_j)) \tag{17}$$

Notice that

$$\sum_{j=1}^{N}\alpha_j \int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{D}_j} |\nabla_\xi f_j|^2(x,\xi)\, d\xi dx < \infty \quad \forall F \in H^1_D(E,\alpha)$$

and

$$\sum_{j=1}^{N}\alpha_j \|f_j\|^2_{L^2(\mathbb{R}^d;H^2(\mathcal{D}_j))} = \sum_{j=1}^{N}\alpha_j \int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{D}_j} \Big(|f_j(x,\xi)|^2 + \sum_{k,l=1}^{d} \Big| \frac{\partial^2 f_j(x,\xi)}{\partial\xi_k\partial\xi_l}\Big|^2 \Big) d\xi dx < \infty$$

for any $F \in H^2_D(E,\alpha)$.

The space $H^{-1}(E,\alpha)$ is defined as the dual space to $H^1_D(E,\alpha)$ in $L^2(E,\alpha)$.

Lemma 3.1. *For any $m > 0$ the operator $(m\mathbf{I} - A, \hat{D}(A))$ is a coercive self-adjoint operator in $L^2(E,\alpha)$.*

Proof. Consider the following quadratic form in $L^2(E,\alpha)$

$$\begin{aligned}\Gamma(F,F) = \alpha_0 \int\limits_{\mathbb{R}^d} \Theta\nabla f_0(x)\cdot\nabla f_0(x)\, dx\\ + \sum_{j=1}^{N}\alpha_j \int\limits_{\mathbb{R}^d}\int\limits_{\mathcal{D}_j} |\nabla_\xi f_j|^2(x,\xi)\, d\xi dx + m\|F\|^2_{L^2(E,\alpha)}\end{aligned} \tag{18}$$

with a domain $D(\Gamma) = H^1_c(E,\alpha)$. Notice that $f_j(x,\cdot)\big|_{\partial\mathcal{D}_j} = f_0(x)$ for any $F \in D(\Gamma)$. According to [23, Theorem VIII.15] there exists a unique self-adjoint operator $\tilde{A}_m$ that has the following properties:
- its domain $D(\tilde{A}_m)$ is dense in $L^2(E,\alpha)$;
- $D(\tilde{A}_m)$ belongs to $D(\Gamma)$;
- $(\tilde{A}_m F, F)_{L^2(E,\alpha)} = \Gamma(F,F)$ for any $F \in D(\tilde{A}_m)$.

We are going to show that $\tilde{A}_m$ coincides with $m\mathbf{I} - A$. First we prove that $D(\tilde{A}_m) \subset \hat{D}(A)$. Separating the first component f_0 in (10) we will use the notation $F = (f_0, V)$. Taking $F \in D(\tilde{A}_m)$ and $U = (0, U_1) \in D(\Gamma)$ with $U_1 \in C_0^\infty(\mathbb{R}^d; C_0^\infty(\boldsymbol{\mathcal{D}}))$, and using the relation $(\tilde{A}_m F, U)_{L^2(E,\alpha)} = \Gamma(F,U)$, we obtain

$$(\tilde{A}_m F, U)_{L^2(E,\alpha)} = \Gamma(F,U) = \sum_{j=1}^{N} \alpha_j ((m-\Delta_\xi)V_j, U_{1,j}),$$

where the terms $(-\Delta_\xi V_j, U_{1,j})$ on the right-hand side are understood as a pairing between $L^2(\mathbb{R}^d; H^{-1}(\mathcal{D}_j))$ and $L^2(\mathbb{R}^d; H^1_0(\mathcal{D}_j))$. This implies that $(m-\Delta_\xi)V_j \in L^2(\mathbb{R}^d \times \mathcal{D}_j)$ and $\big(0, \{(m-\Delta_\xi)V_j\}_{j\ge 1}\big) \in L^2(E,\alpha)$. Therefore, $(0,V) \in H^2_c(E,\alpha)$. Choosing now $U = (u_0(x), 0)$ with $u_0 \in C_0^\infty(\mathbb{R}^d)$ and considering the fact that $\sum\limits_{j=1}^{N} \alpha_j \int\limits_{\mathcal{D}_j} \triangle_\xi V_j(\cdot,\xi) d\xi \in L^2(\mathbb{R}^d)$, we get $mf_0 - \operatorname{div}(\Theta \nabla f_0) \in L^2(\mathbb{R}^d)$. Therefore, $f_0 \in H^2(\mathbb{R}^d)$, and $D(\tilde{A}_m) \subset \hat{D}(A)$.

Moreover, $\tilde{A}_m F = (m-A)F$ for any $F \in D(\tilde{A}_m)$. Since $\tilde{A}_m$ is self-adjoint, $D(\tilde{A}_m) = \hat{D}(A)$. This yields the desired statement. □

We define the following set of functions:

$$D_A = \big\{ f_0(x) \in C_0^\infty(\mathbb{R}^d),\ f_j(x,\xi) - f_0(x) \in C_0^\infty(\mathbb{R}^d, \mathbb{D}_j(\Delta)) \big\}. \tag{19}$$

Notice that $f_j(x,\xi)\big|_{\xi\in\partial\mathcal{D}_j} = f_0(x)$ for any $F = \{f_j\}_{j\ge 0} \in D_A$.

Corollary 3.2. *The set $D_A \subset L^2(E,\alpha)$ defined in (19) is a core of A, i.e. D_A is a dense subset of $L^2(E,\alpha)$ and $A = \overline{A\big|_{D_A}}$, see [10] for the details.*

Proof. Clearly, D_A is a dense subset in $L^2(E,\alpha)$. In order to show that D_A is a core of A we should also check that for some $m > 0$ the set $\{(m-A)F,\ F \in D_A\}$ is dense in $L^2(E,\alpha)$. Denote $J^\infty = \{(u_0, U) = (u_0(x), U_j(x,\xi)) : u_0 \in C_0^\infty(\mathbb{R}^d),\ U_j \in C_0^\infty(\mathbb{R}^d; C_0^\infty(\mathcal{D}_j))\}$. Observe that J^∞ is dense in $L^2(E,\alpha)$. By Lemma 3.1 for an arbitrary $U \in J^\infty$ and for any $m > 0$ the equation

$$mF - AF = U \tag{20}$$

has a unique solution $F = (f_0, V) \in \hat{D}(A)$. Then the equation for V can be rewritten as

$$\begin{aligned} &(m-\Delta_\xi)(V(x,\xi) - f_0(x)) = U(x,\xi) - mf_0(x) \ \text{ in } \boldsymbol{\mathcal{D}}, \\ &(V(x,\xi) - f_0(x))\big|_{\xi\in\partial\boldsymbol{\mathcal{D}}} = 0, \end{aligned} \tag{21}$$

or, in the coordinate form,

$$\begin{aligned} &(m-\Delta_\xi)(V_j(x,\xi) - f_0(x)) = U_j(x,\xi) - mf_0(x) \ \text{ in } \mathcal{D}_j, \\ &(V_j(x,\xi) - f_0(x))\big|_{\xi\in\partial\mathcal{D}_j} = 0, \quad j \ge 1. \end{aligned} \tag{22}$$

From this equation we derive the following relation:

$$V_j(x,\xi) = V_j^I(x,\xi) + mf_0(x)V_j^0(\xi) + f_0(x) \tag{23}$$

with $V_j^I = (m\mathbf{I}-\Delta_\xi)^{-1}U_j \in C_0^\infty(\mathbb{R}^d;\ \mathbb{D}_j(\Delta))$ and $V_j^0 = (m\mathbf{I}-\Delta_\xi)^{-1}1 \in \mathbb{D}_j(\Delta)$.

Substituting the right-hand side of (23) for V into the first equation in (20) yields

$$mf_0 - \Theta\cdot\nabla\nabla f_0 - cmf_0 = w_0,$$

where

$$w_0 = u_0 - \sum_{j=1}^N \alpha_j \int\limits_{\mathcal{D}_j} \Delta_\xi(m\mathbf{I}-\Delta_\xi)^{-1}U_j(\cdot,\xi)\,d\xi, \quad c = -\sum_{j=1}^N \alpha_j \int\limits_{\mathcal{D}_j} \Delta_\xi(m\mathbf{I}-\Delta_\xi)^{-1}1\,d\xi.$$

Under our assumptions on U we have $w_0 \in C_0^\infty(\mathbb{R}^d)$. Also, it is straightforward to check that $c<1$ for any $m>0$. Consequently, f_0 is a Schwartz class function in $\mathbb{R}^d$. Taking a proper sequence of smooth cut-off functions φ_n we conclude that $(m-A)(\varphi_n f_0, V+\varphi_n f_0)$ converges in $L^2(E,\alpha)$ to U. Since $(\varphi_n f_0, V+\varphi_n f_0)\in D_A$, this yields the desired statement. □

Summarizing the above arguments we conclude that A is a generator of a strongly continuous, positive, contraction semigroup $T(t)$ on $L^2(E,\alpha)$ with core D_A defined by (19).

4. **The semigroup convergence.**

In this section we prove the convergence of semigroups acting in different spaces following the methods developed in [10].

Define a bounded linear transformation $\pi_\varepsilon^\omega : L^2(E,\alpha)\to L^2(\mathbb{R}^d)$ for every $\varepsilon\in(0,1)$ and every $\omega\in\Omega$ as follows:

$$(\pi_\varepsilon^\omega F)(x) \;=\; \begin{cases} f_0(x), & \text{if } x\in\varepsilon\mathcal{G}_0^\omega; \\ \hat f_j(\varepsilon\hat x_j^{\omega,i},\, \mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat x_j^{\omega,i})), & \text{if } x\in\varepsilon\mathcal{G}_j^{\omega,i}, \end{cases} \tag{24}$$

where$(\mathcal{S}_j^{\omega,i})^{-1}$ and $\hat x_j^{\omega,i}$ are respectively the rotation and the vector that define the translation from $\mathcal{D}_j$ to its copy $\mathcal{G}_j^{\omega,i}$, and

$$\hat f_j(\varepsilon\hat x_j^{\omega,i},\xi) = \frac{1}{\varepsilon^d|\mathcal{D}_j|}\int_{\varepsilon\mathcal{D}j} f_j(\varepsilon\hat x_j^{\omega,i} + (\mathcal{S}_j^{\omega,i})^{-1}\eta,\xi)\,d\eta. \tag{25}$$

Lemma 4.1. *Almost surely the linear operators π_ε^ω are uniformly bounded in the operator norm for all $\varepsilon\in(0,1)$, that is*

$$\|\pi_\varepsilon^\omega F\|_{L^2(\mathbb{R}^d)} \le C\|F\|_{L^2(E,\alpha)} \tag{26}$$

for any $F\in L^2(E,\alpha)$; the constant C is deterministic and does not depend on ε. Moreover, for each $F\in L^2(E,\alpha)$ the following relation holds a.s.

$$\|\pi_\varepsilon^\omega F\|^2_{L^2(\mathbb{R}^d)} \;\to\; \|F\|^2_{L^2(E,\alpha)} \qquad as\ \ \varepsilon\to 0. \tag{27}$$

Proof. For every $x\in\mathbb{R}^d$ and every ω

$$\sum_{j=0}^N \chi_{\mathcal{G}_j^\omega}(x) \;=\; 1,$$

where χ_D is the characteristic function of $D \subset \mathbb{R}^d$. Then we get

$$\begin{aligned}\|\pi_\varepsilon^\omega F\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} \big(\pi_\varepsilon^\omega F(x)\big)^2 dx = \sum_{j=0}^{N} \int_{\mathbb{R}^d} \big(\pi_\varepsilon^\omega F(x)\big)^2 \chi_{\mathcal{G}_j^\omega}(\frac{x}{\varepsilon}) dx \\ &= \int_{\mathbb{R}^d} f_0^2(x)\, \chi_{\mathcal{G}_0^\omega}(\frac{x}{\varepsilon}) dx \\ &\quad + \sum_{j=1}^{N} \sum_{i\in\mathbb{N}} \int_{\mathbb{R}^d} \Big(\hat f_j\big(\varepsilon \hat x_j^{\omega,i}, \mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon} - \hat x_j^{\omega,i})\big)\Big)^2 \chi_{\mathcal{G}_j^{\omega,i}}(\frac{x}{\varepsilon}) dx.\end{aligned} \tag{28}$$

By the Jenssen inequality and the definition of $\hat f_j$ in (25) this implies that

$$\|\pi_\varepsilon^\omega F\|^2_{L^2(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} f_0^2(x)\, dx + \sum_{j=1}^{N} \frac{1}{|\mathcal{D}^j|} \int_{\mathbb{R}^d} \int_{\mathcal{D}_j} \big(f_j(x,\xi)\big)^2 d\xi\, dx \le \check C \|F\|_{L^2(E,\alpha)}$$

with $\check C = \max_j\{(\alpha_j^o)^{-1}\}$. This yields (26).

We turn to (27) and consider the set of functions in $L^2(E,\alpha)$ which are piece-wise constant and compactly supported with respect to the first variable x. We denote this set by $\mathcal{E}$ and notice that it is dense in $L^2(E,\alpha)$. If $F \in \mathcal{E}$ then (27) holds by the Birkhoff ergodic theorem. Then, taking into account (26) we conclude that (27) holds for any $F \in L^2(E,\alpha)$. □

Now we are ready to formulate the main result of the work.

Theorem 4.2 (Main theorem). *For every $F \in L^2(E,\alpha)$ ω-a.e.*

$$T_\varepsilon^\omega(t)\pi_\varepsilon^\omega F \to\ T(t)F, \quad i.e.\ \ \|T_\varepsilon^\omega(t)\pi_\varepsilon^\omega F - \pi_\varepsilon^\omega\, T(t)F\|_{L^2(\mathbb{R}^d)} \to 0 \ \ for\ all\ \ t \ge 0 \tag{29}$$

as $\varepsilon \to 0$.

The proof of the semigroup convergence in (29) relies on the following approximation theorem [10, Theorem 6.1, Ch.1].

Theorem (see [10]). *For $\varepsilon \in (0,1]$, let $T_\varepsilon(t)$ and $T(t)$ be strongly continuous contraction semigroups on Banach space $\mathcal{L}_\varepsilon$ and $\mathcal{L}$, with generators A_ε and A. Let D be a core for A. Assume that $\pi_\varepsilon : \mathcal{L} \mapsto \mathcal{L}_\varepsilon$ are bounded linear transformations with $\sup_\varepsilon \|\pi_\varepsilon\| < +\infty$. Then the following are equivalent:*

a) For each $f \in \mathcal{L}$, $T_\varepsilon(t)\pi_\varepsilon f \ \to\ T(t)f$ as $\varepsilon \to 0$, for all $t \ge 0$.

b) For each $f \in D$, there exists a family $f_\varepsilon \in \mathcal{L}_\varepsilon$, $\varepsilon \in (0,1]$, such that $f_\varepsilon \to f$ and $A_\varepsilon f_\varepsilon \to Af$ as $\varepsilon \to 0$.

According to this theorem the semigroups convergence (29) is a consequence of the following statement:

Theorem 4.3. *Let the generators A and A_ε^ω of the strongly continuous, positive, contraction semigroups $T(t)$ and $T_\varepsilon^\omega(t)$ be defined by (12) and (1), (2), (8), respectively, and assume that a core $D_A \subset L^2(E,\alpha)$ for the generator A is defined by (19), and that a bounded linear transformation $\pi_\varepsilon^\omega :\ L^2(E,\alpha) \to L^2(\mathbb{R}^d)$ is defined by (24) for every $\varepsilon \in (0,1)$.*

Then there exists a positive definite symmetric constant matrix Θ *such that a.s. for every* $F \in D_A$*, there exists* $F_\varepsilon^\omega \in D(A_\varepsilon^\omega)$ *such that*

$$\|F_\varepsilon^\omega - \pi_\varepsilon^\omega F\|_{L^2(\mathbb{R}^d)} \to 0 \quad \textit{and} \quad \|A_\varepsilon^\omega F_\varepsilon^\omega - \pi_\varepsilon^\omega AF\|_{L^2(\mathbb{R}^d)} \to 0 \quad \textit{as } \varepsilon \to 0. \tag{30}$$

Proof. The proof relies on the correctors technique. For any $F \in D_A$, $F = (f_0, \{f_j\})$, where

$$f_0(x) \in C_0^\infty(\mathbb{R}^d), \quad f_j(x,\xi) \in C_0^\infty\big(\mathbb{R}^d; C^2(\overline{\mathcal{D}_j})\big),$$

with

$$f_j(x,\xi)\big|_{\xi\in\partial\mathcal{D}_j} = f_0(x), \quad x \in \mathbb{R}^d, \ \forall\, j = 1,\dots,N, \tag{31}$$

we construct the following family of functions F_ε^ω depending on the realization ω of random environment:

$$F_\varepsilon^\omega(x) = \begin{cases} f_0(x) + \varepsilon(\nabla f_0(x), h_\varepsilon^\omega(\frac{x}{\varepsilon})) \\ \quad + \varepsilon^2(\nabla\nabla f_0(x), g_\varepsilon^\omega(\frac{x}{\varepsilon})) + \varepsilon^2 q_\varepsilon^\omega(\frac{x}{\varepsilon}), & x \in \mathbb{R}^d \backslash G_\varepsilon^\omega, \\ \sum\limits_{i\in\mathbb{N}} \big[f_1(x, \mathcal{S}_1^{\omega,i}(\frac{x}{\varepsilon} - \hat{x}_1^{\omega,i})) \\ \quad + \varepsilon\phi_1^\omega(x, \mathcal{S}_1^{\omega,i}(\frac{x}{\varepsilon} - \hat{x}_1^{\omega,i}))\big]\chi_{\mathcal{G}_1^{\omega,i}}(\frac{x}{\varepsilon}), & x \in \varepsilon\mathcal{G}_1^\omega, \\ \dots \\ \sum\limits_{i\in\mathbb{N}} \big[f_N(x, \mathcal{S}_N^{\omega,i}(\frac{x}{\varepsilon} - \hat{x}_N^{\omega,i})) \\ \quad + \varepsilon\phi_N^\omega(x, \mathcal{S}_N^{\omega,i}(\frac{x}{\varepsilon} - \hat{x}_N^{\omega,i}))\big]\chi_{\mathcal{G}_N^{\omega,i}}(\frac{x}{\varepsilon}), & x \in \varepsilon\mathcal{G}_N^\omega. \end{cases} \tag{32}$$

Here $h_\varepsilon^\omega(\xi)$, $g_\varepsilon^\omega(\xi)$, $q_\varepsilon^\omega(\xi)$ are random functions of ξ (so-called correctors) that also depend on ε; $h_\varepsilon^\omega(\xi)$ is the random vector function whose gradient does not depend on ε, $g_\varepsilon^\omega(\xi)$ is the random matrix function. In what follows we drop both indices ω and ε when refer to these functions. Correctors $\phi_j^\omega(x,\xi)$ has been introduced in order to ensure the continuity of the function F_ε^ω and the fluxes on the boundary $\partial\big(\varepsilon\mathcal{G}_j^\omega\big)$ of the corresponding inclusions.

Observe that for any $F \in D_A$ as well as for any $F \in C(E)$ and any $x \in \varepsilon\mathcal{G}_j^{\omega,i}$ we have:

$$(\pi_\varepsilon^\omega F)(x) = \hat{f}_j\big(\varepsilon\hat{x}_j^{\omega,i}, \mathcal{S}_j^{\omega,i}\big(\frac{x}{\varepsilon} - \hat{x}_j^{\omega,i}\big)\big) = f_j\big(x, \mathcal{S}_j^{\omega,i}\big(\frac{x}{\varepsilon} - \hat{x}_j^{\omega,i}\big)\big) + O(\varepsilon), \tag{33}$$

where the L^∞ norm of $O(\varepsilon)$ does not exceed $C\varepsilon$. Our goal is to choose the correctors in such a way that the function F_ε^ω defined in (32) belongs to $D(A_\varepsilon^\omega)$, and both relations in (30) are fulfilled. Denote by B_0 the ball in $\mathbb{R}^d$ centered at 0 that contains the supports in x of all the functions f_j, $j = 0,1,\dots,N$.

In order to introduce the correctors in (32) we substitute for F_ε^ω in the expression $A_\varepsilon^\omega F_\varepsilon^\omega - \pi_\varepsilon^\omega AF$ the right-hand side of (32). Using repeatedly the formula

$$\frac{\partial}{\partial x} f(x, \frac{x}{\varepsilon}) = \Big(\frac{\partial}{\partial x} f(x,\xi) + \frac{1}{\varepsilon}\frac{\partial}{\partial \xi} f(x,\xi)\Big)\Big|_{\xi=\frac{x}{\varepsilon}}, \tag{34}$$

for $x \in \mathbb{R}^d \backslash G_\varepsilon^\omega$ after straightforward computation we obtain

$$\begin{aligned} (A_\varepsilon^\omega F_\varepsilon^\omega)(x) &= \triangle_x\Big(f_0(x) + \varepsilon\nabla f_0(x)\cdot h(\frac{x}{\varepsilon}) + \varepsilon^2\nabla\nabla f_0(x)\cdot g(\frac{x}{\varepsilon}) + \varepsilon^2 q(\frac{x}{\varepsilon})\Big) \\ &= \Big(\triangle f_0(x) + 2\nabla\nabla f_0(x)\nabla_\xi h(\xi) + \frac{1}{\varepsilon}\nabla f_0(x)\ \triangle_\xi h(\xi) \\ &\quad + \nabla\nabla f_0(x)\triangle_\xi g(\xi) + \varepsilon^2\triangle_x q(\frac{x}{\varepsilon}) + \Xi_\varepsilon^\omega(x,\xi)\Big)\Big|_{\xi=\frac{x}{\varepsilon}}\ , \end{aligned} \tag{35}$$

with

$$\Xi_\varepsilon^\omega(x,\xi) = \Delta\nabla f_0(x)\cdot\varepsilon h(\xi) + 2\nabla\nabla\nabla f_0(x)\cdot\varepsilon\nabla_\xi g(\xi) + \Delta\nabla\nabla f_0(x)\cdot\varepsilon^2 g(\xi)$$

In a similar way for $x\in\varepsilon\mathcal{G}_j^{\omega,i}$, we have

$$\begin{aligned}(A_\varepsilon^\omega F_\varepsilon^\omega)(x) &= \varepsilon^2\triangle_x\Big(f_j\big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\big) + \varepsilon\phi_j^\omega\big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\big)\Big)\\ &= \big(\triangle_\xi f_j(x,\mathcal{S}_j^{\omega,i}(\xi-\hat{x}_j^{\omega,i})) + \Psi_j^\omega(x,\mathcal{S}_j^{\omega,i}(\xi-\hat{x}_j^{\omega,i}))\big)\big|_{\xi=\frac{x}{\varepsilon}},\quad \xi\in\mathcal{G}_j^{\omega,i},\end{aligned}\tag{36}$$

with

$$\begin{aligned}\Psi_j^\omega(x,\hat\xi) &= \varepsilon^2\Delta_x f_j(x,\hat\xi) + 2\varepsilon\nabla_x\cdot\nabla_\xi f_j(x,\hat\xi)\\ &\quad+\varepsilon^3\Delta_x\phi_j^\omega(x,\hat\xi) + 2\varepsilon^2\nabla_x\cdot\nabla_\xi\phi_j^\omega(x,\hat\xi) + \varepsilon\Delta_\xi\phi_j^\omega(x,\hat\xi),\\ \hat\xi &= \mathcal{S}_j^{\omega,i}(\xi-\hat{x}_j^{\omega,i})\in\mathcal{D}_j.\end{aligned}$$

In order to make F_ε^ω belong to $D(A_\varepsilon^\omega)$ we should design it in such a way that the following conditions are fulfilled on $\partial G_\varepsilon^\omega$:
1) continuity condition on $\partial(\varepsilon\mathcal{G}_j^{\omega,i})$

$$\begin{aligned}&\Big(f_0(x)+\varepsilon\nabla f_0(x)\cdot h(\frac{x}{\varepsilon}) + \varepsilon^2\nabla\nabla f_0(x)\cdot g(\frac{x}{\varepsilon}) + \varepsilon^2 q(\frac{x}{\varepsilon})\Big)\Big|_{x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})}\\ &\quad= \Big(f_j\Big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\Big) + \varepsilon\phi_j^\omega\big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\big)\Big)\Big|_{x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})};\end{aligned}\tag{37}$$

2) continuity of the normal fluxes condition

$$\begin{aligned}&\nabla_x\Big(f_0(x)+\varepsilon\nabla f_0(x)\cdot h(\frac{x}{\varepsilon}) + \varepsilon^2\nabla\nabla f_0(x)\cdot g(\frac{x}{\varepsilon}) + \varepsilon^2 q(\frac{x}{\varepsilon})\Big)\Big|_{x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})}\cdot n^-\\ &= -\varepsilon^2\nabla_x\Big(f_j\Big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\Big) + \varepsilon\phi_j^\omega\big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\big)\Big)\Big|_{x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})}\cdot n^+,\end{aligned}\tag{38}$$

The main purpose of the functions $\phi_j^\omega(x,\xi)$ is to compensate the discrepancy between the inner and outer expansions for the function F_ε^ω at the boundary $\partial G_\varepsilon^\omega$, see Proposition 4.6 below. It follows from (31) that continuity condition (37) leads to the relation

$$\begin{aligned}&\phi_j^\omega\big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\big)\Big|_{x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})}\\ &\quad= \Big(\nabla f_0(x)\cdot h(\frac{x}{\varepsilon}) + \varepsilon\nabla\nabla f_0(x)\cdot g(\frac{x}{\varepsilon}) + \varepsilon q(\frac{x}{\varepsilon})\Big)\Big|_{x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})}.\end{aligned}\tag{39}$$

Notice that equality (39) defines the functions $\phi_j^\omega\big(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i})\big)$ only for $x\in\partial(\varepsilon\mathcal{G}_j^{\omega,i})$.

With the help of (34) the relation (38) can be rewritten as

$$\begin{aligned}&\Big(\nabla f_0(x)+\varepsilon\nabla\nabla f_0(x)h(\xi) + \nabla_\xi(\nabla f_0(x)h(\xi)) + \varepsilon^2\nabla\nabla\nabla f_0(x)g(\xi)\\ &\qquad+\varepsilon\nabla_\xi(\nabla\nabla f_0(x)g(\xi)) + \varepsilon\nabla_\xi q(\xi)\Big)\Big|_{\xi=\frac{x}{\varepsilon}\in\partial\,\mathcal{G}_j^{\omega,i}}\cdot n^-\\ &= -\big(\varepsilon^2\nabla_x f_j(x,\hat\xi) + \varepsilon\nabla_\xi f_j(x,\hat\xi) + \varepsilon^2\nabla_\xi\phi_j^\omega(x,\hat\xi) + \varepsilon^3\nabla_x\phi_j^\omega(x,\hat\xi)\big)\Big|_{\hat\xi\in\partial\mathcal{D}_j}\cdot n^+\end{aligned}\tag{40}$$

with $\hat\xi = \mathcal{S}_j^{\omega,i}(\xi-\hat{x}_j^{\omega,i})\in\mathcal{D}_j$.

We first consider the ansatz in (32) in the set $\varepsilon\mathcal{G}_0^\omega$. Collecting power-like terms in (35) and (40) and considering the terms of order ε^{-1} in (35) and of order ε^0 in (40), we conclude that $h(\cdot)$ should satisfy the equation

$$\nabla f_0(x)\,\triangle_\xi h(\xi)=0,\ \xi\in\mathcal{G}_0^\omega,\qquad \big(\nabla f_0(x)+\nabla f_0(x)\nabla_\xi h(\xi)\big)\cdot n_\xi^-=0,\ \xi\in\partial\mathcal{G}_0^\omega;$$

here x is a parameter. Since f_0 does not depend on ξ, this problem can be rewritten as follows:

$$\triangle_\xi h(\xi)=0,\ \ \xi\in\mathcal{G}_0^\omega,\qquad \nabla_\xi h(\xi)\cdot n_\xi^-=-n_\xi^-,\ \ \xi\in\partial\mathcal{G}_0^\omega. \tag{41}$$

This suggests the choice of h, it should coincide with the standard corrector used for homogenization of the Neumann problem in a random perforated domain, see [13]. We recall that the gradient of $h(\xi)$ is a statistically homogeneous matrix function that does not depend on ε, and h satisfies equation (41). Moreover, h shows a sublinear growth in L^2. Namely, assuming that $\int_{B_0}h(\frac{x}{\varepsilon})dx=0$, we have

$$\big\|\varepsilon h\big(\frac{\cdot}{\varepsilon}\big)\big\|_{L^2(B_0)}\longrightarrow 0,\quad \text{a.s. as } \varepsilon\to 0. \tag{42}$$

We also have

$$\big\|\nabla_\xi h\big(\frac{\cdot}{\varepsilon}\big)\big\|_{L^2(B_0)}\le C$$

a.s. with a constant C that does not depend on ε, see [13].

The matrix Θ in (12) is then defined by

$$\Theta\ =\ \mathbb{E}\big[(\mathbb{I}+\nabla_\xi h(\xi))\,\chi_{\mathcal{G}_0^\omega}(\xi)\big],\quad \text{i.e.}\ \ \Theta^{ij}\ =\ \mathbb{E}\big[(\delta_{ij}+\nabla_\xi^i h^j(\xi))\,\chi_{\mathcal{G}_0^\omega}(\xi)\big], \tag{43}$$

where $\chi_{\mathcal{G}_0^\omega}(\cdot)$ is the characteristic function of $\mathcal{G}_0^\omega$. It is proved in [13] that Θ is positive definite.

At the next step we collect the terms of order ε^0 on the right-hand side of (35) and equate them to $\Theta\cdot\nabla\nabla f_0(x)+\Upsilon(x)$ in order to make the difference $(A_\varepsilon^\omega F_\varepsilon^\omega-\pi_\varepsilon^\omega AF)=\big(A_\varepsilon^\omega F_\varepsilon^\omega(x)-(\Theta\cdot\nabla\nabla f_0(x)+\Upsilon(x))\big)$ small in $L^2(\varepsilon\mathcal{G}_0^\omega)$ norm. This yields

$$\begin{aligned}&\Big(\triangle f_0(x)+2\nabla\nabla f_0(x)\cdot\nabla_\xi h(\xi)+\nabla\nabla f_0(x)\cdot\triangle_\xi g(\xi)+\triangle_\xi q(\xi)\Big)\Big|_{\xi=\frac{x}{\varepsilon}}\\&\quad=\Theta\cdot\nabla\nabla f_0(x)+\Upsilon(x),\end{aligned} \tag{44}$$

where $x\in(\varepsilon\mathcal{G}_0^\omega\cap B_0)$; the function $\Upsilon(x)$ is defined in (14). We also collect the terms of order ε^1 in (40):

$$\begin{aligned}&\varepsilon\Big(\nabla\nabla f_0(x)h(\xi)+\nabla_\xi(\nabla\nabla f_0(x)\cdot g(\xi))+\nabla_\xi q(\xi)\Big)\Big|_{\xi=\frac{x}{\varepsilon}\in\partial\mathcal{G}_j^{\omega,i}}\cdot n^-\\&\quad=-\varepsilon\nabla_\xi f_j(x,\mathcal{S}_j^{\omega,i}(\xi-\hat{x}_j^{\omega,i}))\Big|_{\xi=\frac{x}{\varepsilon}\in\partial\mathcal{G}_j^{\omega,i}}\cdot n^+.\end{aligned} \tag{45}$$

Selecting all the terms in (44)-(45) that contain the second order derivatives of f_0, we arrive at the following problem for the random matrix valued function $g(\frac{x}{\varepsilon})=\{g_{ij}(\frac{x}{\varepsilon})\}$:

$$\begin{aligned}&\Big(\triangle f_0(x)+2\nabla\nabla f_0(x)\cdot\nabla_\xi h(\xi)+\nabla\nabla f_0(x)\cdot\triangle_\xi g(\xi)\Big)\Big|_{\xi=\frac{x}{\varepsilon}}=\Theta\cdot\nabla\nabla f_0(x),\\&\qquad\qquad x\in\varepsilon\mathcal{G}_0^\omega\cap B_0,\\&\nabla_\xi g(\xi)\cdot n^-\big|_{\xi=\frac{x}{\varepsilon}}=-h(\xi)\otimes n^-\big|_{\xi=\frac{x}{\varepsilon}},\quad x\in\varepsilon\partial\mathcal{G}_0^\omega\cap B_0.\end{aligned} \tag{46}$$

In addition to these two equations we impose the homogeneous Dirichlet boundary condition on the boundary of B_0

$$g\Big(\frac{x}{\varepsilon}\Big)=0\quad \text{on } \partial B_0.$$

Finally, $g(\frac{x}{\varepsilon})$ is introduced as a solution to the following problem:

$$\begin{aligned}\varepsilon^2\triangle_x g(\frac{x}{\varepsilon}) &= \mathbb{E}\big[(\mathbf{I}+\nabla_\xi h(\xi))\,\chi^\omega(\xi)\big]-\mathbf{I}-2\varepsilon\nabla_x h(\frac{x}{\varepsilon}),\quad x\in V^\varepsilon:=\varepsilon\mathcal{G}_0^\omega\cap B_0,\\ \varepsilon\nabla_x g(\frac{x}{\varepsilon})\cdot n^- &= -h(\frac{x}{\varepsilon})\otimes n^-,\quad x\in\varepsilon\partial\mathcal{G}_0^\omega\cap B_0,\\ g(\frac{x}{\varepsilon}) &= 0,\quad x\in\partial B_0;\end{aligned}\tag{47}$$

here $\mathbf{I}$ stands for the unit $d\times d$ matrix.

Lemma 4.4. *Problem* (47) *has a unique solution. Moreover, a.s.*

$$\lim_{\varepsilon\to 0}\|\varepsilon^2 g\big(\tfrac{\cdot}{\varepsilon}\big)\|_{H^1(\varepsilon\mathcal{G}_0^\omega\cap B_0)}=0.\tag{48}$$

The proof of this lemma is provided in Appendix 1, Section 6.

Next, collecting the remaining terms in (44) and (45), we arrive at the following problem for the function $q(\frac{x}{\varepsilon})$:

$$\begin{aligned}&\varepsilon^2\triangle_x q(\frac{x}{\varepsilon})=\Upsilon(x),\quad x\in\varepsilon\mathcal{G}_0^\omega\cap B_0,\\ &\nabla_\xi q(\xi)\cdot n^-\big|_{\xi=\frac{x}{\varepsilon}}=-\nabla_\xi f_j(x,\mathcal{S}_j^{\omega,i}(\xi-\hat{x}_j^{\omega,i}))\cdot n^+\big|_{\xi=\frac{x}{\varepsilon}},\quad x\in\varepsilon\partial\mathcal{G}_j^{\omega,i}\cap B_0,\end{aligned}\tag{49}$$

where the function $\Upsilon(x)\in C_0^\infty(\mathbb{R}^d)$ is defined in (14). We then equip system (49) with the homogeneous Dirichlet boundary condition at ∂B_0:

$$q(\frac{x}{\varepsilon})=0\quad\text{for } x\in\partial B_0.\tag{50}$$

Denote $\Phi_\varepsilon(x)=\varepsilon^2 q(\frac{x}{\varepsilon})$. Let $\phi_\Phi(\cdot)\in C_0^\infty(B_0)$ be a function such that

$$\begin{aligned}&\phi_\Phi\geq 0,\quad\text{and}\quad\phi_\Phi=1\text{ for all}\\ &x\in\{x\in\mathbb{R}^d:\ \text{there exist } j\text{ and }\xi\text{ such that } f_j(x,\xi)\neq 0\}.\end{aligned}\tag{51}$$

Proposition 4.5. *The following limit relations hold a.s.:*

$$\lim_{\varepsilon\to 0}\|\Phi_\varepsilon\|_{H^1(\varepsilon\mathcal{G}_0^\omega\cap B_0)}=0,\tag{52}$$

$$\lim_{\varepsilon\to 0}\|\phi_\Phi\Phi_\varepsilon\|_{H^1(\varepsilon\mathcal{G}_0^\omega\cap B_0)}=0.\tag{53}$$

Moreover,

$$\lim_{\varepsilon\to 0}\|\Delta_x\big(\phi_\Phi\Phi_\varepsilon\big)-\Upsilon\|_{L^2(\varepsilon\mathcal{G}_0^\omega\cap B_0)}=0.\tag{54}$$

The proof of this statement is given in Appendix 2.

We now turn to the correctors $\varepsilon\phi_j^\omega$, $j=1,\dots,N$. Our goal is to define them in such a way that

$$\begin{aligned}&\hat{f}_j\big(\hat{x}_j^{\omega,i},\mathcal{S}_j^{\omega,i}\big(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}\big)\big)+\varepsilon\phi_j^\omega(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\\ &=f_0(x)+\varepsilon(\nabla f_0(x),h(\tfrac{x}{\varepsilon}))+\varepsilon^2(\nabla\nabla f_0(x),g(\frac{x}{\varepsilon}))+\varepsilon^2 q(\frac{x}{\varepsilon})\quad\text{on }\varepsilon\partial\mathcal{G}_j^{\omega,i};\end{aligned}\tag{55}$$

$$\begin{aligned}&-\varepsilon^2\nabla\Big[\hat{f}_j\big(\hat{x}_j^{\omega,i},\mathcal{S}_j^{\omega,i}\big(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}\big)\big)+\varepsilon\phi_j^\omega(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\Big]\cdot n^+\\ &=\nabla\Big[f_0(x)+\varepsilon(\nabla f_0(x),h(\tfrac{x}{\varepsilon}))+\varepsilon^2(\nabla\nabla f_0(x),g(\frac{x}{\varepsilon}))+\varepsilon^2 q(\frac{x}{\varepsilon})\Big]\cdot n^-\\ &\qquad\text{on }\varepsilon\partial\mathcal{G}_j^{\omega,i}\end{aligned}\tag{56}$$

and

$$\begin{aligned} &\varepsilon \Big\| \sum_i \chi_{\mathcal{G}_j^{\omega,i}}(\frac{x}{\varepsilon}) \phi_j^\omega(x, \mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon} - \hat{x}_j^{\omega,i})) \Big\|_{L^2(\varepsilon \mathcal{G}_j^\omega)} \\ &+\varepsilon^3 \Big\| \sum_i \chi_{\mathcal{G}_j^{\omega,i}}(\frac{x}{\varepsilon}) \Delta_x \phi_j^\omega(x, \mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon} - \hat{x}_j^{\omega,i})) \Big\|_{L^2(\varepsilon \mathcal{G}_j^\omega)} \longrightarrow 0 \end{aligned} \tag{57}$$

as $\varepsilon \to 0$.

Proposition 4.6. *There exists a family of functions ϕ_j^ω with $j = 1, \dots, N$ and $\varepsilon \in (0,1)$ such that the relations in (55)–(57) are fulfilled.*

For the proof, see Appendix 2.

We turn back to the *Proof of Theorem* 4.3. The statement of this Theorem is now a straightforward consequence of (41), (42), Lemma 4.4 and Propositions 4.5 - 4.6. Indeed, due to (55) and (56), we have $F_\varepsilon^\omega \in D(A_\varepsilon^\omega)$. Then the convergence

$$\|F_\varepsilon^\omega - \pi_\varepsilon^\omega F\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0$$

follows from (33), (42), (48), (53) and (57). Finally, by (46), (54) and (57) we obtain

$$\|A_\varepsilon^\omega F_\varepsilon^\omega - \pi_\varepsilon^\omega A F\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

This completes the proof of Theorem 4.3. □

5. **Spectrum of the limit operator.** We proceed with the description of the spectrum of the limit operator A given by (13), and then using the strong convergence of Markov semigroup $T_\varepsilon^\omega(t)$ in $L^2(E)$ we describe the limit behavior of the spectra of operators A_ε^ω, as $\varepsilon \to 0$ almost surely.

Remind that each component $f_j(x,\xi)$ of $F \in D_A$ can be written as the sum

$$f_j(x,\xi) = f_0(x) + r_j(x,\xi) \quad \text{with} \quad r_j(x,\xi)\big|_{\xi \in \partial \mathcal{D}_j} = 0 \quad \forall\, j = 1, \dots, N.$$

Then (13) takes the form

$$(-AF)(x,\xi) = \begin{pmatrix} -\Theta \cdot \nabla\nabla f_0(x) + \frac{1}{\alpha_0} \sum_{j=1}^N \alpha_j \int\limits_{\mathcal{D}_j} \triangle_\xi r_j(x,\xi) d\xi \\ \\ -\triangle_\xi r_1(x,\xi) \\ \cdots \\ -\triangle_\xi r_N(x,\xi) \end{pmatrix}. \tag{58}$$

For each j the operator $-\triangle_\xi$ on $\mathcal{D}_j$ with homogeneous Dirichlet boundary condition has a discrete positive spectrum $\{\beta_m^j\}_{m \in \mathbb{N}}$, $\beta_m^j > 0$, $\beta_m^j \to \infty$ as $m \to \infty$. We denote by $\varkappa_m^j(\xi)$, $m = 1, 2, \dots$, the corresponding normalized eigenfunctions and by $\mathbb{M}$ the set of all indices (j,m). We introduce the set $\mathbb{M}^* \subset \mathbb{M}$ of indices (j,m) such that $\int\limits_{\mathcal{D}_j} \varkappa_m^j(\xi) d\xi = \langle \varkappa_m^j \rangle \neq 0$. Let $\mathbb{B}$ be a (countable) set of all β_m^j:

$$\mathbb{B} = \bigcup_{(j,m) \in \mathbb{M}} \beta_m^j,$$

and

$$b_1 = \min_{(j,m) \in \mathbb{M}} \beta_m^j = \min_{(j,m) \in \mathbb{M}^*} \beta_m^j, \quad b_1 > 0.$$

Lemma 5.1. *The continuous spectrum $\sigma_{cont}(-A)$ of the operator $-A$ is a countable set of non-overlapping segments*

$$\sigma_{cont}(-A) \ = \bigcup_{(j,m) \in \mathbb{M}^*} [\hat{\lambda}_m^j, \beta_m^j],$$

where $\hat\lambda_1 = 0$*, and* $\hat\lambda^j_m < \beta^j_m$ *is the nearest to* β^j_m *solution of equation*

$$\frac{1}{\alpha_0}\sum_{j=1}^{N}\alpha_j\sum_{m=1}^{\infty}\frac{(u^j_m)^2\ \beta^j_m}{\beta^j_m-\lambda}+1=0 \quad \text{with} \quad u^j_m=\langle\varkappa^j_m\rangle.$$

The point spectrum of the operator $-A$ *is the union of eigenvalues* β^j_m *with* $(j,m)\in\mathbb{M}\setminus\mathbb{M}^*$*:*

$$\sigma_p(-A) \ = \bigcup_{(j,m)\in\mathbb{M}\setminus\mathbb{M}^*}\beta^j_m.$$

Each eigenvalue $\beta^j_m\in\sigma_p(-A)$ *has infinite multiplicity, so that* $\sigma_p(-A)$ *belongs to the essential spectrum of* $-A$*.*

Proof. Each line in the equation $-AF=\lambda F$ except of the first one reads

$$-A(f_0(x)+r_j(x,\xi)) = -\triangle_\xi r_j(x,\xi) = \lambda(f_0(x)+r_j(x,\xi)), \quad \xi\in\mathcal{D}_j. \tag{59}$$

The function $f_0(x)$ does not depend on ξ, its Fourier series w.r.t. $\{\varkappa^j_m(\xi)\}$ for every j takes the form

$$f_0(x)\cdot 1 = f_0(x)\sum_m u^j_m\varkappa^j_m(\xi), \quad \text{with } \ u^j_m=\int_{\mathcal{D}_j}\varkappa^j_m(\xi)\,d\xi. \tag{60}$$

Denoting by $\gamma^j_m=\gamma^j_m(x)$ the Fourier coefficients of r_j, from (59) - (60) we get

$$-\triangle_\xi r_j(x,\xi)=\sum_m\beta^j_m\gamma^j_m\varkappa^j_m(\xi)=\lambda f_0(x)\sum_m u^j_m\varkappa^j_m(\xi)+\lambda\sum_m\gamma^j_m\varkappa^j_m(\xi).$$

Consequently, for any $\lambda\notin\mathbb{B}$ we have $\gamma^j_m=\lambda f_0(x)\,\frac{u^j_m}{\beta^j_m-\lambda}$, and thus the function

$$r_j(x,\xi)=\sum_m\gamma^j_m\varkappa^j_m(\xi)=\lambda f_0(x)\sum_m\frac{u^j_m}{\beta^j_m-\lambda}\varkappa^j_m(\xi), \tag{61}$$

is a solution of equation $-A(f_0+r_j)=-\triangle_\xi r_j=\lambda(f_0+r_j)$ for any j and any $\lambda\notin\mathbb{B}$.

Inserting (61) in the first line of the equation $-AF=\lambda F$ with $-A$ given by (58) yields

$$-\Theta\cdot\nabla\nabla f_0(x)-\lambda f_0(x)\frac{1}{\alpha_0}\sum_j\alpha_j\sum_m\frac{u^j_m\ \beta^j_m}{\beta^j_m-\lambda}\int_{\mathcal{D}_j}\varkappa^j_m(\xi)d\xi=\lambda f_0(x).$$

Consequently

$$-\Theta\cdot\nabla\nabla f_0(x)=\lambda f_0(x)\left(\frac{1}{\alpha_0}\sum_j\alpha_j\sum_m\frac{(u^j_m)^2\ \beta^j_m}{\beta^j_m-\lambda}+1\right). \tag{62}$$

Since the spectrum of the operator $-\Theta\cdot\nabla\nabla$ fills up the positive half-line, we obtain that all $\lambda>0$ such that

$$\frac{1}{\alpha_0}\sum_j\alpha_j\sum_m\frac{(u^j_m)^2\ \beta^j_m}{\beta^j_m-\lambda}+1\geq 0$$

belong to the spectrum of the operator $-A$. One can easily check that the segment $[0,\beta_1]$, $b_1=\min_{(j,m)\in\mathbb{M}^*}\beta^j_m>0$ belongs to the continuous spectrum of $-A$. This implies the desired statement on $\sigma_{cont}(-A)$.

Recall that $\langle\varkappa^j_m\rangle=0$ for all $(j,m)\in\mathbb{M}\setminus\mathbb{M}^*$. It is straightforward to check that for any $(j,m)\in\mathbb{M}\setminus\mathbb{M}^*$ the function $F^{(j,m)}=(0,\dots,\varphi(x)\,\varkappa^j_m(\xi),0,\dots,0)$ with

$\varphi(x) \in L^2(\mathbb{R}^d)$, is the eigenfunction of $-A$ with the corresponding eigenvalue β_m^j. This completes the proof. □

Notice that the operators A_ε^ω for every ε have statistically homogeneous coefficients, i.e. they are metrically transitive with respect to the unitary group of the space translations in $\mathbb{R}^d$. Then from the general results, see e.g. [20], it follows that the spectra of the operators A_ε^ω are non-random for a.e. ω.

Proposition 5.2. *For any $\lambda \in \sigma(-A)$ a.s. there exists a sequence λ_ε, $\lambda_\varepsilon \in \sigma(A_\varepsilon^\omega)$, that converges to λ as $\varepsilon \to 0$.*

Proof. Since $\lambda \in \sigma(-A)$, there exist functions $F_n \in D_A$, $\|F_n\|_{L^2(E,\alpha)} = 1$ such that $\|(A+\lambda)F_n\|_{L^2(E,\alpha)} \to 0$ as $n \to \infty$. Using Theorem 4.3 we additionally have that for any $F_n \in D_A$ there exists $F_{n,\varepsilon}^\omega \in D(A_\varepsilon^\omega)$ for a.e. ω such that

$$\|F_{n,\varepsilon}^\omega - \pi_\varepsilon^\omega F_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{and} \quad \|A_\varepsilon^\omega F_{n,\varepsilon}^\omega - \pi_\varepsilon^\omega A F_n\|_{L^2(\mathbb{R}^d)} \to 0 \quad \text{as } \varepsilon \to 0.$$

Thus using Lemma 4.1 we obtain that for any (small) $\delta > 0$ there exists $\varepsilon_0 = \varepsilon_0(\lambda, \delta) > 0$ such that for all $\varepsilon < \varepsilon_0$ there exists $F_{n,\varepsilon}^\omega \in L^2(\mathbb{R}^d)$ with $\|F_{n,\varepsilon}^\omega\| = 1$, and

$$\|A_\varepsilon^\omega F_{n,\varepsilon}^\omega + \lambda F_{n,\varepsilon}^\omega\|_{L^2(\mathbb{R}^d)} < \delta. \tag{63}$$

This implies that there is a point of the spectrum of $-A_\varepsilon^\omega$ in the δ-neighbourhood of λ. □

6. **Conclusions.** We have presented in this work new homogenization results for diffusion processes in high contrast random statistically homogeneous porous media that is defined by its generator (1) - (2). Our approach relies on introducing an extended state space for the process. Namely, we equip the coordinate process with extra components characterising the behaviour of the process inside the slow diffusion inclusions. In the extended space the limit process remains Markov, however, its first component need not be Markov. This explains the appearance of the memory in the limit dynamics of the coordinate process. This memory effect is governed by random times that the process spends in the inclusions.

The extension of these results to diffusion-convection equations is straightforward under the natural assumption that the transport coefficient is of order ε in $\mathbb{R}^d \setminus G^\omega$ and of order ε^2 in G^ω. The study still needs to be improved by developing a general approach that would allow us to incorporate the case of complex processes that include passive solute transport through highly heterogeneous random media. In this case, the flow is governed by a coupled system of an elliptic equation and a linear convection-diffusion concentration equation with a diffusion term small with respect to the convection, i.e., with a relatively high Peclet number. Our future study will focus on extension of the homogenization results obtained in [3, 6] for the periodic case to the case of high contrast random media.

Appendix 1. The second corrector g. Proof of Lemma 4.4. Recall that the matrix valued function $g(\frac{x}{\varepsilon})$ was defined as a solution to the following problem:

$$\begin{aligned} \varepsilon^2 \triangle_x g(\frac{x}{\varepsilon}) &= \mathbb{E}\big[(\mathbf{I} + \nabla_\xi h(\xi))\,\chi^\omega(\xi)\big] - \mathbf{I} - 2\varepsilon \nabla_x h(\frac{x}{\varepsilon}), \\ & \quad x \in V^\varepsilon := \varepsilon \mathcal{G}_0^\omega \cap B_0, \\ \varepsilon \nabla_x g(\frac{x}{\varepsilon}) \cdot n^- &= -h(\frac{x}{\varepsilon}) \otimes n^-, \quad x \in \varepsilon \partial \mathcal{G}_0^\omega \cap B_0, \\ g_\varepsilon^\omega(\frac{x}{\varepsilon}) &= 0, \quad x \in \partial B_0, \end{aligned} \tag{64}$$

where B_0 is a ball that is centered at the origin and contains the supports of f_j, $j = 0, 1, \dots, N$. In this section we will use notation $\chi^\omega(\cdot) = \chi_{\mathcal{G}_0^\omega}(\cdot)$ for the characteristic function of the random set $\mathcal{G}_0^\omega$. In the coordinate form the above problem reads

$$\begin{aligned} \varepsilon^2 \triangle_x g^{ij}(\frac{x}{\varepsilon}) &= \mathbb{E}\big[(\delta_{ij} + \nabla_\xi^i h^j(\xi))\,\chi^\omega(\xi)\big] - \delta_{ij} - 2\varepsilon \nabla_x^j h^i(\frac{x}{\varepsilon}), \quad x \in V^\varepsilon, \\ \varepsilon \nabla_x g^{ij}(\frac{x}{\varepsilon}) \cdot n^- &= -h^i(\frac{x}{\varepsilon})\,(n^-)^j, \quad x \in \partial V^\varepsilon \cap B_0, \\ g^{ij}(\frac{x}{\varepsilon}) &= 0, \quad x \in \partial B_0. \end{aligned} \tag{65}$$

Each component of $g(\frac{x}{\varepsilon}) = \{g^{ij}(\frac{x}{\varepsilon})\}$ can be considered separately and in what follows we omit a super index ij.

Denote $\Psi_\varepsilon(\frac{x}{\varepsilon}) = \varepsilon^2 g(\frac{x}{\varepsilon})$. Our first goal is to prove that the set of functions $\Psi_\varepsilon(\frac{x}{\varepsilon})$ is bounded in $H^1(V^\varepsilon)$. Integrating by parts and using the second equality in (65) we get

$$\begin{aligned} \int\limits_{V^\varepsilon} \varepsilon^2 \triangle_x g(\frac{x}{\varepsilon})\,\varepsilon^2 g(\frac{x}{\varepsilon}) dx &= -\varepsilon^4 \int\limits_{V^\varepsilon} \big|\nabla_x g(\frac{x}{\varepsilon})\big|^2 dx + \varepsilon^4 \int\limits_{\partial V^\varepsilon} \frac{\partial g(\frac{x}{\varepsilon})}{\partial n}\, g(\frac{x}{\varepsilon})\, d\sigma(x) \\ = -\varepsilon^4 \int\limits_{V^\varepsilon} \big|\nabla_x g(\frac{x}{\varepsilon})\big|^2 dx &- \varepsilon^3 \int\limits_{\partial V^\varepsilon} h(\frac{x}{\varepsilon})\, n^-\, g(\frac{x}{\varepsilon})\, d\sigma(x). \end{aligned} \tag{66}$$

On the other hand, using the first equality in (65) and integrating by parts we transform the left hand side of (66) as follows:

$$\begin{aligned} &\int\limits_{V^\varepsilon} \varepsilon^2 \triangle_x g(\frac{x}{\varepsilon})\,\varepsilon^2 g(\frac{x}{\varepsilon}) dx \\ &= \varepsilon^2 \int\limits_{V^\varepsilon} \mathbb{E}\big[(\mathbb{I} + \nabla_\xi h)\chi^\omega\big]\, g(\frac{x}{\varepsilon}) dx - \varepsilon^2 \int\limits_{V^\varepsilon} \big(\mathbb{I} + \nabla_\xi h(\frac{x}{\varepsilon})\big) g(\frac{x}{\varepsilon}) dx \\ &\quad -\varepsilon^3 \int\limits_{V^\varepsilon} \nabla_x h(\frac{x}{\varepsilon}) g(\frac{x}{\varepsilon}) dx \\ &= \varepsilon^2 \int\limits_{V^\varepsilon} \Big(\mathbb{E}\big[(\mathbb{I} + \nabla_\xi h)\chi^\omega\big] - (\mathbb{I} + \nabla_\xi h(\frac{x}{\varepsilon}))\Big)\, g(\frac{x}{\varepsilon}) dx \\ &\quad -\varepsilon^3 \int\limits_{\partial V^\varepsilon} h(\frac{x}{\varepsilon}) n^- g(\frac{x}{\varepsilon}) d\sigma(x) + \varepsilon^3 \int\limits_{V^\varepsilon} h(\frac{x}{\varepsilon})\, \mathrm{div}_x g(\frac{x}{\varepsilon}) dx. \end{aligned} \tag{67}$$

Thus, (66) - (67) imply

$$\begin{aligned} \varepsilon^4 \int\limits_{V^\varepsilon} \big|\nabla_x g(\frac{x}{\varepsilon})\big|^2 dx = -\varepsilon^2 \int\limits_{V^\varepsilon} \Big(\mathbb{E}\big[(\mathbb{I} + \nabla_\xi h)\chi^\omega\big] - (\mathbb{I} + \nabla_\xi h(\frac{x}{\varepsilon}))\Big) g(\frac{x}{\varepsilon}) dx \\ -\varepsilon^3 \int\limits_{V^\varepsilon} h(\frac{x}{\varepsilon})\, \mathrm{div}_x g(\frac{x}{\varepsilon}) dx. \end{aligned} \tag{68}$$

We get from (68) that

$$\|\nabla_x \Psi_\varepsilon\|^2_{L^2(V^\varepsilon)} \le A \|\Psi_\varepsilon\|_{L^2(V^\varepsilon)} + \|\varepsilon h\|_{L^2(V^\varepsilon)}\, \|\nabla_x \Psi_\varepsilon\|_{L^2(V^\varepsilon)}. \tag{69}$$

We have used here the fact that $\Lambda_\varepsilon^\omega = \mathbb{E}\big[(\mathbb{I}+\nabla_\xi h^\omega)\chi^\omega\big] - (\mathbb{I}+\nabla_\xi h^\omega(\frac{x}{\varepsilon}))\chi^\omega(\frac{x}{\varepsilon})$ is a stationary random field with finite second moment $\mathbb{E}(\Lambda_\varepsilon^\omega)^2 < +\infty$. Moreover, by the Birkhoff' theorem

$$\lim_{\varepsilon\to 0}\mathbb{E}(\Lambda_\varepsilon^\omega) = 0, \tag{70}$$

and thus $\Lambda_\varepsilon^\omega$ a.s. weakly converges to zero in $L^2(B_0)$ as $\varepsilon \to 0$.

Next we apply the results on extensions in random perforated domains, see [1], [13]. According to these results there exists a liner extension operator $L : H^1(V^\varepsilon) \to H^1(B_0)$ such that for any $f \in H^1(V^\varepsilon)$

$$Lf|_{H^1(V^\varepsilon)} = f, \quad \|Lf\|_{L^2(B_0)} \le C\,\|f\|_{L^2(V^\varepsilon)}, \quad \|\nabla(Lf)\|_{L^2(B_0)} \le \tilde C\,\|\nabla f\|_{L^2(V^\varepsilon)},$$

where the constants C and $\tilde C$ do not depend on ε. Keeping for the extended function $L\Psi_\varepsilon$ the same notation Ψ_ε and considering the Dirichlet boundary condition on ∂B_0 in (65), by the Friedrichs inequality we obtain

$$\|\Psi_\varepsilon\|^2_{L^2(V^\varepsilon)} \le \|\Psi_\varepsilon\|^2_{L^2(B_0)} \le c_1\|\nabla_x\Psi_\varepsilon\|^2_{L^2(B_0)} \le C\|\nabla_x\Psi_\varepsilon\|^2_{L^2(V^\varepsilon)}. \tag{71}$$

Combining this with (69) yields

$$\|\nabla_x\Psi_\varepsilon\|_{L^2(V^\varepsilon)} \le A_1, \qquad \|\Psi_\varepsilon\|_{L^2(V^\varepsilon)} \le A_2 \tag{72}$$

with the constants A_1 and A that do not depend on ε. Thus a.s. the family of functions $\{\Psi_\varepsilon\}$ is bounded in $H^1(V^\varepsilon)$ and in $H^1(B_0)$. Due to the compactness of embedding of $H^1(B_0)$ in $L^2(B_0)$ we can pass to the limit in the product $\Lambda_\varepsilon^\omega\,\Psi_\varepsilon$ as $\varepsilon \to 0$. Thus the integral

$$\begin{aligned}&\varepsilon^2\int\limits_{V^\varepsilon}\Big(\mathbb{E}\big[(\mathbb{I}+\nabla_\xi h)\chi^\omega\big] - (\mathbb{I}+\nabla_\xi h(\frac{x}{\varepsilon}))\Big)g(\frac{x}{\varepsilon})dx\\ &= \int\limits_{B_0}\Big(\mathbb{E}\big[(\mathbb{I}+\nabla_\xi h)\chi^\omega\big] - (\mathbb{I}+\nabla_\xi h(\frac{x}{\varepsilon}))\Big)\chi^\omega\Big(\frac{x}{\varepsilon}\Big)\Psi_\varepsilon(\frac{x}{\varepsilon})dx\end{aligned}$$

tends to zero as $\varepsilon \to 0$ a.s. Taking into account relations (42) we derive from (68) that

$$\|\nabla_x\Psi_\varepsilon\|_{L^2(B_0)} \ \to\ 0 \tag{73}$$

and, by the Friedrichs inequality,

$$\|\Psi_\varepsilon\|_{L^2(B_0)} \ \to\ 0. \tag{74}$$

Appendix 2. Proofs of Propositions 4.5 and 4.6. We begin this section by proving Proposition 4.5. Denote $\Phi_\varepsilon(x) := \varepsilon^2 q(\frac{x}{\varepsilon})$. Then $\Phi_\varepsilon(x)$ is a solution of the following problem:

$$\begin{aligned}\triangle_x\Phi_\varepsilon(x) &= \Upsilon(x), \quad x \in V^\varepsilon = \varepsilon\mathcal{G}_0^\omega \cap B_0,\\ \nabla_x\Phi_\varepsilon(x)\cdot n^- &= -\varepsilon^2\nabla_x f_j(x, \mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon} - \hat x_j^{\omega,i}))\cdot n^+, \quad x \in \varepsilon\partial\mathcal{G}_j^{\omega,i}\cap B_0,\\ \Phi_\varepsilon(x) &= 0, \quad x \in \partial B_0.\end{aligned} \tag{75}$$

In what follows we will use the following notations:

$$\nabla f(x, \frac{x}{\varepsilon}) = \nabla_x f(x, \frac{x}{\varepsilon}), \quad \nabla h(\xi) = \nabla_\xi h(\xi), \quad \nabla\Phi_\varepsilon(x) = \nabla_x\Phi_\varepsilon(x).$$

In order to show that the functions $\Phi_\varepsilon(x)$ are bounded in $H^1(V^\varepsilon)$ we follow the line of the proof in the previous section. Multiplying the equation in (75) by $\Phi_\varepsilon(x)$ and integrating the resulting relation over V^ε after integration by parts we obtain

$$\begin{aligned}\int\limits_{V^\varepsilon} \Upsilon(x)\,\Phi_\varepsilon(x)dx &= \int\limits_{V^\varepsilon} \triangle\Phi_\varepsilon(x)\,\Phi_\varepsilon(x)dx \\ &= \int\limits_{\partial V^\varepsilon} \Phi_\varepsilon(x)\,\nabla\Phi_\varepsilon(x)\cdot n^-\,d\sigma(x) - \int\limits_{V^\varepsilon} \big|\nabla\Phi_\varepsilon(x)\big|^2 dx \\ &= -\int\limits_{V^\varepsilon} \big|\nabla\Phi_\varepsilon(x)\big|^2 dx \\ &\quad -\sum_j\sum_i \int\limits_{\varepsilon\partial\mathcal{G}_j^{\omega,i}} \Phi_\varepsilon(x)\,\varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\cdot n^+\,d\sigma(x).\end{aligned} \tag{76}$$

By the Friedrichs inequality

$$\Big|\int\limits_{V^\varepsilon} \triangle\Phi_\varepsilon(x)\,\Phi_\varepsilon(x)dx\Big| \le C_1\|\Upsilon(x)\|_{L^2(B_0)}\,\|\nabla\Phi_\varepsilon(x)\|_{L^2(V^\varepsilon)}. \tag{77}$$

with a constant C_1 that does not depend on ε.

To estimate the second integral on the right-hand side of (76) we extend the functions Φ_ε on B_0, denote the extended functions by $\bar{\Phi}_\varepsilon(x)$ and apply the Stokes formula. This yields

$$\begin{aligned}&\int\limits_{\varepsilon\partial\mathcal{G}_j^{\omega,i}} \varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\cdot n^+\Phi_\varepsilon(x)\,d\sigma(x) \\ &= \int\limits_{\varepsilon\mathcal{G}_j^{\omega,i}} \varepsilon^2\triangle f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\,\bar{\Phi}_\varepsilon(x)\,dx \\ &\quad + \int\limits_{\varepsilon\mathcal{G}_j^{\omega,i}} \varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\cdot\nabla\bar{\Phi}_\varepsilon(x)\,dx.\end{aligned} \tag{78}$$

From this relation by the Friedrichs inequality we derive the following upper bound:

$$\sum_j\sum_i\Big|\int\limits_{\varepsilon\partial\mathcal{G}_j^{\omega,i}} \varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat{x}_j^{\omega,i}))\cdot n^+\,\Phi_\varepsilon(x)\,d\sigma(x)\Big| \le C\,\|\nabla\Phi_\varepsilon\|_{L^2(V^\varepsilon)}. \tag{79}$$

Finally, (76), (77) and (79) imply the desired upper bound:

$$\|\nabla\Phi_\varepsilon\|_{L^2(V^\varepsilon)} \le \tilde{C}, \tag{80}$$

i.e. the functions $\Phi_\varepsilon(x)$ are bounded in $H^1(V^\varepsilon)$. Consequently, the extensions $\bar{\Phi}_\varepsilon(x)$ are also bounded in $H^1(B_0)$ and form a compact set in $L^2(B_0)$. Thus, there exists $\Phi_0 \in H^1(B_0)$ such that, for a subsequence,

$$\|\bar{\Phi}_\varepsilon - \Phi_0\|_{L^2(B_0)} \to 0.$$

Our goal is to prove that $\Phi_0 \equiv 0$, or equivalently

$$\|\Phi_\varepsilon\|_{L^2(B_0)} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{81}$$

For an arbitrary $\hat{\psi} \in C^\infty(G_\varepsilon^\omega)$ with a compact support in B_0 we have

$$\int_{V^\varepsilon} \triangle\Phi_\varepsilon(x)\,\hat\psi(x)\,dx = -\int_{V^\varepsilon}\nabla\Phi_\varepsilon(x)\cdot\nabla\hat\psi(x)\,dx + \sum_j\sum_i \int_{\varepsilon\partial\mathcal{G}_j^{\omega,i}} \varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat x_j^{\omega,i}))\cdot n^+\,\hat\psi(x)\,d\sigma(x). \tag{82}$$

On the other hand,

$$\int_{V^\varepsilon}\triangle_x\Phi_\varepsilon(x)\,\hat\psi(x)dx = \int_{V^\varepsilon}\Upsilon(x)\hat\psi(x)\,dx = \int_{B_0}\Upsilon(x)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x)\,\hat\psi(x)\,dx. \tag{83}$$

Therefore,

$$\begin{aligned}&\int_{B_0}\nabla\Phi_\varepsilon(x)\cdot\nabla\hat\psi(x)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x)\,dx\\ &=\sum_j\sum_i\int_{\varepsilon\partial\mathcal{G}_j^{\omega,i}}\varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat x_j^{\omega,i}))\cdot n^+\,\hat\psi(x)\,d\sigma(x)-\int_{V^\varepsilon}\Upsilon(x)\hat\psi(x)\,dx.\end{aligned} \tag{84}$$

For an arbitrary $\psi\in C_0^\infty(B_0)$, substituting in the last relation the function $\psi(x)+\varepsilon h\big(\frac{x}{\varepsilon}\big)\nabla\psi(x)$ for $\hat\psi$ we obtain

$$\begin{aligned}&\int_{B_0}\nabla\Phi_\varepsilon(x)\cdot\big(\nabla\psi(x)+\nabla h\big(\frac{x}{\varepsilon}\big)\nabla\psi(x)\big)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x)\,dx+o(1)\\ &=\sum_j\sum_i\int_{\varepsilon\partial\mathcal{G}_j^{\omega,i}}\varepsilon^2\nabla f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat x_j^{\omega,i}))\cdot n^+\,\hat\psi(x)\,d\sigma(x)-\int_{V^\varepsilon}\Upsilon(x)\hat\psi(x)dx\\ &=-\sum_j\sum_i\int_{\varepsilon\mathcal{G}_j^{\omega,i}}\varepsilon^2\triangle f_j(x,\mathcal{S}_j^{\omega,i}(\frac{x}{\varepsilon}-\hat x_j^{\omega,i}))\psi(x)\,dx-\int_{V^\varepsilon}\Upsilon(x)\psi(x)dx+o(1),\end{aligned} \tag{85}$$

where $o(1)$ a.s. tends to zero as $\varepsilon\to0$; here we have used the inequality

$$\|\varepsilon h\big(\frac{\cdot}{\varepsilon}\big)\|_{H^1(\varepsilon\mathcal{G}_0^\omega)}\le C$$

and the fact that $\|\varepsilon h\big(\frac{\cdot}{\varepsilon}\big)\|_{L^2(\varepsilon\mathcal{G}_0^\omega)}$ vanishes as $\varepsilon\to0$. Using representation (14) of the function $\Upsilon(x)$, the Stokes formula and the Birkhoff ergodic theorem we conclude that the right-hand side in (85) tends to 0 as $\varepsilon\to0$ for any $\psi\in C_0^\infty(B_0)$ and thus

$$\begin{aligned}&\lim_{\varepsilon\to0}\int_{B_0}\nabla\Phi_\varepsilon(x)\cdot\Big(\nabla\psi(x)+\nabla h\big(\frac{x}{\varepsilon}\big)\nabla\psi(x)\Big)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x)\,dx=0\\ &\Phi_\varepsilon(x)\big|_{\partial B_0}=0.\end{aligned} \tag{86}$$

The subsequence of $\nabla\Phi_\varepsilon$ converges weakly in $(L^2(B_0))^d$ to $\nabla\Phi_0$, as $\varepsilon\to0$. By the definition of matrix Θ the sequence $\big(\nabla\psi+\nabla h\big(\frac{\cdot}{\varepsilon}\big)\nabla\psi\big)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}$ converges weakly in $(L^2(B_0))^d$ to $\Theta\nabla\psi$. Since the function $h(\cdot)$ satisfies equation (41), we have

$$\operatorname{div}\big[\big(\nabla\psi(x)+\nabla h\big(\frac{x}{\varepsilon}\big)\nabla\psi(x)\big)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x)\big]=\big(\Delta\psi(x)+\nabla h\big(\frac{x}{\varepsilon}\big)\nabla\nabla\psi(x)\big)\,\chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x).$$

The right-hand side here is bounded in $L^2(B_0)$ and thus compact in $H^{-1}(B_0)$. By the compensated compactness theorem, see [17], we obtain

$$0 = \lim_{\varepsilon\to 0} \int\limits_{B_0} \nabla\Phi_\varepsilon(x) \cdot \Big(\nabla\psi(x) + \nabla h\big(\frac{x}{\varepsilon}\big)\nabla\psi\Big)\, \chi_{\{\varepsilon\mathcal{G}_0^\omega\}}(x)\, dx = \int_{B_0} \nabla\Phi_0 \cdot \Theta\nabla\psi\, dx.$$

Since $\Phi_0 = 0$ on ∂B_0, this implies that $\Phi_0 = 0$, and (81) follows. This convergence to $\Phi_0 = 0$ holds for the whole family Φ_ε^ω.

The proof of other statements of Proposition 4.5 is now straightforward.

We turn to the proof of Proposition 4.6. Denote

$$\tilde\Xi^\varepsilon\big(x\big) = f_0(x) + \varepsilon(\nabla f_0(x), h(\frac{x}{\varepsilon})) + \varepsilon^2(\nabla\nabla f_0(x), g(\frac{x}{\varepsilon})) + \varepsilon^2 q(\frac{x}{\varepsilon}).$$

For each (j,i) with $j \in \{1,\dots,N\}$ and $i \in \mathbb{N}$ we define an open set $\varepsilon\mathcal{Q}^\varkappa_{\omega,ji} = \{x \in \mathbb{R}^d : \operatorname{dist}(x, \varepsilon\partial\mathcal{G}_j^{\omega,i}) < \varepsilon\varkappa\}$ and introduce in this set coordinates y such that $y' = (y_2,\dots,y_d)$ are smooth coordinates on $\varepsilon\partial\mathcal{G}_j^{\omega,i}$, and y_1 is directed along the exterior normal, $y_1 = -\operatorname{dist}(x, \varepsilon\partial\mathcal{G}_j^{\omega,i})$ if $x \in \varepsilon\mathcal{G}_j^{\omega,i}$ and $y_1 = \operatorname{dist}(x, \varepsilon\partial\mathcal{G}_j^{\omega,i})$ if $x \in \mathbb{R}^d \setminus \varepsilon\mathcal{G}_j^{\omega,i}$. Under our assumptions on the geometry of $\mathcal{G}_j^{\omega,i}$ there exists $\varkappa > 0$ such that

- $\varepsilon\mathcal{Q}^\varkappa_{\omega,ji}$ do not intersect with $\varepsilon\mathcal{Q}^\varkappa_{\omega,km}$, if $(j,i) \neq (k,m)$.
- Coordinates $y = y(x)$ are well defined in $\varepsilon\mathcal{Q}^\varkappa_{\omega,ji}$, that is $y = y(x)$ is an invertible diffeomorphism.

Letting $\Xi^\varepsilon\big(y\big) = \tilde\Xi^\varepsilon\big(x(y)\big)$ and $\check f^\varepsilon(y) = \hat f_j(\hat x_j^{\omega,i}, \mathcal{S}_j^{\omega,i}(\frac{x(y)}{\varepsilon} - \hat x_j^{\omega,i}))$, we define in $\varepsilon\mathcal{Q}^\varkappa_{\omega,ji} \cap \varepsilon\mathcal{G}_j^{\omega,i}$ the function

$$\varepsilon\phi_j^\omega(y) = \Big(\Xi^\varepsilon\big(0,y'\big) - \check f_j^\varepsilon(0,y') + y_1\big[\frac{\partial}{\partial y_1}\Xi^\varepsilon\big(0,y'\big) - \frac{\partial}{\partial y_1}\check f_j^\varepsilon\big(0,y'\big)\big] - \overline{\Xi}\Big)\theta\big(-\frac{y_1}{\varepsilon}\big) + \overline{\Xi},$$

where $\theta(s)$ is a C^∞ cut-off function such that $0 \le \theta \le 1$, $\theta = 1$ for $s < \frac{\varkappa}{3}$ and $\theta = 0$ for $s > \frac{2\varkappa}{3}$; $\overline{\Xi}$ is the mean value of Ξ^ε over $\varepsilon\mathcal{Q}^\varkappa_{\omega,ji} \cap (\mathbb{R}^d \setminus \varepsilon\mathcal{G}_j^{\omega,i})$.

By construction (55) and (56) are fulfilled. Relation (57) follows from the properties of the correctors and elliptic estimates, see [11, Chapter X].

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