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MATHEMATICS

Spectral Problem with Steklov Condition on a Thin Perforated Interface

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Abstract—A two-dimensional Steklov-type spectral problem for the Laplacian in a domain divided into two parts by a perforated interface with a periodic microstructure is considered. The Steklov boundary condition is set on the lateral sides of the channels, a Neumann condition is specified on the rest of the interface, and a Dirichlet and Neumann condition is set on the outer boundary of the domain. Two-term asymptotic expansions of the eigenvalues and the corresponding eigenfunctions of this spectral problem are constructed.

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1. INTRODUCTION

We study a spectral problem with a Steklov boundary condition specified on a thin perforated interface with a periodic microstructure. Starting with [1] for perforated domains and with [2] for operators with rapidly oscillating coefficients, much attention has been given in the mathematical literature to various aspects of the homogenization of spectral problems (see, for example, [3–5]).

The problem is considered in a two-dimensional domain divided into two parts by an interface with periodically located channels. The thickness of the interface and the period of the channels are identical and equal to ε. The thickness of the channels is $a\varepsilon$, where $a < 1$ is a constant. Here and below, ε is a small parameter defined as $\varepsilon = \dfrac{1}{2\mathcal{N} + 1}$, where $\mathcal{N} \gg 1$ is a positive integer. Assume that a Steklov-type spectral condition is set on the boundaries of the channels, a homogeneous Dirichlet boundary condition is specified on the outer boundary of the domain, and a homogeneous Neumann condition is set on the rest of the interface. A limiting (homogenized) spectral problem for this problem was obtained in [5]. The goal of this paper is to construct the leading terms in the asymptotic expansions of eigenpairs and to justify the constructed asymptotic expansions.

2. FORMULATION OF THE PROBLEM AND PRELIMINARIES

Let Ω be a domain in $\mathbb{R}^2$ with a smooth boundary Γ that coincides with the line segments $\left\{x \in \mathbb{R}^2: x_1 = -\dfrac{1}{2}\right\}$ and $\left\{x \in \mathbb{R}^2: x_1 = \dfrac{1}{2}\right\}$ near the endpoints of the interval $\Gamma_1 = -\left[-\dfrac{1}{2}; \dfrac{1}{2}\right]$ on the horizontal axis, respectively. Denote the nonempty boundary segment $\Gamma_2 = \{x \in \Gamma: |x_2| > c\}$ for some $c > 0$, and let $\Gamma_3 = \Gamma\backslash\Gamma_2$.

Let Q be the rectangle $\left\{x \in \mathbb{R}^2: x_1 \in \left(-\dfrac{1}{2}; \dfrac{1}{2}\right), x_2 \in -\left(-\dfrac{h\varepsilon}{2}; \dfrac{h\varepsilon}{2}\right)\right.$ and B be the rectangle $\{\xi \in \mathbb{R}^2: \xi_1 \in \left(-\dfrac{a}{2}; \dfrac{a}{2}\right), \xi_2 \in \left(-\dfrac{h}{2}; \dfrac{h}{2}\right)$. Recall that $\varepsilon = (2\mathcal{N} + 1)^{-1}$, where $\mathcal{N} \in \mathbb{N}$. Define $B_\varepsilon^j = \{x \in \Omega: \varepsilon^{-1}(x_1 - j, x_2) \in B\}$, $j \in \mathbb{Z}$, and $B_\varepsilon = \bigcup\limits_j B_\varepsilon^j$ and consider a strip with channels $Q_\varepsilon := Q\backslash\overline{B_\varepsilon}$. The vertical boundary of the channels

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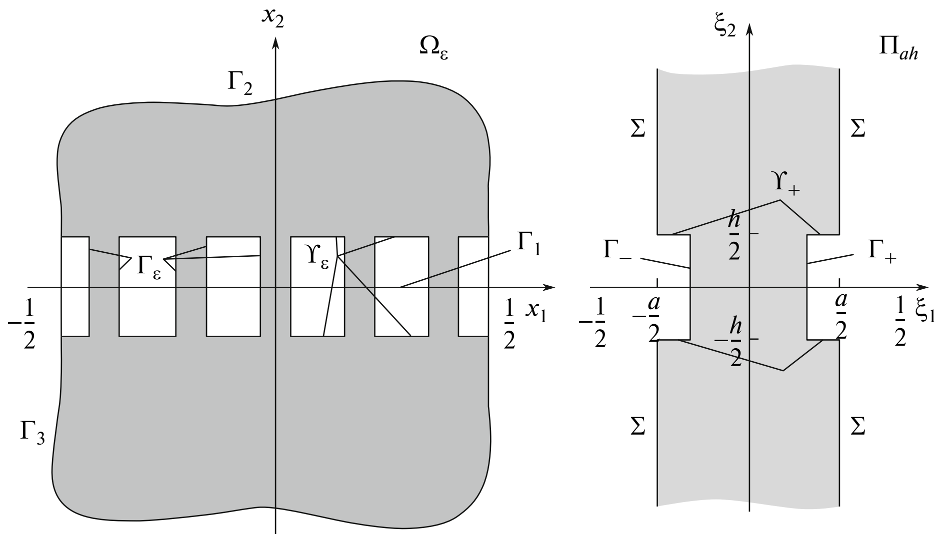

Structure of the domain Ω_ε and the periodicity cell Π_{ah}.

is denoted by $\Gamma_\varepsilon = \partial B_\varepsilon \cap Q$. The domain Ω_ε is defined as $\Omega\backslash\overline{Q}_\varepsilon$ (see figure). Let $\Gamma_3^\varepsilon = \left\{x \in \Gamma_3 : |x_2| > \frac{h\varepsilon}{2}\right\}$ and $\Upsilon_\varepsilon = \left\{x \in \partial Q_\varepsilon : |x_2| = \frac{h\varepsilon}{2}\right\}$.

The space $H^1(\Omega_\varepsilon, \Gamma_2)$ is defined as the closure of the set $C^\infty(\overline{\Omega}_\varepsilon)$ of functions vanishing near Γ_2 with respect to the $H^1(\Omega_\varepsilon)$ norm.

Consider the Steklov-type spectral problem

$$\begin{aligned} -\Delta u_\varepsilon &= 0 \text{ in } \Omega_\varepsilon, \\ u_\varepsilon &= 0 \text{ on } \Gamma_2, \\ \frac{\partial u_\varepsilon}{\partial \nu} &= 0 \text{ on } \Upsilon_\varepsilon \cup \Gamma_3^\varepsilon, \\ \frac{\partial u_\varepsilon}{\partial \nu} &= \lambda_\varepsilon u_\varepsilon \text{ on } \Gamma_\varepsilon. \end{aligned} \tag{1}$$

Here and below, ν denotes an outward normal vector.

According to [6], problem (1) has a discrete spectrum $\lambda_{\varepsilon,1}, \ldots, \lambda_{\varepsilon,j}, \ldots \to \infty$. Let $u_{\varepsilon,1}, \ldots, u_{\varepsilon,j}, \ldots$ denote the corresponding eigenfunctions orthonormalized in $L_2(\Gamma_\varepsilon)$.

It was shown in [5] that the homogenized problem for (1) has the form

$$\begin{aligned} -\Delta u_0 &= 0 \text{ in } \Omega, \\ u_0 &= 0 \text{ on } \Gamma_2, \\ \frac{\partial u_0}{\partial \nu} &= 0 \text{ on } \Gamma_3, \\ [u_0] &= 0 \text{ on } \Gamma_1, \\ \left[\frac{\partial u_0}{\partial x_2}\right] &= -2h\lambda_0 u_0 \text{ on } \Gamma_1, \end{aligned} \tag{2}$$

where $[\cdot]$ denotes the jump in a function across Γ_1. This problem has a discrete spectrum. The corresponding eigenvalues $\lambda_{0,k}$ numbered counting multiplicity tend to $+\infty$ as $k \to \infty$. Moreover, the following result was proved in [5].

Proposition 1. *Let the multiplicity of the eigenvalue* $\lambda_0 = \lambda_{0,j}$ *of the boundary value problem* (2) *be equal to n; i.e.,* $\lambda_{0,j} = \ldots = \lambda_{0,j+n-1}$. *Then the boundary value problem* (1) *has exactly n eigenvalues* $\lambda_\varepsilon^{(l)} = \lambda_{\varepsilon,j+l-1}$, $l = 1, 2, \ldots, n$, *converging to* λ_0 *as* $\varepsilon \to 0$.

Let $u_\varepsilon^{(l)}$ *be the* $L_2(\Omega_\varepsilon)$-*orthonormalized eigenfunctions of problem* (1) *corresponding to* $\lambda_\varepsilon^{(l)}$. *Then any sequence* $\varepsilon_q \xrightarrow[q\to\infty]{} 0$ *contains a subsequence such that on it*

$$\left\|u_{\varepsilon_{q_i}}^{(l)} - u_*^{(l)}\right\|_{H^1\left(\Omega_{\varepsilon_{q_i}}\right)} \xrightarrow[i\to\infty]{} 0, \tag{3}$$

where $u_*^{(l)}$ *are the* $L_2(\Omega)$-*orthonormalized eigenfunctions of problem* (2) *corresponding to* λ_0 (*which, generally speaking, depend on the choice of the sequence* $\varepsilon_q \xrightarrow[q\to\infty]{} 0$ *and its subsequence*).

Remark 1. In what follows, the index j will be omitted whenever possible; i.e., we will use the notation λ_0 and $\lambda_\varepsilon^{(l)}$, $l = 1, 2, \ldots, n$.

Below, the cases of a simple and a multiple eigenvalue λ_0 are considered. We construct two-term asymptotic expansions of eigenvalues of problem (1) and the leading terms of asymptotic expansions of the corresponding eigenfunctions.

3. FORMULATION OF THE MAIN RESULT

Let $\Pi = \left\{\xi \in \mathbb{R}^2\colon -\frac{1}{2} < \xi_1 < \frac{1}{2}\right\}$ and $\Pi_{ah} = \Pi \setminus \left\{\left\{\left[-\frac{1}{2}, -\frac{a}{2}\right] \cup \left[\frac{a}{2}, \frac{1}{2}\right]\right\} \times \left[-\frac{h}{2}, \frac{h}{2}\right]\right\}$. Define $\Upsilon = \Upsilon_+ \cup \Upsilon_-$, $\Upsilon_\pm = \left\{\xi \in R^2\colon \xi_1 \in \left[-1, -\frac{a}{2}\right] \cup \left[\frac{a}{2}, 1\right], \xi_2 = \pm\frac{h}{2}\right\}$, and $\Gamma_\pm = \left\{\xi \in R^2\colon \xi_1 = \pm\frac{a}{2}, \xi_2 \in \left[-\frac{h}{2}, \frac{h}{2}\right]\right\}$. The rest of the boundary of the strip is denoted by Σ, i.e., $\Sigma = \partial\Pi_{ah} \setminus (\Upsilon \cup \Gamma_+ \cup \Gamma_-)$ (see figure).

Lemma 1. *The problems*

$$\Delta_\xi X_0 = 0 \quad in \quad \Pi_{ah},$$
$$\frac{\partial X_0}{\partial \nu_\xi} = 0 \quad on \quad \Sigma \cup \Upsilon,$$
$$\frac{\partial X_0}{\partial \nu_\xi} = 1 \quad on \quad \Gamma_\pm;$$

$$\Delta_\xi Y_0 = 0 \quad in \quad \Pi_{ah},$$
$$\frac{\partial Y_0}{\partial \nu_\xi} = 0 \quad on \quad \Upsilon,$$
$$Y_0 = 0 \quad on \quad \Sigma,$$
$$\frac{\partial Y_0}{\partial \nu_\xi} = \pm 1 \quad on \quad \Gamma_\pm,$$

have solutions with asymptotics

$$X_0(\xi) = -h|\xi_2| + \mathcal{O}(e^{-\pi|\xi_2|}),$$
$$Y_0(\xi) = \mathcal{O}(e^{-\pi|\xi_2|}) \quad as \quad \xi_2 \to \pm\infty.$$

Let

$$\mathcal{A}_{ah} = \frac{4}{h(1-a)} \int_{-\frac{h}{2}}^{\frac{h}{2}} Y_0\left(\frac{a}{2}, \xi_2\right) d\xi_2,$$
$$\mathcal{B}_{ah} = \frac{2}{h(1-a)} \int_{-\frac{h}{2}}^{\frac{h}{2}} X_0\left(\frac{a}{2}, \xi_2\right) d\xi_2. \tag{4}$$

Define

$$\langle u \rangle(x_1) := \frac{u(x_1 + 0) + u(x_1 - 0)}{2}.$$

Let $u_0^{(l)}$ be the eigenfunctions of problem (2) that correspond to the eigenvalue λ_0 of multiplicity n and satisfy the conditions

$$\|u_0^{(l)}\|_{L_2(\Gamma_1)} = 1, \quad (u_0^{(l)}, u_0^{(k)})_{L_2(\Gamma_1)} = 0,$$
$$\left(\left\langle \frac{\partial u_0^{(l)}}{\partial x_2} \right\rangle, \left\langle \frac{\partial u_0^{(k)}}{\partial x_2} \right\rangle\right)_{L_2(\Gamma_1)} \tag{5}$$
$$+ (1 - 3\mathcal{A}_{ah})\left(\frac{\partial u_0^{(l)}}{\partial x_1}, \frac{\partial u_0^{(k)}}{\partial x_1}\right)_{L_2(\Gamma_1)} = 0 \quad \text{for} \quad l \neq k.$$

Define

$$\lambda_1^{(l)} = \frac{1-a}{2}\left(\mathcal{B}_{ah}\lambda_0^2 + \left\|\left\langle \frac{\partial u_0^{(l)}}{\partial x_2} \right\rangle\right\|^2_{L_2(\Gamma_1)} + (1 - 3\mathcal{A}_{ah})\left\|\frac{\partial u_0^{(l)}}{\partial x_1}\right\|^2_{L_2(\Gamma_1)}\right), \tag{6}$$

where $\mathcal{A}_{ah}$ and $\mathcal{B}_{ah}$ are the constants given by (4). If $\lambda_1^{(l)} = \lambda_1^{(l+1)} = \ldots = \lambda_1^{(l+n_l-1)}$, while the other $\lambda_1^{(k)}$ are other than $\lambda_1^{(l)}$, then the eigenvalue $\lambda_1^{(l)}$ is said to be of multiplicity n_l. The linear subspace spanned by the corresponding eigenfunctions $u_0^{(l)}$, $u_0^{(l+1)}$, …, $u_0^{(l+n_l-1)}$ is called the eigenspace of $\lambda_1^{(l)}$.

By Proposition 1, problem (1) has n eigenvalues $\lambda_\varepsilon^{(l)}$ (counting multiplicity) that converge to λ_0. The corresponding $L^2(\Gamma_\varepsilon)$-orthonormalized eigenfunctions are denoted by $u_\varepsilon^{(l)}$. The functions $u_\varepsilon^{(j)}$ are extended to Ω so that

$$\|u_\varepsilon^{(j)}\|_{H_1(\Omega)} \le C\|u_\varepsilon^{(j)}\|_{H_1(\Omega_\varepsilon)}$$

with a constant C independent of ε. This extension is possible according to [7] and by the above assumptions on the structure of Ω_ε. The same notation is retained for the extended functions.

Theorem 1. *Let λ_0 be an eigenvalue of problem* (2) *of multiplicity n, and let $u_0^{(l)}$ be the corresponding eigenfunctions normalized by conditions* (5). *Then the asymptotics of the eigenvalues $\lambda_\varepsilon^{(l)}$ of problem* (1) *converging to λ_0 as $\varepsilon \to 0$ have the form*

$$\lambda_\varepsilon^{(l)} = \lambda_0 + \varepsilon\lambda_1^{(l)} + o\left(\varepsilon^{\frac{3}{2}-\mu}\right), \tag{7}$$

where $\lambda_1^{(l)}$ is given by (6) *and μ is an arbitrarily small positive number.*

If the multiplicity of $\lambda_1^{(l)}$ is n_l, then the total multiplicity of the eigenvalues $\lambda_\varepsilon^{(l)}$ of problem (1) *that have* *asymptotics* (7) *is also* n_l *and the subspace spanned by the corresponding eigenfunctions* $u_\varepsilon^{(l)}$ *converges to the eigenspace of* $\lambda_1^{(l)}$ *in* $L_2(\Gamma_1)$.

4. CONSTRUCTION OF ASYMPTOTICS

The asymptotics are constructed by applying the method of matched asymptotic expansions (see [8, 9] and also [10–13] for a multiple eigenvalue).

Before proceeding to the construction procedure, we note that the convergence in (3) is equivalent to the convergence

$$\left\| u_\varepsilon^{(l)} - u_*^{(l)} \right\|_{L_2(\Gamma_1)} \to 0,$$

where $u_*^{(l)}$ is an element of the eigenspace in $L_2(\Gamma_1)$ generated by $u_0^{(1)}, \ldots, u_0^{(n)}$.

Let $\mathbf{u}_0$ denote the vector function with components $u_0^{(1)}, \ldots, u_0^{(n)}$.

By the properties of solutions to problem (2), for the function $\mathbf{u}_0(x)$, we have

$$\mathbf{u}_0(x) = \boldsymbol{\alpha}_0(x_1) + \boldsymbol{\alpha}_{1,\pm}(x_1)x_2 - \boldsymbol{\alpha}_0''(x_1)\frac{x_2^2}{2} + \mathcal{O}(x_2^3) \quad \text{as } x_2 \to \pm 0, \tag{8}$$

where

$$\boldsymbol{\alpha}_0(x_1) := \mathbf{u}_0(x_1, 0), \quad \boldsymbol{\alpha}_{1,\pm}(x_1) := \frac{\partial \mathbf{u}_0}{\partial x_2}(x_1, \pm 0). \tag{9}$$

Making the substitution $x_2 = \varepsilon\xi_2$ in (8) yields

$$\mathbf{u}_0(x) = \boldsymbol{\alpha}_0(x_1) + \varepsilon\boldsymbol{\alpha}_{1,\pm}(x_1)\xi_2 - \varepsilon^2\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} + \mathcal{O}(\varepsilon^3\xi_2^3)$$
$$\text{as } x_2 = \varepsilon\xi_2 \to \pm 0.$$

Following the method of matched asymptotic expansions, we conclude that the leading terms of the inner expansion near Γ_1 have the form

$$\hat{\mathbf{v}}_\varepsilon(x) = \mathbf{v}_0(\xi; x_1) + \varepsilon\mathbf{v}_1(\xi; x_1) + \varepsilon^2\mathbf{v}_2(\xi; x_1), \tag{10}$$

where $\xi = \dfrac{x}{\varepsilon}$ and

$$\mathbf{v}_0(\xi; x_1) \sim \boldsymbol{\alpha}_0(x_1) \quad \text{as } \xi_2 \to \pm\infty, \tag{11}$$

$$\mathbf{v}_1(\xi; x_1) \sim \boldsymbol{\alpha}_{1,\pm}(x_1)\xi_2 \quad \text{as } \xi_2 \to \pm\infty, \tag{12}$$

$$\mathbf{v}_2(\xi; x_1) \sim -\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} \quad \text{as } \xi_2 \to \pm\infty. \tag{13}$$

Here, x_1 is a slow variable and $\xi = (\xi_1, \xi_2)$ is a fast variable.

Written in terms of $(\xi; x_1)$, the Laplacian and the boundary operator become

$$\Delta = -\varepsilon^{-2}\Delta_\xi - 2\varepsilon^{-1}\frac{\partial^2}{\partial x_1 \partial \xi_1} - \frac{\partial^2}{\partial x_1^2},$$
$$\frac{\partial}{\partial \nu} = (\nu, \varepsilon^{-1}\nabla_\xi) + (\nu, \nabla_x). \tag{14}$$

Remark 2. Since the boundary Ω_ε is ε-periodic, the functions $\mathbf{v}_j(\xi; x_1)$ near the interface Γ_1 are sought in the form of 1-periodic functions of ξ_1.

Define

$$\hat{\Lambda}_\varepsilon = \lambda_0 E + \varepsilon\Lambda_1, \tag{15}$$

where Λ_1 is a diagonal $n \times n$ matrix with as yet arbitrary elements $\lambda_1^{(1)}, \ldots, \lambda_1^{(n)}$ and E is the identity matrix. The diagonal elements of the matrix $\hat{\Lambda}_\varepsilon$ are denoted by $\hat{\lambda}_\varepsilon^{(l)}$, $l = 1, 2, \ldots, n_1$. Then, in view of Remark 2, substituting (10), (15), and (14) into (1) and equating the coefficients of like powers of ε in the resulting equations and boundary conditions, we obtain the following boundary value problems for $\mathbf{v}_j$:

$$\Delta_\xi \mathbf{v}_0 = 0 \quad \text{in } \Pi_{ah},$$
$$\frac{\partial \mathbf{v}_0}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon \cup \Gamma_\pm; \tag{16}$$

$$\Delta_\xi \mathbf{v}_1 = -2\frac{\partial^2 \mathbf{v}_0}{\partial \xi_1 \partial x_1} \quad \text{in } \Pi_{ah},$$
$$\frac{\partial \mathbf{v}_1}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon, \tag{17}$$
$$\frac{\partial \mathbf{v}_1}{\partial \nu_\xi} = -\frac{\partial \mathbf{v}_0}{\partial \nu_x} + \lambda_0 \mathbf{v}_0 \quad \text{on } \Gamma_\pm;$$

and

$$\Delta_\xi \mathbf{u}_2 = -2\frac{\partial^2 \mathbf{v}_1}{\partial \xi_1 \partial x_1} - \frac{\partial^2 \mathbf{v}_0}{\partial x_1^2} \quad \text{in } \Pi_{ah},$$
$$\frac{\partial \mathbf{v}_2}{\partial \nu_\xi} = 0 \quad \text{on } \Upsilon, \tag{18}$$
$$\frac{\partial \mathbf{v}_2}{\partial \nu_\xi} = -\frac{\partial \mathbf{v}_1}{\partial \nu_x} + \lambda_0 \mathbf{v}_1 + \Lambda_1 \mathbf{v}_0 \quad \text{on } \Gamma_\pm.$$

Obviously, the function

$$\mathbf{v}_0(\xi; x_1) \equiv \boldsymbol{\alpha}_0(x_1)$$

is a solution of problem (16) and has the required behavior (11) as $\xi_2 \to \pm\infty$. In view of this identity, problem (17) can be rewritten as

$$\Delta_\xi \mathbf{v}_1 = 0 \text{ in } \Pi_{ah},$$

$$\frac{\partial \mathbf{v}_1}{\partial \nu_\xi} = 0 \text{ on } \Upsilon,$$

$$\frac{\partial \mathbf{v}_1}{\partial \nu_\xi} = \mp \boldsymbol{\alpha}_0'(x_1) + \lambda_0 \boldsymbol{\alpha}_0(x_1) \text{ on } \Gamma_\pm.$$

It turns out that this problem has a solution with asymptotics

$$\mathbf{v}_1(\xi; x_1) = \boldsymbol{\alpha}_{1,\pm}(x_1)\xi_2 \pm \mathcal{C}_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2}\right\rangle(x_1) + \boldsymbol{\beta}(x_1) + \mathcal{O}(e^{-\pi|\xi_2|}) \text{ as } \xi_2 \to \pm\infty, \tag{19}$$

where $\mathcal{C}_{ah}$ is a constant. This asymptotics is more accurate than (12).

Rewriting the asymptotics of sum (10) as $\xi_2 \to \pm\infty$ in terms of x_2 and taking into account (19), we conclude that the leading terms of the outer expansion of eigenfunctions have to be sought in the form

$$\hat{\mathbf{u}}_\varepsilon(x) = \mathbf{u}_0(x) + \varepsilon \mathbf{u}_1(x), \tag{20}$$

where

$$\mathbf{u}_1(x) \sim \pm \mathcal{C}_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2}\right\rangle(x_1) + \boldsymbol{\beta}(x_1) \text{ as } x_2 \to \pm 0.$$

Obviously, these conditions are equivalent to

$$[\mathbf{u}_1](x_1) = 2\mathcal{C}_{ah}\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2}\right\rangle(x_1), \tag{21}$$

$$\boldsymbol{\beta}(x_1) = \langle \mathbf{u}_1 \rangle(x_1). \tag{22}$$

Rewriting the asymptotics of function (20) as $x_2 \to 0$ in terms of the internal variables ξ, we improve asymptotics (13) of $\mathbf{v}_2(\xi; x_1)$ at infinity:

$$\mathbf{v}_2(\xi; x_1) \sim -\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} + \frac{\partial \mathbf{u}_1}{\partial x_2}(x_1, \pm 0)\xi_2 \quad \text{as } \xi_2 \to \pm\infty. \tag{23}$$

For any function $\boldsymbol{\varphi}(x_1)$, problem (18) has a solution with asymptotics

$$\mathbf{v}_2(\xi; x_1) = -\boldsymbol{\alpha}_0''(x_1)\frac{\xi_2^2}{2} + \Big(\boldsymbol{\alpha}_0''(x_1)(3\mathcal{A}_{ah} - 1)\frac{h}{2}(1-a) + \lambda_0^2 \boldsymbol{\alpha}_0(x_1)\mathcal{B}_{ah}\frac{h}{2}(1-a) - h\Lambda_1\boldsymbol{\alpha}_0(x_1) - h\lambda_0\boldsymbol{\beta}(x_1)\Big)|\xi_2| + \boldsymbol{\varphi}(x_1)\xi_2 \pm \mathcal{C}_{ah}\boldsymbol{\varphi}(x_1) \pm \lambda_0\left\langle \frac{\partial \mathbf{u}_0}{\partial x_2}\right\rangle(x_1)\mathcal{C}_{ah} + \mathcal{O}(e^{-\pi|\xi_2|}) \quad \text{as } |\xi_2| \to \infty.$$

Comparing this equality with (23) and taking into account (4), we obtain

$$\left[\frac{\partial \mathbf{u}_1}{\partial x_2}\right](x_1) = h(1-a)(\boldsymbol{\alpha}_0''(x_1)(3\mathcal{A}_{ah} - 1) + \lambda_0^2\boldsymbol{\alpha}_0(x_1)\mathcal{B}_{ah}) - 2h(\Lambda_1\boldsymbol{\alpha}_0(x_1) + \lambda_0\boldsymbol{\beta}(x_1)),$$

$$\left\langle \frac{\partial \mathbf{u}_1}{\partial x_2}\right\rangle(x_2) = \boldsymbol{\varphi}(x_1).$$

In turn, combining these relations with (9) and (22) yields

$$\left[\frac{\partial \mathbf{u}_1}{\partial x_2}\right](x_1) = -2h\lambda_0\langle \mathbf{u}_1\rangle - 2h\Lambda_1\mathbf{u}_0(x_1, 0) + h(1-a)\left(\frac{\partial^2 \mathbf{u}_0}{\partial x_1^2}(x_1, 0)(3\mathcal{A}_{ah} - 1) + \lambda_0^2\mathbf{u}_0(x_1, 0)\mathcal{B}_{ah}\right). \tag{24}$$

Substituting (20) into (1), we obtain the following equation and boundary conditions for $\mathbf{u}_1$:

$$-\Delta \mathbf{u}_1 = 0 \text{ in } \Omega\backslash\Gamma_1,$$

$$\mathbf{u}_1 = 0 \text{ on } \Gamma_2,$$

$$\frac{\partial \mathbf{u}_1}{\partial \nu} = 0 \text{ on } \Gamma_3.$$

The solvability condition for problem (2) and relations (21) and (24) imply that the elements of the diagonal matrix Λ_1 are defined by (6). This completes the formal derivation of formulas (6) and (7).

The constructed asymptotics of the eigenvalues and the convergence of the corresponding eigenfunctions to those of the homogenized problem with normalization conditions (5) can be rigorously proved relying on standard arguments (see, e.g., [13]).

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