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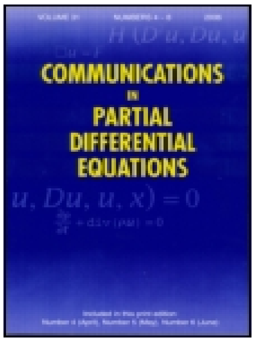


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# On the first eigenpair of singularly perturbed operators with oscillating coefficients

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**ABSTRACT**
The paper deals with a Dirichlet spectral problem for a singularly perturbed second order elliptic operator with rapidly oscillating locally periodic coefficients. We study the limit behavior of the first eigenpair (ground state) of this problem. The main tool in deriving the limit (effective) problem is the viscosity solutions technique for Hamilton-Jacobi equations. The effective problem need not have a unique solution. We study the non-uniqueness issue in a particular case of zero potential and construct the higher order term of the ground state asymptotics.

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## 1. Introduction

Given a singularly perturbed elliptic operator of the form

$$\mathcal{L}_\varepsilon u = \varepsilon^2 a^{ij}(x, x/\varepsilon^\alpha)\frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x, x/\varepsilon^\alpha)\frac{\partial u}{\partial x_j} + c(x, x/\varepsilon^\alpha)u \tag{1.1}$$

with a small parameter $\varepsilon > 0$, we consider a Dirichlet spectral problem

$$\mathcal{L}_\varepsilon u = \lambda u, \quad u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$

stated in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. We assume that the coefficients $a^{ij}(x,y)$, $b^j(x,y)$ and $c(x,y)$ are sufficiently regular functions periodic in $y$ variable, and that $a^{ij}(x,y)$ satisfy the uniform ellipticity condition. Finally, $\alpha > 0$ is a fixed positive parameter. Let us remark that in the underlying convection-diffusion model $\varepsilon$ represents characteristic ratio between the diffusion and convection coefficients, while $\varepsilon^\alpha$ refers to the microstructure period.

As well known, the operator $\mathcal{L}_\varepsilon$ has a discrete spectrum, and the first eigenvalue $\lambda_\varepsilon$ (the eigenvalue with the maximal real part) is real and simple; the corresponding eigenfunction $u_\varepsilon$ can be chosen to satisfy $u_\varepsilon > 0$ in $\Omega$. The goal of this work is to study the asymptotic behavior of $\lambda_\varepsilon$ and $u_\varepsilon$ as $\varepsilon \to 0$.

The first eigenpair (ground state) of (1.1) plays a crucial role when studying the large time behavior of solutions to the corresponding parabolic initial boundary problem. The first eigenvalue characterizes an exponential growth or decay of a typical solution, as $t \to \infty$, while the corresponding eigenfunction describes the limit profile of a normalized solution.

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Also, since in a typical case the first eigenfunction shows a singular behavior, as $\varepsilon \to 0$, in many applications it is important to know the set of concentration points of $u_\varepsilon$, the so-called hot spots. This concentration set might consist of one point, or finite number of points, or a surface of positive codimension, or it might have more complicated structure. An interesting discussion on hot spots can be found in [43].

Boundary value problems for singularly perturbed elliptic operators have been widely studied in the existing literature. An important contribution to this topic has been done in the classical work [47] that deals with singular perturbed operators with smooth non-oscillating coefficients under the assumption that for $\varepsilon = 0$ the problem remains (in a certain sense) well-posed.

The Dirichlet problem for a convection-diffusion operator with a small diffusion and with a convection directed outward at the domain boundary was studied for the first time in [22]. The approach developed in that work relies on large deviation results for trajectories of a diffusion process being a solution of the corresponding stochastic differential equation.

The probabilistic interpretation of solutions and the aforementioned large deviation principle have also been used in [18, 26, 27], where the first eigenvalue is studied for a second order elliptic operator being a singular perturbation of a first order operator.

There are two natural approaches that can be used for studying the logarithmic asymptotics of the principal eigenfunction of a second order singularly perturbed operator. One of them relies on the above mentioned large deviation results for diffusion processes with a small diffusion coefficients. This method was used in [39] for studying operators with smooth coefficients on a compact Riemannian manifolds.

We follow yet another (deterministic) approach based on the viscosity solution techniques for nonlinear PDEs. In the context of linear singularly perturbed equations, these techniques were originally developed in [21] and followed by [6, 10, 11, 24, 37] and other works (see also a review in [5]). Since $u_\varepsilon > 0$ in $\Omega$, we can represent $u_\varepsilon$ as $u_\varepsilon(x) = e^{-W_\varepsilon(x)/\varepsilon}$ to find that $W_\varepsilon$ satisfies

$$- \varepsilon a^{ij}(x, x/\varepsilon^\alpha)\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} + H(\nabla W_\varepsilon, x, x/\varepsilon^\alpha) = \lambda_\varepsilon \tag{1.3}$$

with $H(p, x, y) = a^{ij}(x, y)p_i p_j - b^j(x, y)p_j + c(x, y)$, and the Dirichlet boundary condition for $u_\varepsilon$ yields $W_\varepsilon = +\infty$ on $\partial\Omega$. Using perturbed test functions we pass to the limit in (1.3) and get the limit Hamilton-Jacobi equation of the form

$$\overline{H}(\nabla W(x), x) = \lambda \quad \text{in } \Omega. \tag{1.4}$$

with an effective Hamiltonian $\overline{H}(p, x)$ whose definition depends on whether $\alpha > 1$, $\alpha = 1$ or $0 < \alpha < 1$. We show that in the limit $\varepsilon \to 0$ the boundary condition $W_\varepsilon = +\infty$ on $\partial\Omega$ in conjunction with (1.3) yield

$$\overline{H}(\nabla W(x), x) \geq \lambda \quad \text{on } \partial\Omega. \tag{1.5}$$

The latter condition is known [14, 45] as the state constraint boundary condition. Both equation (1.4) and boundary condition (1.5) are understood in viscosity sense.[1]

[1] When referring to the boundary condition (1.5) hereafter we always assume continuous in $\overline{\Omega}$ functions satisfying (1.5) in viscosity sense which (in general) is stronger than simply pointwise inequality in (1.5). The latter fact is sometimes a source of confusions.

We recall that a continuous in $\Omega$ function $W$ is a viscosity solution of equation (1.4) if for any $x \in \Omega$ and any $C^\infty$ function $\varphi$ such that $W - \varphi$ has a maximum (minimum) at $x$ one has

$$\overline{H}(\nabla\varphi(x), x) \le \lambda \quad (\overline{H}(\nabla\varphi(x), x) \ge \lambda).$$

A function $W \in C(\overline{\Omega})$ satisfies boundary condition (1.5) if for any $x \in \partial\Omega$ and any $C^\infty$ function $\varphi$ such that the minimum of $W - \varphi$ in $\overline{\Omega}$ is attained at $x$, it holds

$$\overline{H}(\nabla\varphi(x), x) \ge \lambda.$$

Equations of type (1.3) have been extensively studied in the existing literature. One can find a short review of state of the art in [28, 33] and in more recent works [2, 8], see also references therein.

Earlier, singularly perturbed KPP-type reaction-diffusion equations were studied in [35] where, in particular, equations with rapidly oscillating coefficients were considered. It was shown that the classical Huygens principle might fail to work in this case.

In the case when the equation coefficients do not depend on "slow" variable, the homogenization of singularly perturbed spectral problems have been studied in a number of works. In [12] spectral problems for operators with periodic coefficients were considered, the study relied on factorization principle. Similar periodic homogenization results for weakly coupled systems were obtained in [1, 13]. The case of a weakly coupled elliptic system with statistically homogeneous rapidly oscillating coefficients has been considered in the recent work [3].

In the present work, deriving the effective problem (1.4)–(1.5) relies on the idea of perturbed test functions originally proposed in [19]. We strongly believe that with the help of the techniques developed recently in [2, 29, 33, 34] this result can be extended to a more general almost periodic setting as well as random stationary ergodic setting. In other words, the periodicity assumption can be replaced with the assumption that the coefficients in (1.1) are almost periodic or random statistically homogeneous and ergodic with respect to the fast variable, at least in the case $\alpha = 1$. The case $\alpha \neq 1$ looks more difficult and might require some extra assumptions. We refer to [2, 8, 9, 29, 32, 44, 46] for (far not complete list of) various results on almost periodic and random homogenization of nonlinear PDEs. However, the essential novelty of this work comes in the (logically) second part of the paper devoted to the improved ground state asymptotics and resolving the non-uniqueness issue for (1.4)–(1.5). The generalization of this part to non-periodic settings is an open problem.

Problem (1.4)–(1.5) is known as ergodic or additive eigenvalue problem. Its solvability was first proved in [32] in periodic setting, more recent results are contained, e.g., in [25] as well as in [16], where stationary ergodic Hamiltonians were considered. There exists the unique additive eigenvalue $\lambda$ of (1.4)–(1.5) while the eigenfunction $W$ need not be unique even up to an additive constant. This non-uniqueness issue is intimately related to the structure of the so-called Aubry set of effective Hamiltonian which play the role of a hidden boundary for (1.4)–(1.5). The non-uniqueness in (1.4)–(1.5) appears when the Aubry set is not connected. By contrast, for every $\varepsilon > 0$ the eigenfunction $u_\varepsilon$ is unique up to a normalization, and it is natural to try to select the solution of (1.4)–(1.5) that coincides with the limit of $W_\varepsilon = -\varepsilon \log u_\varepsilon$. This challenging problem is addressed in a particular case of (1.1) with $c(x, y) = 0$, $\alpha \ge 1$. Following [39] we introduce the effective drift (convection) and assume that it has a finite number of hyperbolic fixed points in $\Omega$, and that the Aubry set of the effective Hamiltonian coincides with this finite collection of points. Notice that any fixed point of the effective drift in $\overline{\Omega}$ belongs to the Aubry set of the effective Hamiltonian. It follows from our results that in this case $\lambda_\varepsilon$ tends to zero as $\varepsilon \to 0$. We show that $\lambda_\varepsilon/\varepsilon$ has a finite limit that can be determined in terms of eigenvalues of Ornstein-Uhlenbeck operators in $\mathbb{R}^N$ obtained via local analysis of (1.1) at the scale $\sqrt{\varepsilon}$ in the vicinity of aforementioned fixed points. This, in turn, enables fine selection of the additive eigenfunction corresponding to $\lim_{\varepsilon\to 0} W_\varepsilon$.

## 2. Main results

We begin with standing hypotheses which are assumed to hold throughout this paper. We assume that $\Omega$ is connected and has $C^2$ boundary $\partial\Omega$; the coefficients $a^{ij}(x,y)$, $b^j(x,y)$, $c(x,y) \in C^1(\overline{\Omega} \times \mathbb{R}^N)$ are $Y$-periodic in $y$ functions, where $Y = (0,1)^N$. The matrix $(a^{ij})_{i,j=\overline{1,N}}$ is uniformly positive definite:

$$a^{ij}(x,y)\zeta_i\zeta_j \geq m|\zeta|^2 > 0 \quad \forall\zeta \neq 0, \tag{2.1}$$

and, without loss of generality, we can assume the symmetry $a^{ij} = a^{ji}$.

The first eigenfunction $u_\varepsilon$ of the operator (1.1) can be normalized to satisfy

$$1 = \max_{\Omega} u_\varepsilon \quad (u_\varepsilon > 0 \text{ in } \Omega), \tag{2.2}$$

then its scaled logarithmic transformation

$$W_\varepsilon := -\varepsilon \log u_\varepsilon$$

is a nonnegative function vanishing at the points of maxima of $u_\varepsilon$.

The asymptotic behavior of $\lambda_\varepsilon$ and $W_\varepsilon$ is described in

**Theorem 1.** *The eigenvalues $\lambda_\varepsilon$ converge as $\varepsilon \to 0$ to the limit $\lambda$, which is the unique real number for which problem (1.4), (1.5) has a continuous viscosity solution. The functions $W_\varepsilon$ converge (up to extracting a subsequence) to a limit $W$ uniformly on compacts in $\Omega$ (see Remark 2 below), and every limit function $W$, extended by continuity onto $\overline{\Omega}$, is a viscosity solution of (1.4), (1.5).*

*The effective Hamiltonian $\overline{H}(p,x)$ in (1.4) is given by the following formulas, depending on the parameter $\alpha$.*

(i) *If $\alpha > 1$ then*

$$\overline{H}(p,x) = \int_Y H(p,x,y)\vartheta(y)\,dy \tag{2.3}$$

*where*

$$H(p,x,y) = a^{ij}(x,y)p_ip_j - b^j(x,y)p_j + c(x,y),$$

*and $\vartheta(y)$ is the unique $Y-$periodic solution of the equation $\frac{\partial^2}{\partial y_i \partial y_j}(a^{ij}(x,y)\vartheta) = 0$ normalized by $\int_Y \vartheta(y)\,dy = 1$.*

(ii) *If $\alpha = 1$ then $\overline{H}(p,x)$ is the first eigenvalue (eigenvalue with the maximal real part) of the problem*

$$a^{ij}(x,y)\frac{\partial^2\vartheta}{\partial y_i \partial y_j} + (b^j(x,y) - 2a^{ij}(x,y)p_i)\frac{\partial\vartheta}{\partial y_j} + H(p,x,y)\vartheta = \overline{H}(p,x)\vartheta, \quad \vartheta(y) \quad \text{is } Y\text{-periodic}. \tag{2.4}$$

*According to the Krein-Rutman theorem $\overline{H}(p,x)$ is real.*

(iii) *If* $0 < \alpha < 1$ *then* $\overline{H}(p,x)$ *is the unique number such that the problem*

$$H(p + \nabla\vartheta(y), x, y) = \overline{H}(p,x) \tag{2.5}$$

*has a* $Y-$*periodic viscosity solution* $\vartheta(y)$*; here* $p \in \mathbb{R}^N$ *and* $x \in \overline{\Omega}$ *are parameters.*

**Remark 2.** Although, for each $\varepsilon > 0$, $W_\varepsilon(x)$ tends to $+\infty$ as $x$ approaches the boundary $\partial\Omega$, it can be shown (see Lemma 10 in Section 4) that for every $\beta > 0$ there exists a constant $C_\beta$, independent of $\varepsilon$, such that

$$|W_\varepsilon(x) - W_\varepsilon(z)| \leq C_\beta |x - z|$$

if $x, z \in \Omega$ and $\min\{\mathrm{dist}(x, \partial\Omega), \mathrm{dist}(z, \partial\Omega)\} \geq \beta\varepsilon$. This implies that, for a subsequence, $W_\varepsilon$ converges in $\Omega$ to a Lipschitz continuous function. The latter can be extended to $\overline{\Omega}$ by continuity.



We note that the effective Hamiltonian $\overline{H}(p,x)$ is continuous on $\mathbb{R}^N \times \overline{\Omega}$, convex in $p$ and coercive, moreover $\overline{H}(p,x) \geq m_1|p|^2 - C$, $m_1 > 0$. The viscosity solutions theory for such Hamiltonians is well established. Following [14, 25] and [36], we provide here various representation formulas for the solutions of problem (1.4)–(1.5).

Let us rewrite problem (1.4)–(1.5) in the form

$$\overline{H}(\nabla W(x), x) \leq \lambda \quad \text{in } \Omega \tag{2.6}$$

$$\overline{H}(\nabla W(x), x) \geq \lambda \quad \text{in } \overline{\Omega}, \tag{2.7}$$

i.e. (2.6) requires that $W$ is a viscosity subsolution in $\Omega$ while (2.7) means that $W$ is a viscosity supersolution in $\overline{\Omega}$. It is known that there exists a unique number $\lambda = \lambda_{\overline{H}}$ (additive eigenvalue), for which (1.4)–(1.5) has a solution $W$. For the reader convenience we formulate here the first item of Theorem VIII.1 in [14].

**Theorem 3** (see [14]). *Let* $\overline{H}(p,x)$ *be continuous in* $\mathbb{R}^N \times \overline{\Omega}$*, and suppose that* $\overline{H}(p,x) \to \infty$*, as* $|p| \to \infty$*, uniformly in* $x \in \overline{\Omega}$*. Then there is a unique* $\lambda = \lambda_{\overline{H}}$ *such that problem (2.6)–(2.7) has a solution.*

According to [36, Sect. 3] the number $\lambda_{\overline{H}}$ is given by

$$\lambda_{\overline{H}} = \inf\{\lambda;\ (2.6) \text{ has a solution } W \in C(\overline{\Omega})\}. \tag{2.8}$$

It can also be expressed in terms of action minimization (see [14, Theorem X.1. item (3)]),

$$\lambda_{\overline{H}} = -\lim_{t\to\infty} \frac{1}{t} \inf \int_0^t \overline{L}(\dot{\eta}, \eta)\, d\tau,$$

where the infimum is taken over absolutely continuous curves $\eta : [0,t] \to \overline{\Omega}$, and $\overline{L}(v,x)$ is the Legendre transform of $\overline{H}(p,x)$,

$$\overline{L}(v,x) = \max\{v \cdot p - \overline{H}(p,x)\}.$$

Let us define now the distance function

$$d_{\overline{H}-\lambda_{\overline{H}}}(x,y) = \sup\{W(x) - W(y);\ W \in C(\overline{\Omega}) \text{ is a solution of (2.6) for } \lambda = \lambda_{\overline{H}}\}. \tag{2.9}$$

It is known (see, e.g., [25, Theorem 1.4]) that $d_{\overline{H}-\lambda_{\overline{H}}}(x,x) = 0$, $d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ is Lipschitz continuous, $d_{\overline{H}-\lambda_{\overline{H}}}(x,y) \leq d_{\overline{H}-\lambda_{\overline{H}}}(x,z) + d_{\overline{H}-\lambda_{\overline{H}}}(z,y)$. Besides, for every $y \in \overline{\Omega}$ the function $d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ is a solution of (2.6) for $\lambda = \lambda_{\overline{H}}$ and, according to [25, Lemma 6.3], $\overline{H}(\nabla_x d_{\overline{H}-\lambda_{\overline{H}}}(x,y),x) \geq \lambda_{\overline{H}}$ in $\overline{\Omega}\setminus\{y\}$. The number $\lambda_{\overline{H}}$ is such that the *Aubry* set $\mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$,

$$\mathcal{A}_{\overline{H}-\lambda_{\overline{H}}} = \{y \in \overline{\Omega};\ d_{\overline{H}-\lambda_{\overline{H}}}(x,y) \text{ is a solution of (2.7) for } \lambda = \lambda_{\overline{H}}\}, \tag{2.10}$$

is nonempty, see [36, Proposition 6.4]. Note also that, by [25, Proposition 1.6], the distance function $d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ admits the representation

$$d_{\overline{H}-\lambda_{\overline{H}}}(x,y) = \inf\left\{\int_0^t (\overline{L}(\dot\eta,\eta) + \lambda_{\overline{H}})\mathrm{d}\tau\ ;\ \eta(0) = y, \eta(t) = x, t > 0\right\}, \tag{2.11}$$

and the Aubry set can be characterized by

$$y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}} \Longleftrightarrow \sup_{\delta>0}\inf\left\{\int_0^t (\overline{L}(\dot\eta,\eta) + \lambda_{\overline{H}})\,\mathrm{d}\tau\ ;\ \eta(0) = \eta(t) = y, t > \delta\right\} = 0. \tag{2.12}$$

The infimum in (2.11) and (2.12) is taken over absolutely continuous curves $\eta : [0,t] \to \overline{\Omega}$. Since we did not succeed to find the proof of (2.12) in the existing literature, we prove it in Appendix A.

According to the definition of $d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$, every solution $W$ of (1.4)–(1.5) satisfies $W(x) - W(y) \leq d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$; this inequality holds, in particular, for all $x, y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$. Conversely, given a function $g(x)$ on $\mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$ which satisfies the compatibility condition $g(x) - g(y) \leq d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ $\forall x, y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$, by [25, Proposition 7.1 and Theorem 7.2], the function

$$W(x) = \min\{d_{\overline{H}-\lambda_{\overline{H}}}(x,y) + g(y);\ y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}\} \tag{2.13}$$

is the unique solution of (1.4)–(1.5) for $\lambda = \lambda_{\overline{H}}$ that satisfies $W(x) = g(x)$ on $\mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$. In Appendix A we show the following simple uniqueness criterion for problem (1.4)–(1.5): a solution $W$ (for $\lambda = \lambda_{\overline{H}}$) is unique up to an additive constant if and only if $S_{\overline{H}-\lambda_{\overline{H}}}(x,y) = 0$ $\forall x, y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$, where $S_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ denotes the symmetrized distance, $S_{\overline{H}-\lambda_{\overline{H}}}(x,y) = d_{\overline{H}-\lambda_{\overline{H}}}(x,y) + d_{\overline{H}-\lambda_{\overline{H}}}(y,x)$.

Since an additive eigenfunction of the limit (homogenized) problem need not be unique, an important issue, in the case of non-uniqueness, is to select a solution being responsible for the first eigenpair asymptotics in (1.2). Under some additional conditions this problem can be solved by studying the higher order terms in the asymptotic expansion of $\lambda_\varepsilon$. This question is rather delicate, and we mainly focus in this work on a particular case when $c(x,y) = 0$ and $\alpha = 1$, so that operator (1.1) takes the form

$$\mathcal{L}_\varepsilon u = \varepsilon^2 a^{ij}(x,x/\varepsilon)\frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x,x/\varepsilon)\frac{\partial u}{\partial x_j}. \tag{2.14}$$

Moreover, we assume that $\lambda_{\overline{H}} = 0$ and that the corresponding Aubry set $\mathcal{A}_{\overline{H}}$ has a special structure.

The analogous result for $\alpha > 1$ is established in Section 8. In this case,

$$\mathcal{L}_\varepsilon u = \varepsilon^2 a^{ij}(x,x/\varepsilon^\alpha)\frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x,x/\varepsilon^\alpha)\frac{\partial u}{\partial x_j}, \quad \alpha > 1. \tag{2.15}$$

For $\alpha \geq 1$ the effective Hamiltonian $\overline{H}(p,x)$ is a $C^\infty$ strictly convex function in $p$ variable, i.e. $\left(\frac{\partial^2}{\partial p_i \partial p_j}\overline{H}(p,x)\right)_{i,j=\overline{1,N}}$ is positive definite for all $p \in \mathbb{R}^N$ and $x \in \overline{\Omega}$, see [12], or [17] for $\alpha = 1$, while for $\alpha > 1$ the Hamiltonian $\overline{H}(p,x)$ is a quadratic function in $p$. Note also that if $c(x,y) = 0$ then $\overline{H}(0,x) = 0$. Therefore, for $\alpha \geq 1$ and $c(x,y) = 0$, the Lagrangian $\overline{L}(v,x)$ is strictly convex and $\overline{L}(v,x) = \max\{p \cdot v - \overline{H}(p,x)\} \geq -\overline{H}(0,x) = 0$. Thus we have

$$\overline{L}(v,x) \geq 0, \quad \text{and } \overline{L}(v,x) = 0 \iff v_j = \frac{\partial \overline{H}}{\partial p_j}(0,x).$$

On the other hand direct calculations show that

$$-\frac{\partial \overline{H}}{\partial p_j}(0,x) = \bar{b}^j(x) := \int_Y b^j(x,y)\theta^*(x,y)\mathrm{d}y, \tag{2.16}$$

the functions $\bar{b}^j(x)$ being components of the so-called effective drift $\bar{b}(x)$ defined by the right hand side of (2.16) via the $Y$-periodic solution $\theta^*$ of

$$\frac{\partial^2}{\partial y_i \partial y_j}\left(a^{ij}(x,y)\theta^*\right) - \frac{\partial}{\partial y_j}\left(b^j(x,y)\theta^*\right) = 0 \tag{2.17}$$

normalized by $\int_Y \theta^* \mathrm{d}y = 1$. (Note that $\theta^* > 0$ and it is a $C^2$ function.) Thus the Lagrangian $\overline{L}(v,x)$ admits the following representation:

$$\overline{L}(v,x) = \kappa \sum (v_j + \bar{b}^j(x))^2 + \widetilde{L}(v,x), \quad \text{with } 0 \leq \widetilde{L}(v,x) \leq \tilde{\kappa} \sum (v_j + \bar{b}^j(x))^2,\ 0 < \kappa < \tilde{\kappa}. \tag{2.18}$$

This implies, in view of (2.12), that the Aubry set $\mathcal{A}_{\overline{H}}$ of the Hamiltonian $\overline{H}$ coincides with that of the Hamiltonian $\sum p_j^2 - \bar{b}^j(x)p_j$ whose corresponding Lagrangian is $\frac{1}{4}\sum(v_j + \bar{b}^j(x))^2$. In particular, the additive eigenvalue $\lambda_{\overline{H}}$ is zero if and only if there is an orbit $\eta : \mathbb{R} \to \overline{\Omega}$, $\dot{\eta} = -\bar{b}(\eta)$ which in turn holds if and only if $\mathcal{A}_{\overline{H}} \neq \emptyset$. We study the eigenvalue problem for operator (2.14) under the following conditions:

$$\begin{aligned} &\mathcal{A}_{\overline{H}} \neq \emptyset \text{ and } \mathcal{A}_{\overline{H}} \subset \Omega, \\ &\mathcal{A}_{\overline{H}} \text{ is a finite set of hyperbolic fixed points } \xi \text{ of the ODE } \dot{x} = -\bar{b}(x). \end{aligned} \tag{2.19}$$

Under these assumptions we first obtain the leading term of the asymptotic expansion of $\lambda_\varepsilon$ which is vanishing as $\varepsilon \to 0$, because $\lambda_{\overline{H}} = 0$. In fact, this term is of order $\varepsilon$. This in turn allows us to select, among solutions of homogenized problem (1.4)–(1.5), the solution that is equal to $\lim_{\varepsilon \to 0} W_\varepsilon$.

We introduce the matrices $B(\xi)$ and $Q(\xi)$ with entries

$$B^{ji}(\xi) = \frac{\partial \bar{b}^j}{\partial x_i}(\xi), \quad Q^{ij}(\xi) = \frac{1}{2}\frac{\partial^2 \overline{H}}{\partial p_i \partial p_j}(0,\xi).$$

For fixed points $\xi$ of the ODE $\dot{x} = \bar{b}(x)$ the matrix $-B(\xi)$ corresponds to the linearized effective drift. Then, for $\xi \in \mathcal{A}_{\overline{H}}$, we define $\sigma(\xi)$ as the sum of negative real parts of the eigenvalues of $-B(\xi)$. Since every fixed point $\xi$ is assumed to be hyperbolic, $-B(\xi)$ has no eigenvalues with zero real part. We also denote by $\Pi_s$ and $\Pi_u$ the spectral projectors on the invariant subspaces of the matrix $B$ that corresponds to the eigenvalues with positive and negative real parts, respectively (stable and unstable subspaces of the system $\dot{z}_i = -B^{ij}z_j$).

**Theorem 4.** *Let $\alpha = 1$ and $c(x, y) = 0$. Then, under conditions (2.19) we have*

$$\lambda_\varepsilon = \varepsilon\bar\sigma + \bar o(\varepsilon), \quad \text{where } \bar\sigma = \max\{\sigma(\xi);\ \xi \in \mathcal{A}_{\overline{H}}\}. \tag{2.20}$$

*Moreover, if the maximum in (2.20) is attained at exactly one $\xi = \overline{\xi}$, then*

(i) *the scaled logarithmic transformations $W_\varepsilon = -\varepsilon \log u_\varepsilon$ of eigenfunctions $u_\varepsilon$ normalized by (2.2) converge to $W(x) = d_{\overline{H}}(x, \overline{\xi})$ uniformly on compacts in $\Omega$, i.e. $W$ is the maximal viscosity solution of $\overline{H}(\nabla W(x), x) = 0$ in $\Omega$, $\overline{H}(\nabla W(x), x) \geq 0$ on $\partial\Omega$, such that $W(\bar\xi) = 0$;*

(ii) *$u_\varepsilon(\overline{\xi} + \sqrt{\varepsilon} z)/u_\varepsilon(\bar\xi) \to u(z)$ in $C(K)$ and weakly in $H^1(K)$ for every compact $K \subset \mathbb{R}^N$, and the limit $u$ is the unique positive eigenfunction of the Ornstain-Uhlenbeck operator,*

$$Q^{ij}\frac{\partial^2 u}{\partial z_i \partial z_j} + z_i B^{ji}\frac{\partial u}{\partial z_j} = \overline{\sigma} u \quad \text{in } \mathbb{R}^N, \tag{2.21}$$

*normalized by $u(0) = 1$ and satisfying the following condition: $u(z)e^{\mu|\Pi_s z|^2 - \nu|\Pi_u z|^2}$ is bounded in $\mathbb{R}^N$ for some $\mu > 0$ and every $\nu > 0$; the existence and the uniqueness of such a positive eigenfunction are granted by Lemma 16 proved in Section 7. The coefficients in (2.21) are given by $B^{ji} = B^{ji}(\bar\xi)$, $Q^{ji} = Q^{ji}(\bar\xi)$.*

**Remark 5.** I. Condition (2.19) is satisfied, in particular, when the vector field $b(x, y)$ is a $C^1$-small perturbation of a gradient field $\nabla P(x)$ with $C^2$ potential $P(x)$ having the following properties:

- the set $\{x \in \overline{\Omega};\ \nabla P(x) = 0\}$ is formed by a finite collection of points in $\Omega$,
- the Hessian matrix $\left(\frac{\partial^2}{\partial x_i \partial x_j} P(x)\right)_{i,j=\overline{1,N}}$ at every such a point is nonsingular

(for the proof see Appendix B).

II. Condition (2.19) is satisfied if and only if the vector field $\bar b$ possesses the following properties:

- $\bar b$ has a finite number of fixed points in $\overline{\Omega}$, say $\xi^1, \ldots \xi^n$. All of them are hyperbolic, and none of them is situated on $\partial\Omega$.
- $\forall\, y \in \overline{\Omega}$, either $\sup\{t < 0 : x^y(t) \notin \overline{\Omega}\} > -\infty$, or $\lim_{t\to-\infty} x^y(t) = \xi^j$ for some $j \in \{1, \ldots, n\}$, where $x^y$ is a solution of the ODE $\dot x^y = -\bar b(x^y)$, $x^y(0) = y$.
- there is no any closed path $\xi^{j_1}, \xi^{j_2}, \ldots, \xi^{j_k} = \xi^{j_1}$ with $k \geq 2$ such that for any two consecutive points $\xi^{j_s}$ and $\xi^{j_{s+1}}$ there is a solution of the equation $\dot x = -\bar b(x)$ with $\lim_{t\to-\infty} x(t) = \xi^{j_s}$ and $\lim_{t\to+\infty} x(t) = \xi^{j_{s+1}}$. Note that $\xi^{j_1}$ might coincide with $\xi^{j_2}$.

**Remark 6.** It is not hard to show that under condition (2.19) we have $S_{\overline{H}}(\xi, \xi') > 0$ for all $\xi, \xi' \in \mathcal{A}_{\overline{H}}$, $\xi \neq \xi'$. This means that problem (1.4), (1.5) does have many solutions unless $\mathcal{A}_{\overline{H}}$ is a single point.

Note that condition (2.19) of Theorem 4 assumes, in particular, that all $\omega$(and $\alpha$)-limit points of the ODE $\dot x = -\overline{b}(x)$ are fixed points. Another important case, when the ODE $\dot x = -\overline{b}(x)$ has limit cycles in $\Omega$ (which is also the case of general position) is considered in the companion paper [40].

## 3. Singularly perturbed operators on the periodicity cell

In this section we deal with an auxiliary cell spectral problem for singularly perturbed elliptic operators of the form

$$\mathcal{L}_\varepsilon^{(\mathrm{per})} u = \varepsilon^2 a^{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \varepsilon b^j(x) \frac{\partial u}{\partial x_j} + c(x) u, \tag{3.1}$$

with $Y$-periodic coefficients $a^{ij}$, $b^j$, $c \in C^1(\mathbb{R}^N)$, $u$ also being $Y$-periodic. This problem plays a crucial role in the proof of Theorem 1 in the case $\alpha < 1$. We assume the uniform ellipticity condition $a^{ij}(x)\zeta_i\zeta_j \geq m|\zeta|^2 > 0$ for any $\zeta \in \mathbb{R}^N \setminus \{0\}$, and the symmetry $a^{ij} = a^{ji}$. Similarly to the case of the Dirichlet boundary condition, the first eigenvalue $\mu_\varepsilon$ of $\mathcal{L}_\varepsilon^{(\mathrm{per})}$ (eigenvalue with the maximal real part) is real and simple, the corresponding eigenfunction $u_\varepsilon$ can be chosen to satisfy $0 < u_\varepsilon(x) \leq \max u_\varepsilon = 1$. The asymptotic behavior of $\mu_\varepsilon$ and $u_\varepsilon$, as $\varepsilon \to 0$, was studied in [39] using a combination of large deviation and variational techniques. We recover hereafter the results of [39] by means of vanishing viscosity approach and establish as a bi-product some bounds for derivatives of functions $W_\varepsilon(x) = -\varepsilon \log u_\varepsilon(x)$ that are essential in the proof of Theorem 1.

First we derive the *a priori* bounds for the eigenvalues.

**Lemma 7.** *For every $\varepsilon > 0$ the eigenvalue $\mu_\varepsilon$ of $L_\varepsilon^{(\mathrm{per})}$ satisfies the inequalities*

$$\min c(x) \leq \mu_\varepsilon \leq \max c(x). \tag{3.2}$$

*Proof.* Let $x'$ be a maximum point of $u_\varepsilon$, we have

$$\nabla u_\varepsilon(x') = 0, \quad \varepsilon^2 a^{ij}(x') \frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j}(x') \leq 0,$$

therefore $c(x')u_\varepsilon(x') \geq \mu_\varepsilon u_\varepsilon(x')$, i.e. $\mu_\varepsilon \leq \max c(x)$. Similarly, if $x''$ is a minimum point of $u_\varepsilon$ then $\mu_\varepsilon u_\varepsilon(x'') \geq c(x'')u_\varepsilon(x'')$ and therefore $\mu_\varepsilon \geq \min c(x)$. □

Since $u_\varepsilon = e^{-W_\varepsilon(x)/\varepsilon}$, we have

$$-\varepsilon a^{ij}(x) \frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} + a^{ij}(x) \frac{\partial W_\varepsilon}{\partial x_i} \frac{\partial W_\varepsilon}{\partial x_j} - b^j(x) \frac{\partial W_\varepsilon}{\partial x_j} + c(x) = \mu_\varepsilon. \tag{3.3}$$

The bounds for the first and second derivatives of $W_\varepsilon(x)$ are obtained in the following

**Lemma 8.** *There is a constant $C$, independent of $\varepsilon$, such that*

$$\max |\nabla W_\varepsilon| \leq C, \quad \max |\partial^2 W_\varepsilon / \partial x_i \partial x_j| \leq C/\varepsilon. \tag{3.4}$$

*Proof.* The proof of the first bound in (3.4) is borrowed from [21]. Let $D_1(x) := |\nabla W_\varepsilon(x)|^2$ and $D_2(x) := \sum |\partial^2 W_\varepsilon(x)/\partial x_i \partial x_j|^2$ . From (3.3) in conjunction with (3.2) we get $mD_1 \leq C(\varepsilon D_2^{1/2} + D_1^{1/2} + 1)$, this in turn implies that

$$D_1 \leq C(\varepsilon D_2^{1/2} + 1). \tag{3.5}$$

Assume that $D_1$ attains its maximum at a point $x'$, then we have $\nabla D_1(x') = 0$ and $a^{ij}(x')\frac{\partial^2 D_1}{\partial x_i \partial x_j}(x') \le 0$ or

$$\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}(x')\frac{\partial W_\varepsilon}{\partial x_k}(x') = 0 \tag{3.6}$$

and

$$\varepsilon \sum_k a^{ij}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}\frac{\partial^2 W_\varepsilon}{\partial x_j \partial x_k} \le -\varepsilon \sum_k a^{ij}\frac{\partial^3 W_\varepsilon}{\partial x_i \partial x_j \partial x_k}\frac{\partial W_\varepsilon}{\partial x_k} \quad \text{at } x'. \tag{3.7}$$

In order to bound the right hand side of (3.7) we take derivatives of (3.3), this yields

$$-\varepsilon a^{ij}\frac{\partial^3 W_\varepsilon}{\partial x_i \partial x_j \partial x_k} = \varepsilon \frac{\partial a^{ij}}{\partial x_k}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j} - 2a^{ij}\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}\frac{\partial W_\varepsilon}{\partial x_j} + b^i\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k} + \frac{\partial b^i}{\partial x_k}\frac{\partial W_\varepsilon}{\partial x_i} - \frac{\partial c}{\partial x_k}. \tag{3.8}$$

Then we multiply (3.8) by $\partial W_\varepsilon/\partial x_k$, sum up the resulting relations in $k$ and insert the result into (3.7) to obtain

$$\varepsilon m D_2(x') \le \varepsilon \sum_k a^{ij}(x')\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_k}(x')\frac{\partial^2 W_\varepsilon}{\partial x_j \partial x_k}(x') \le C\Big(\varepsilon D_1^{1/2}(x')D_2^{1/2}(x') + D_1(x') + D_1^{1/2}(x')\Big).$$

Next we use (3.5) to get that $D_2(x') \le C/\varepsilon$, and exploiting once more (3.5) we obtain the first bound in (3.4).

To show the second bound in (3.4) we use the following interpolation inequality

$$\|\nabla u\|^2_{L^\infty} \le C(\big\|a^{ij}\tfrac{\partial^2 u}{\partial x_i \partial x_j}\big\|_{L^\infty} + \|u\|_{L^\infty})\|u\|_{L^\infty}, \tag{3.9}$$

which holds for every $Y$-periodic $u$ with a constant $C$ independent of $u$. The proof of this inequality follows the lines of one in the Appendix of [7] (here it is important that the coefficients $a^{ij}$ are Lipschitz continuous). We apply (3.9) to (3.8) to obtain

$$\Big\|\frac{\partial^2 W_\varepsilon}{\partial x_l \partial x_k}\Big\|^2_{L^\infty} \le \frac{C}{\varepsilon}\Big(\sum \Big\|\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j}\Big\|_{L^\infty} + 1\Big) \quad \forall l, k, \tag{3.10}$$

here we have also used the first bound in (3.4). From (3.10) one easily derives the second bound in (3.4). □

It follows from Lemma 7 that $\mu_\varepsilon \to \mu$, up to extracting a subsequence. Due to Lemma 8 the family of functions $W_\varepsilon(x)$ is equicontinuous, moreover $\min W_\varepsilon(x) = 0$ therefore passing to a further subsequence (if necessary) we have $W_\varepsilon(x) \to W(x)$ uniformly. The standard arguments (see, e.g., [15]) show that the pair $\mu$ and $W$ satisfies the equation

$$a^{ij}(x)\frac{\partial W}{\partial x_i}\frac{\partial W}{\partial x_j} - b^j(x)\frac{\partial W}{\partial x_j} + c(x) = \mu \tag{3.11}$$

in the viscosity sense.

The number $\mu$ for which (3.11) has a periodic viscosity solution is unique (see [20, 32]), therefore the entire sequence $\mu_\varepsilon$ converges to $\mu$ as $\varepsilon \to 0$.

## 4. *A priori* bounds

In this section we show that the eigenvalues $\lambda_\varepsilon$ of (1.1) are uniformly bounded and the functions $W_\varepsilon$ (given by (2)) uniformly converge on compacts in $\Omega$ as $\varepsilon \to 0$, up to extracting a subsequence. We also establish existence of continuous up to the boundary relaxed semi-limits, which is important for deriving the homogenized boundary condition (1.5).

Because of the Dirichlet boundary condition on the boundary $\partial\Omega$ and fast oscillations of the coefficients the arguments here are more involved than those in the periodic case.

**Lemma 9.** *There is a constant $\Lambda$ independent of $\varepsilon$ and such that*

$$-\Lambda \le \lambda_\varepsilon \le \sup c(x,y). \tag{4.1}$$

*Proof.* The proof of the upper bound follows by the maximum principle as in Lemma 7.

To derive a lower bound for $\lambda_\varepsilon$ we construct a function $v_\varepsilon$ and choose a number $\Lambda > 0$ such that $v_\varepsilon = 0$ on $\partial\Omega$, and

$$\mathcal{L}_\varepsilon v_\varepsilon - \lambda v_\varepsilon > 0 \quad \text{in } \Omega \tag{4.2}$$

for every $\lambda < -\Lambda$, $0 < \varepsilon < 1$. There is a function $V \in C^2(\overline{\Omega})$ satisfying the following conditions, $V > 0$ in $\Omega$ and $V = 0$ on $\partial\Omega$, $|\nabla V| > 1$ in a neighborhood of $\partial\Omega$. Set $v_\varepsilon(x) := e^{\kappa V(x)/\varepsilon} - 1$, where $\kappa$ is a positive parameter to be chosen later. We assume that $-\Lambda \le \min c(x,y)$ so that $\lambda < \min c(x,y)$. Then we have

$$\mathcal{L}_\varepsilon v_\varepsilon - \lambda v_\varepsilon \ge \big(m\kappa^2 - \kappa(M_1 + \varepsilon M_2) + (c(x,x/\varepsilon^\alpha) - \lambda)\big)e^{\kappa V(x)/\varepsilon} - (c(x,x/\varepsilon^\alpha) - \lambda) > 0 \quad \text{in } \Omega'$$

when $\kappa > \kappa_1 := (M_1 + M_2)/m$. Here $\Omega' = \{x \in \Omega;\ |\nabla V| \ge 1\}$, $M_1 = \max\left|b^i(x,y)\frac{\partial V}{\partial x_i}(x)\right|$, $M_2 = \max\left|a^{ij}(x,y)\frac{\partial^2 V}{\partial x_i \partial x_j}(x)\right|$. On the other hand, $\delta := \inf\{V(x);\ x \in \Omega\backslash\Omega'\} > 0$. Therefore, $e^{\kappa V(x)/\varepsilon} > 2$ in $\Omega\backslash\Omega'$, when $\kappa > \kappa_2 := (\log 2)/\delta$. Assuming additionally that $\min c(x,y) - \lambda > 2\kappa(M_1 + M_2)$, we have

$$\mathcal{L}_\varepsilon v_\varepsilon - \lambda v_\varepsilon \ge (-\kappa(M_1 + \varepsilon M_2) + (c(x,x/\varepsilon^\alpha) - \lambda))\exp(\kappa\gamma/\varepsilon) - (c(x,x/\varepsilon^\alpha) - \lambda) > 0 \quad \text{in } \Omega\backslash\Omega'.$$

Thus, setting $\kappa := \max\{\kappa_1, \kappa_2\}$ and $\Lambda := 2\kappa(M_1 + M_2) - \min c(x,y)$, we get (4.2).

Now note that $\lambda_\varepsilon$ is also the first eigenvalue (the eigenvalue with the maximal real part) of the adjoint operator $\mathcal{L}^*_\varepsilon u = \varepsilon^2 \frac{\partial^2}{\partial x_i \partial x_j}(a^{ij}u) - \varepsilon\frac{\partial}{\partial x_i}(b^i u) + cu$, and the corresponding eigenfunction $u^*_\varepsilon$ can be chosen positive in $\Omega$. Therefore, if $\lambda_\varepsilon < -\Lambda$ then $(\mathcal{L}_\varepsilon v_\varepsilon - \lambda_\varepsilon v_\varepsilon)u^*_\varepsilon > 0$ in $\Omega$. This contradicts the Fredholm theorem. □

The following two results show that, up to extracting a subsequence, functions $W_\varepsilon$ converge uniformly on compacts in $\Omega$. For brevity introduce the notation $d(x) = \text{dist}(x, \partial\Omega)$.

**Lemma 10.** *For every $\beta > 0$ there is a constant $C_\beta$, independent of $\varepsilon$, such that*

$$\|\nabla W_\varepsilon\|_{L^\infty(\Omega_{\beta\varepsilon})} \le C_\beta, \tag{4.3}$$

*where $\Omega_{\beta\varepsilon} = \{x \in \Omega;\ d(x) > \beta\varepsilon\}$.*

*Proof.* As in Lemma 8 we use the Bernstein method, but its local version because $W_\varepsilon(x)$ tends to $+\infty$ as $x \to \partial\Omega$. For more details see for example [2]. Let $\xi$ be an arbitrary point in $\Omega_{\beta\varepsilon}$.

Introduce a smooth cutoff function $\varphi \geq 0$ compactly supported in the ball $B_\beta = \{x;\ |x| < \beta\}$ and such that $\varphi = 1$ in $B_{\beta/2}$. Consider the functions

$$D_1(x) := \varphi^4((x-\xi)/\varepsilon)|\nabla W_\varepsilon|^2 \quad \text{and} \quad D_2(x) := \sum \Big|\frac{\partial^2 W_\varepsilon}{\partial x_i \partial x_j}\Big|^2.$$

Following the line of Lemma 8 one can show that if $D_1$ attains its nonzero maximum at a point $x'$ then

$$\varphi_\varepsilon^4 D_2 \leq \frac{C}{\varepsilon^2}\Big(1 + \varphi_\varepsilon^2|\nabla W_\varepsilon|^2 + \varepsilon\varphi_\varepsilon^3 D_2^{1/2}|\nabla W_\varepsilon| + \varepsilon\varphi_\varepsilon^3|\nabla W_\varepsilon|^3\Big), \quad \varphi_\varepsilon(x) = \varphi((x-\xi)/\varepsilon)$$

where $C$ is independent of $\varepsilon$ and $\xi$. This bound in conjunction with the pointwise inequality $|\nabla W_\varepsilon|^2 \leq C_1(\varepsilon D_2^{1/2}+1)$ yields $\varphi^4((x'-\xi)/\varepsilon)D_2(x') \leq C_3/\varepsilon^2$. Thus $\max\{|\nabla W_\varepsilon(x)|;\ |x-\xi| \leq \beta\varepsilon/2\} \leq C_4$ with $C_4 = C_4(\beta, m, \|a^{ij}\|_{C^1}, \|b^i\|_{C^1}, \|c\|_{C^1}, \Omega)$. □

Lemma 10 shows that if we renormalize $W_\varepsilon$ by subtracting a proper constant (e.g. $W_\varepsilon(x_0)$ for a fixed $x_0 \in \Omega$) then functions $W_{\varepsilon_n}$ converge locally uniformly to a function $W(x)$ along a subsequence $\varepsilon_n \to 0$. The latter function can be extended by continuity to a Lipschitz continuous function on $\overline{\Omega}$, moreover we have

$$\sup_{x\in\Omega_{\beta\varepsilon_n}} |W_{\varepsilon_n}(x) - W(x)| \to 0 \quad \text{as } n \to \infty \quad \forall \beta > 0. \tag{4.4}$$

Now for $x \in \overline{\Omega}$ we define a function $\widetilde{W}$ by

$$\widetilde{W}(x) = \liminf_{\substack{x_n \to x\\ x_n \in \Omega}} W_{\varepsilon_n}(x_n). \tag{4.5}$$

It follows from the definition of $\widetilde{W}$ that this function coincides with $W$ in $\Omega$. The following important result shows that actually $\widetilde{W} = W$ on $\overline{\Omega}$ (it is clear that $\widetilde{W} \leq W$, but even boundedness from below of $\widetilde{W}(x)$ on $\partial\Omega$ is not obvious).

**Lemma 11.** *Let $\phi \in C^2(\overline{\Omega})$, then every global minimum point $x_\varepsilon$ of $W_\varepsilon - \phi$ satisfies $d(x) \geq \beta\varepsilon$ with some $\beta > 0$ independent of $\varepsilon$. It follows that the functions $W(x)$ defined via (4.4) and $\widetilde{W}$ given by (4.5) coincide everywhere in $\overline{\Omega}$.*

*Proof.* Consider the function $V_\varepsilon(x) = \phi(x) - \rho_\varepsilon(x)$, where $\rho_\varepsilon = 2d(x) - Kd^2(x)/\varepsilon$ and $K$ is a positive parameter to be chosen later. We claim that $V_\varepsilon$ satisfies

$$-\varepsilon a^{ij}(x, x/\varepsilon^\alpha)\frac{\partial^2 V_\varepsilon}{\partial x_i \partial x_j} + H(\nabla V_\varepsilon, x, x/\varepsilon^\alpha) < -A_K \text{ in } \Omega\backslash\Omega_{\varepsilon/K}, \tag{4.6}$$

where $\Omega_{\varepsilon/K} = \{x \in \Omega;\ d(x) > \varepsilon/K\}$, and $A_K$ is a positive constant which can be chosen as large as we want by choosing an appropriate $K > 0$. Indeed, $|\nabla\rho_\varepsilon| \leq 4$ when $d(x) \leq \varepsilon/K$ while

$$\varepsilon a^{ij}(x, x/\varepsilon^\alpha)\frac{\partial^2 \rho_\varepsilon}{\partial x_i \partial x_j} \leq -2Ka^{ij}(x, x/\varepsilon^\alpha)\frac{\partial d(x)}{\partial x_i}\frac{\partial d(x)}{\partial x_j} + \varepsilon C \leq -2mK + C \quad \text{in } \Omega\backslash\Omega_{\varepsilon/K},$$

where $C$ is independent of $\varepsilon$ and $K$. Thus taking sufficiently large $K$ we get (4.6) with a constant $A_K > -\lambda_\varepsilon$ (by Lemma 9 we have $-\lambda_\varepsilon \leq \Lambda$ with $\Lambda$ independent of $\varepsilon$). Then it follows from (1.3) and (4.6) that the function $W_\varepsilon - V_\varepsilon$ cannot attain its local minimum at any interior point of $\Omega\backslash\Omega_{\varepsilon/K}$, otherwise at such a point $-\varepsilon a^{ij}(x,x/\varepsilon^\alpha)\frac{\partial^2 W_\varepsilon}{\partial x_i\partial x_j}+H(\nabla W_\varepsilon,x,x/\varepsilon^\alpha)\leq -\varepsilon a^{ij}(x,x/\varepsilon^\alpha)\frac{\partial^2 V_\varepsilon}{\partial x_i\partial x_j}+H(\nabla V_\varepsilon,x,x/\varepsilon^\alpha)$ leading to the inequality $A_K\leq-\lambda_\varepsilon$, that contradicts the inequality $A_K>-\lambda_\varepsilon$. Thus, for $x\in\Omega\backslash\Omega_{\varepsilon/K}$ we have

$$W_\varepsilon-\phi+\rho_\varepsilon\geq\min_{\partial\Omega_{\varepsilon/K}}(W_\varepsilon-\phi_\varepsilon+\rho_\varepsilon)=\min_{\partial\Omega_{\varepsilon/K}}(W_\varepsilon-\phi)+\varepsilon/K$$

along with the inequality $\rho_\varepsilon<\varepsilon/K$ in the interior of $\Omega\backslash\Omega_{\varepsilon/K}$, which easily follows from the definition of $\rho_\varepsilon$. This implies, in particular, that the minimum of $W_\varepsilon-\phi$ over $\Omega$ is never attained at a point $x_\varepsilon$ such that $d(x_\varepsilon)<\varepsilon/K$.

To prove the inequality $\widetilde{W}\geq W$ approximate $W$ by functions $w_\delta\in C^2(\overline{\Omega})$ so that $\max_{\overline{\Omega}}|W-w_\delta|\leq\delta$, $\delta>0$. Then

$$\begin{aligned}\widetilde{W}(x)-w_\delta(x)&=\liminf_{n\to\infty,\,\Omega\ni x_n\to x}(W_{\varepsilon_n}(x_n)-w_\delta(x_n))\geq\liminf_{n\to\infty}\min_{\overline{\Omega}}(W_{\varepsilon_n}-w_\delta)\\&=\liminf_{n\to\infty}\inf\{W_{\varepsilon_n}(x)-w_\delta(x);\,x\in\Omega_{\beta\varepsilon_n}\},\end{aligned}\tag{4.7}$$

for some $\beta>0$ which depends on $\delta$ but independent of $n$. By (4.4) the right hand side of (4.7) is $\min_{\overline{\Omega}}(W(x)-w_\delta)$, thus passing to the limit $\delta\to0$ in (4.7) yields $\widetilde{W}(x)\geq W(x)$. □

**Corollary 12.** *There is a constant $\kappa>0$ such that every maximum point $x_\varepsilon$ of $u_\varepsilon$ satisfies $d(x_\varepsilon)\geq\kappa\varepsilon$.*

*Proof.* We simply apply Lemma 11 with $\phi\equiv0$. □

## 5. Vanishing viscosity limit

This section is devoted to the proof of Theorem 1. According to the results of the previous section we can assume that,

$$\lambda_{\varepsilon_n}\to\lambda\tag{5.1}$$

and (4.4) holds with a Lipschitz continuous on $\overline{\Omega}$ function $W$. According to Corollary 12 and Lemma 10 we can return to the normalization (2.2). We are going to show that the pair $\lambda$ and $W$ is a solution of problem (1.4), (1.5).

For brevity we will write $\varepsilon$ in place of $\varepsilon_n$. We follow the same scheme for $\alpha>1,\alpha=1$ and $\alpha<1$. We construct test functions $\phi_\varepsilon$ converging to $\phi$ uniformly in $\overline{\Omega}$, and such that

$$-\varepsilon a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\phi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon)+H\big(\nabla\phi_\varepsilon(x_\varepsilon),x_\varepsilon,x_\varepsilon/\varepsilon^\alpha\big)\to\overline{H}(\nabla\phi(x_0),x_0)\tag{5.2}$$

for every sequence of points $x_\varepsilon\in\Omega$ such that $x_\varepsilon\to x_0$. The existence of such functions $\phi_\varepsilon$ will be established later on.

Consider an arbitrary function $\phi\in C^2(\overline{\Omega})$, and assume that $W-\phi$ attains strict minimum at a point $x_0\in\overline{\Omega}$. Since $W$ can be equivalently defined as the relaxed semi-limit (4.5) ($\widetilde{W}=W$ on $\overline{\Omega}$ by Lemma 11) there exists a sequence $x_\varepsilon\in\Omega$ of minimum points of $W_\varepsilon-\phi_\varepsilon$, converging to $x_0$. We have

$$\nabla\phi_\varepsilon(x_\varepsilon)=\nabla W_\varepsilon(x_\varepsilon)\quad\text{and}\quad a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2W_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon)\geq a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\phi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon).$$

Using (1.3) to get $-\varepsilon a^{ij}(x_\varepsilon, x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\phi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon) + H\big(\nabla\phi_\varepsilon(x_\varepsilon), x_\varepsilon, x_\varepsilon/\varepsilon^\alpha\big) - \lambda_\varepsilon \geq 0$, and then passing to to the limit as $\varepsilon \to 0$ with the help of (5.2) we obtain $\overline{H}(\nabla\phi(x_0), x_0) \geq \lambda$.

If $W - \phi$ attains strict maximum at a point $x_0 \in \Omega$ we argue similarly to derive $\overline{H}(\nabla\phi(x_0), x_0) \leq \lambda$. Thus $W(x)$ is a viscosity solution of (1.4), (1.5).

It remains to construct functions $\phi_\varepsilon$ that satisfy (5.2), and converge to $\phi$ uniformly in $\overline{\Omega}$.

**Case** $\alpha > 1$. We set

$$\phi_\varepsilon(x) = \phi(x) + \varepsilon^{2\alpha-1}\theta(x/\varepsilon^\alpha),$$

where $\theta(y)$ is a $Y$-periodic solution of

$$-a^{ij}(x_0, y)\frac{\partial^2\theta}{\partial y_i\partial y_j} = \overline{H}(\nabla\phi(x_0), x_0) - H(\nabla\phi(x_0), x_0, y). \tag{5.3}$$

Thanks to (2.3) such a solution does exist. Indeed, (2.3) is nothing but the solvability condition for (5.3). Moreover, since the coefficients and the right hand side in (5.3) are Lipschitz continuous, $\theta \in C^{2,1}$ (see, e.g., [23]). Therefore if $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, then we have

$$\begin{aligned}
&-\varepsilon a^{ij}(x_\varepsilon, x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\phi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon) + H\big(\nabla\phi_\varepsilon(x_\varepsilon), x_\varepsilon, x_\varepsilon/\varepsilon^\alpha\big)\\
&\quad= -a^{ij}(x_\varepsilon, x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\theta}{\partial y_i\partial y_j}(x_\varepsilon/\varepsilon^\alpha) + \underline{O}(\varepsilon)\\
&\qquad+ H\big(\nabla\phi(x_\varepsilon) + \underline{O}(\varepsilon^{\alpha-1}), x_\varepsilon, x_\varepsilon/\varepsilon^\alpha\big) = -a^{ij}(x_0, x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\theta}{\partial y_i\partial y_j}(x_\varepsilon/\varepsilon^\alpha)\\
&\qquad+ H\big(\nabla\phi(x_0), x_0, x_\varepsilon/\varepsilon^\alpha\big) + \underline{O}(|x - x_\varepsilon| + \varepsilon + \varepsilon^{\alpha-1}) = \overline{H}(\nabla\phi(x_0), x_0) + \overline{o}(1).
\end{aligned}$$

**Case** $\alpha = 1$. Set $\phi_\varepsilon(x) = \phi(x) + \varepsilon\theta(x/\varepsilon)$, where $\theta(y) = -\log\vartheta(y)$ and $\vartheta(y)$ is the unique (up to multiplication by a positive constant) $Y-$periodic positive solution of

$$a^{ij}(x_0, y)\frac{\partial^2\vartheta}{\partial y_i\partial y_j} + \hat{b}^j(y)\frac{\partial\vartheta}{\partial y_j} + \hat{c}(y))\vartheta = \overline{H}(p, x_0)\vartheta, \tag{5.4}$$

where $p = \nabla\phi(x_0)$, $\hat{b}^j(y) = b^j(x_0, y) - 2a^{ij}(x_0, y)p_i$, $\hat{c}(y) = a^{ij}(x_0, y)p_ip_j - b^j(x_0, y)p_j + c(x_0, y)$. By a standard elliptic regularity result we have $\theta \in C^{2,1}$ (see [23]), and one can easily verify that

$$-\varepsilon a^{ij}(x_\varepsilon, x_\varepsilon/\varepsilon)\frac{\partial^2\phi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon) + H\big(\nabla\phi_\varepsilon(x_\varepsilon), x_\varepsilon, x_\varepsilon/\varepsilon\big) = \overline{H}(\nabla\phi(x_0), x_0) + \overline{o}(1),$$

as soon as $x_\varepsilon \to x_0$ when $\varepsilon \to 0$.

**Case** $\alpha < 1$. Set $\phi_\varepsilon(x) = \phi(x) + \varepsilon^\alpha\theta_\varepsilon(x/\varepsilon^\alpha)$, where $\theta_\varepsilon$ is a $Y$-periodic solution of the equation

$$-\varepsilon^{1-\alpha}a^{ij}(x_0, y)\frac{\partial^2\theta_\varepsilon}{\partial y_i\partial y_j} + H\big(p + \nabla\theta_\varepsilon(y), x_0, y\big) = \overline{H}_\varepsilon(p, x_0) \quad \text{with } p = \nabla\phi(x_0). \tag{5.5}$$

Such a solution exists if $\overline{H}_\varepsilon(p, x_0)$ coincides the first eigenvalue $\mu_\varepsilon$ (eigenvalue with the maximal real part) of the spectral problem

$$\begin{aligned}
&\varepsilon^{2(1-\alpha)}a^{ij}(x_0, y)\frac{\partial^2\vartheta_\varepsilon}{\partial y_i\partial y_j} + \varepsilon^{1-\alpha}\hat{b}^j(y)\frac{\partial\vartheta_\varepsilon}{\partial y_j} + \hat{c}(y)\vartheta_\varepsilon = \mu_\varepsilon\vartheta_\varepsilon,\\
&\vartheta_\varepsilon \quad \text{is } Y\text{-periodic,}
\end{aligned}$$

where $\hat{b}^j$, $\hat{c}$ are as in (5.4). According to the Krein-Rutman theorem $\mu_\varepsilon$ is a real and simple eigenvalue, and the corresponding eigenfunction $\vartheta_\varepsilon$ can be chosen positive. Then a solution of (5.5) is given by $\theta_\varepsilon = -\varepsilon^{1-\alpha}\log\vartheta_\varepsilon$. We invoke now the results obtained in Section 3,

$$\overline{H}_\varepsilon(p,x_0) \to \overline{H}(p,x_0) = \overline{H}(\nabla\phi(x_0),x_0) \tag{5.6}$$

(where the limit $\overline{H}(p,x_0)$ is described in (2.5)),

$$\|\partial^2\vartheta_\varepsilon/\partial y_i\partial y_j\|_{L^\infty} \le C/\varepsilon^{1-\alpha} \tag{5.7}$$

This allows us to obtain (5.2) similarly to other cases considered above,

$$\begin{aligned}
&-\varepsilon a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\phi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon) + H\big(\nabla\phi_\varepsilon(x_\varepsilon),x_\varepsilon,x_\varepsilon/\varepsilon^\alpha\big)\\
&\quad= -\varepsilon^{1-\alpha}a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\theta_\varepsilon}{\partial y_i\partial y_j}(x_\varepsilon/\varepsilon^\alpha)\\
&\qquad+ H\big(\nabla\phi(x_\varepsilon)+\nabla\theta_\varepsilon(x_\varepsilon/\varepsilon^\alpha),x_\varepsilon,x_\varepsilon/\varepsilon^\alpha\big) + \underline{O}(\varepsilon) = -\varepsilon^{1-\alpha}a^{ij}(x_0,x_\varepsilon/\varepsilon^\alpha)\frac{\partial^2\theta_\varepsilon}{\partial y_i\partial y_j}(x_\varepsilon/\varepsilon^\alpha)\\
&\qquad+ H\big(\nabla\phi(x_0)+\nabla\theta_\varepsilon(x_\varepsilon/\varepsilon^\alpha),x_0,x_\varepsilon/\varepsilon^\alpha\big) + \underline{O}(|x-x_\varepsilon|+\varepsilon) = \overline{H}(\nabla\phi(x_0),x_0)+\bar{o}(1).
\end{aligned}$$

Theorem 1 is completely proved.

## 6. Case of zero potential. lower bound for eigenvalues via blow up analysis

The goal of this and the next sections is to prove Theorem 4. Here we consider in details the special case when $\alpha = 1$ and $c(x,y) = 0$, the corresponding convection-diffusion operator is given by (2.14). Under assumptions (2.19) the eigenvalues $\lambda_\varepsilon$ of (1.1) converge to zero as $\varepsilon \to 0$. It is then natural to study the first non-trivial term of the asymptotic expansion for $\lambda_\varepsilon$. We show that, under our standing assumptions, this term is of order $\varepsilon$, and its limit behavior can be characterized by local analysis near points of the Aubry set $\mathcal{A}_{\overline{H}}$ of the effective Hamiltonian.

The main result of this section which establishes a refined lower bound for eigenvalues is given by

**Theorem 13.** *Let $c(x,y) = 0$ and $\alpha = 1$, and assume that (2.19) is fulfilled. Then*

$$\liminf_{\varepsilon\to 0}\lambda_\varepsilon/\varepsilon \ge \bar{\sigma} = \max\{\sigma(\xi);\ \xi\in\mathcal{A}_{\overline{H}}\}, \tag{6.1}$$

*where quantities $\sigma(\xi)$ are defined in Theorem 4.*

*Proof.* Fix a point $\xi \in \mathcal{A}_{\overline{H}}$.

Applying the maximum principle we see that $\lambda_\varepsilon < 0$. In order to obtain a lower bound, we introduce, for any $\delta > 0$, an auxiliary spectral problem $\mathcal{L}_\varepsilon v_\varepsilon - \delta|x-\xi|^2 v_\varepsilon = \varepsilon\tilde{\sigma}_\varepsilon v_\varepsilon$, or

$$\varepsilon a^{ij}(x,x/\varepsilon)\frac{\partial^2 v_\varepsilon}{\partial x_i\partial x_j} + b^j(x,x/\varepsilon)\frac{\partial v_\varepsilon}{\partial x_j} - \frac{\delta|x-\xi|^2}{\varepsilon}v_\varepsilon = \tilde{\sigma}_\varepsilon v_\varepsilon \quad \text{in } \Omega \tag{6.2}$$

with the Dirichlet condition $v_\varepsilon = 0$ on $\partial\Omega$. According to [42] the eigenvalues $\lambda_\varepsilon$ and $\varepsilon\tilde{\sigma}_\varepsilon$ are given by

$$\lambda_\varepsilon = \inf\left\{\sup_{x\in\Omega}\frac{\mathcal{L}_\varepsilon\phi}{\phi}\right\},\quad \varepsilon\tilde{\sigma}_\varepsilon = \inf\left\{\sup_{x\in\Omega}\frac{\mathcal{L}_\varepsilon\phi - \delta|x-\xi|^2\phi}{\phi}\right\},$$

where in both expressions the infimum is taken over the set

$$\{\phi \in C^2(\Omega) \cap C(\overline{\Omega}),\ \phi > 0 \text{ in } \Omega,\ \phi = 0 \text{ on } \partial\Omega\}.$$

Therefore, for any given $\delta > 0$, we have

$$\lambda_\varepsilon \geq \varepsilon\tilde{\sigma}_\varepsilon.$$

We assume hereafter that the first eigenfunction $v_\varepsilon$ of (6.2) is normalized by $v_\varepsilon(\xi) = 1$.

Let us transform (6.2) to a form more convenient for the analysis. First, after changing variables $z = (x - \xi)/\sqrt{\varepsilon}$ and setting $w_\varepsilon(z) = v_\varepsilon(\xi + \sqrt{\varepsilon}z)$ equation (6.2) becomes

$$a^{ij}_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)\frac{\partial^2 w_\varepsilon}{\partial z_i z_j} + \frac{b^j_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)}{\sqrt{\varepsilon}}\frac{\partial w_\varepsilon}{\partial z_j} - \delta|z|^2 w_\varepsilon = \tilde{\sigma}_\varepsilon w_\varepsilon \quad \text{in } (\Omega - \xi)/\sqrt{\varepsilon}. \tag{6.3}$$

Here and in what follows the subscript "$\xi, \xi/\varepsilon$" denotes the shift (translation) by $\xi$ in $x$ and by $\xi/\varepsilon$ in $y$, i.e., for instance, $a^{ij}_{\xi,\xi/\varepsilon}(x,y) = a^{ij}(x + \xi, y + \xi/\varepsilon)$. Next we multiply (6.3) by $\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)$, $\theta^*(x,y)$ being given by (2.17). After simple rearrangements this yields

$$\frac{\partial}{\partial z_i}\Big(\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)a^{ij}_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)\frac{\partial w_\varepsilon}{\partial z_j}\Big) + \frac{S^j_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)}{\sqrt{\varepsilon}}\frac{\partial w_\varepsilon}{\partial z_j}$$
$$+ \Big(\frac{\overline{b}^j(\sqrt{\varepsilon}z + \xi)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}h^j_\varepsilon(z)\Big)\frac{\partial w_\varepsilon}{\partial z_j} = (\tilde{\sigma}_\varepsilon + \delta|z|^2)\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)w_\varepsilon, \tag{6.4}$$

where

$$S^j(x,y) = b^j(x,y)\theta^*(x,y) - \frac{\partial}{\partial y_i}\Big(a^{ij}(x,y)\theta^*(x,y)\Big) - \overline{b}^j(x);$$

$h^j_\varepsilon$ in (6.4) are uniformly bounded functions whose structure is not important. Since $\theta^*$ solves (2.17), the $Y$-periodic vector field $S(x,y) = (S^1(x,y), \ldots, S^N(x,y))$ is divergence free, for every fixed $x$, and, due to the definition of $\overline{b}$, this field has zero mean over the period. Therefore, $S(x,y)$ admits the representation (see, for instance, [12])

$$S^j(x,y) = \frac{\partial}{\partial y_i}T^{ij}(x,y) \text{ with } Y\text{-periodic in } y \text{ skew-symmetric } T^{ij}(x,y)\ (T^{ij} = -T^{ji}).$$

Moreover, functions $T^{ij}$ are continuous with bounded derivatives $\partial T^{ij}/\partial x_k$. We can thus rewrite (6.4) as

$$\frac{\partial}{\partial z_i}\Big(q^{ij}_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)\frac{\partial w_\varepsilon}{\partial z_j}\Big) + \Big(\frac{\overline{b}^j(\sqrt{\varepsilon}z + \xi)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\tilde{h}^j_\varepsilon(z)\Big)\frac{\partial w_\varepsilon}{\partial z_j}$$
$$= (\tilde{\sigma}_\varepsilon + \delta|z|^2)\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)w_\varepsilon, \tag{6.5}$$

where $q^{ij}_{\xi,\xi/\varepsilon}(x,y) = q^{ij}(x + \xi, y + \xi/\varepsilon)$, $q^{ij}(x,y) = \theta^*(x,y)a^{ij}(x,y) + T^{ij}(x,y)$, and $\tilde{h}^j_\varepsilon$ are uniformly bounded functions. Note that on every fixed compact we have

$$\frac{\overline{b}^j(\sqrt{\varepsilon}z + \xi)}{\sqrt{\varepsilon}} = \frac{\overline{b}^j(\sqrt{\varepsilon}z + \xi) - \overline{b}^j(\xi)}{\sqrt{\varepsilon}} \to z_i\frac{\partial \overline{b}^j}{\partial x_i}(\xi)$$

uniformly in $z$ as $\varepsilon \to 0$.

In the following statement we do not suppose condition (2.19) to hold, however we still assume that $c(x,y)=0$ and $\alpha=1$.

**Lemma 14.** *If $\bar{b}(\xi)=0$ for some $\xi\in\overline{\Omega}$ then the first eigenvalue $\lambda_\varepsilon$ of operator (2.14) satisfies the bound $-\Lambda\varepsilon\le\lambda_\varepsilon<0$ with some $\Lambda>0$ independent of $\varepsilon$.*

*Proof.* We know that $\lambda_\varepsilon<0$ and in the proof of the lower bound we assume first that $\xi\in\Omega$. Then (6.5) holds in $B_2=\{z;\ |z|<2\}$ for sufficiently small $\varepsilon$. Letting

$$\mathcal{L}_\varepsilon^{(\mathrm{aux})}w=\frac{\partial}{\partial z_i}\Big(q^{ij}_{\xi,\xi/\varepsilon}\Big(\sqrt{\varepsilon}z,\frac{z}{\sqrt{\varepsilon}}\Big)\frac{\partial w}{\partial z_j}\Big)+\Big(\frac{\overline{b}^j(\sqrt{\varepsilon}z+\xi)}{\sqrt{\varepsilon}}+\sqrt{\varepsilon}\tilde{h}^j_\varepsilon(z)\Big)\times\frac{\partial w}{\partial z_j}-\delta|z|^2\theta^*_{\xi,\xi/\varepsilon}\Big(\sqrt{\varepsilon}z,\frac{z}{\sqrt{\varepsilon}}\Big)w$$

one can rewrite (6.5) in the operator form $\mathcal{L}_\varepsilon^{(\mathrm{aux})}w_\varepsilon=\tilde{\sigma}_\varepsilon\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z,z/\sqrt{\varepsilon}\big)w_\varepsilon$ and consider the parabolic equation for the operator $\mathcal{L}_\varepsilon^{(\mathrm{aux})}$

$$\frac{\partial\tilde{w}_\varepsilon}{\partial t}-\mathcal{L}_\varepsilon^{(\mathrm{aux})}\tilde{w}_\varepsilon=0\quad\text{in }(0,+\infty)\times B_2,$$

subject to the initial condition $\tilde{w}_\varepsilon(0,z)=w_\varepsilon(z)$ and the boundary condition $\tilde{w}_\varepsilon(t,z)=0$ on $(0,+\infty)\times\partial B_2$. The solution $\tilde{w}_\varepsilon$ of this problem satisfies the pointwise bound

$$\tilde{w}_\varepsilon(t,z)\le\exp\big(\tilde{\sigma}_\varepsilon(\min\theta^*)t\big)\,w_\varepsilon(z).\tag{6.6}$$

This follows by the maximum principle applied to

$$\Big(\frac{\partial}{\partial t}-\mathcal{L}_\varepsilon^{(\mathrm{aux})}\Big)\Big(\exp\big(\tilde{\sigma}_\varepsilon(\min\theta^*)t\big)\,w_\varepsilon(z)-\tilde{w}_\varepsilon(t,z)\Big)=\tilde{\sigma}_\varepsilon\Big(\min\theta^*-\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z,z/\sqrt{\varepsilon}\big)\Big)w_\varepsilon(z)\ge 0.$$

On the other hand, since the coefficients of the operator $\mathcal{L}_\varepsilon^{(\mathrm{aux})}$ are uniformly bounded on $B_2$ and the uniform ellipticity bound $\theta^*_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z,z/\sqrt{\varepsilon}\big)a^{ij}_{\xi,\xi/\varepsilon}\big(\sqrt{\varepsilon}z,z/\sqrt{\varepsilon}\big)\zeta^i\zeta^j\ge(\min\theta^*)m\,|\zeta|^2$ holds, by the Aronson estimate (see [4]) we have

$$\min\{\tilde{w}_\varepsilon(1,z);\ z\in B_1\}\ge M\min\{\tilde{w}_\varepsilon(0,z);\ z\in B_1\}$$

with $M>0$ independent of $\varepsilon$, where $B_1$ is the unit ball $B_1=\{z;\ |z|<1\}$. Combining this with (6.6) yields

$$e^{(\min\theta^*)\tilde{\sigma}_\varepsilon}\min_{B_1}w_\varepsilon\ge\min\{\tilde{w}_\varepsilon(1,z);\ z\in B_1\}\ge M\min\{\tilde{w}_\varepsilon(0,z);\ z\in B_1\}=M\min_{B_1}w_\varepsilon,$$

i.e. $\tilde{\sigma}_\varepsilon\ge\log M/\min\theta^*=:-\Lambda$. Thus $\tilde{\sigma}_\varepsilon\ge-\Lambda$ and $\lambda_\varepsilon\ge-\Lambda\varepsilon$.

Finally, in the case $\xi\in\partial\Omega$ we can repeat the above argument taking $\xi_\varepsilon\in\Omega$ in place of $\xi$, with $|\xi_\varepsilon-\xi|=\mathrm{dist}(\xi_\varepsilon,\partial\Omega)=2\sqrt{\varepsilon}$. □

In the proof of Lemma 14 we have got a uniform lower bound for $\tilde{\sigma}_\varepsilon$ which (in conjunction with the obvious inequality $\tilde{\sigma}_\varepsilon<0$) allows one to obtain uniform bounds for the norm of $w_\varepsilon$ in $C^{0,\beta}(K)$ (with $\beta>0$ depending only on bounds for coefficients in (6.5)) and $H^1(K)$, for every compact $K$ (see, e.g., [23, Section 8.9]). Thus, up to extracting a subsequence, $w_\varepsilon\to w$

in $C_{\rm loc}(\mathbb{R}^N)$ and $\tilde\sigma_\varepsilon \to \tilde\sigma$. Moreover, using the standard homogenization techniques based on the div-curl Lemma, one can show that $w$ solves

$$Q^{ij}\frac{\partial^2 w}{\partial z_i \partial z_j} + z_i B^{ji}\frac{\partial w}{\partial z_j} - \delta|z|^2 w = \tilde\sigma w \quad \text{in } \mathbb{R}^N, \tag{6.7}$$

where $Q^{ij} = Q^{ji}$ are homogenized constant coefficients satisfying the ellipticity condition (actually, one has $Q^{ij} = \frac{1}{2}\frac{\partial^2 \overline{H}}{\partial p_i \partial p_j}(0,\xi)$) and

$$B^{ji} = B^{ji}(\xi) = \frac{\partial \bar b^j}{\partial x_i}(\xi).$$

Since we assumed the normalization $w_\varepsilon(0) = 1$, we see that $w(z)$ is a nontrivial solution of (6.7). Moreover, if $z_\varepsilon$ is a maximum point of $w_\varepsilon(z)$ we get from (6.3) $|z_\varepsilon|^2 \le -\tilde\sigma_\varepsilon/\delta$. Therefore, thanks to Lemma 14, $|z_\varepsilon| \le C$. It follows that $w(z)$ is a bounded positive solution of (6.7). Let us now construct an eigenpair $(\sigma', w')$ of (6.7) with $w'$ of the form $w'(z) = e^{-\Gamma_\delta^{ij} z_i z_j}$ and a symmetric positive definite matrix $(\Gamma_\delta^{ij})_{i,j=\overline{1,N}}$. To this end, consider the following matrix Riccati equation

$$4\Gamma_\delta Q \Gamma_\delta - \Gamma_\delta B - B^* \Gamma_\delta - \delta I = 0,$$

where $I$ denotes the unit matrix. It is well-known (see, for instance, [31, Theorem 9.1.5]) that for positive definite $Q$ and $\delta > 0$ this equation has a maximal solution $\Gamma_\delta$, and, moreover, $\Gamma_\delta$ being positive definite. Then $w'(z) = e^{-\Gamma_\delta^{ij} z_i z_j}$ is a positive bounded solution of (6.7) corresponding to the eigenvalue $\sigma' = -2\mathrm{tr}(Q\Gamma_\delta)$. Next observe that, by means of the transformation $\tilde w(z) = e^{-r|z|^2} w(z)$ with $r > 0$, equation (6.7) is reduced to

$$Q^{ij}\frac{\partial^2 \tilde w}{\partial z_i \partial z_j} + (B^{ji} + 4rQ^{ij})z_i\frac{\partial \tilde w}{\partial z_j} + (4r^2 Q^{ij} z_i z_j + 2r\,\mathrm{tr}Q + 2rB^{ji} z_i z_j - \delta|z|^2)\tilde w = \tilde\sigma \tilde w \quad \text{in } \mathbb{R}^N.$$

For sufficiently small $r > 0$ we have $((4r^2 Q^{ij} z_i z_j + 2r\,\mathrm{tr}Q + 2rB^{ji} z_i z_j - \delta|z|^2) \to -\infty$, and, due to the boundedness of $w$, $\tilde w(z) \to 0$ as $|z| \to \infty$. Then, according to [41, Theorem 1], the eigenvalue $\tilde\sigma$ that corresponds to such a positive eigenfunction $\tilde w$ vanishing as $|z| \to \infty$ is unique. Thus $\tilde\sigma = \sigma' = -2\mathrm{tr}(Q\Gamma_\delta)$, and summarizing the above analysis we have $\liminf_{\varepsilon\to 0} \lambda_\varepsilon/\varepsilon \ge -2\mathrm{tr}(Q\Gamma_\delta)$. Finally note that $\Gamma_\delta$ converges to the maximal positive semi-definite solution of the Bernoulli equation (see, e.g., [31, Theorem 11.2.1])

$$4\Gamma Q \Gamma - \Gamma B - B^*\Gamma = 0, \tag{6.8}$$

as $\delta \to +0$. Calculations presented in Appendix C show that $-2\mathrm{tr}(Q\Gamma) = \sigma(\xi)$ with $\sigma(\xi)$ being the sum of negative real parts of the eigenvalues of $-B(\xi)$. Thus, after taking the maximum in $\xi \in \mathcal{A}_{\overline{H}}$, we get the desired lower bound

$$\liminf_{\varepsilon\to 0} \lambda_\varepsilon/\varepsilon \ge \bar\sigma = \max\{\sigma(\xi);\, \xi \in \mathcal{A}_{\overline{H}}\}. \qquad \square$$

## 7. Case of zero potential. upper bound for eigenvalues and selection of the additive eigenfunction

In this section we derive an upper bound for the principal eigenvalue which completes the proof of formula (2.20). Similarly to the previous section we make use of the blow up analysis near points of the Aubry set. We consider here only special (so-called *significant*) points of the Aubry set, where we can control the asymptotic behavior of rescaled eigenfunctions at infinity. We will show that only these special points have an influence on the leading term of the principal eigenvalue and eigenfunction.

To define a significant point we recall that, due to Theorem 1, up to extracting a subsequence, the functions $W_\varepsilon = -\varepsilon \log u_\varepsilon$ converge uniformly on compacts to a viscosity solution $W$ of problem (1.4)–(1.5) with $\lambda = 0$. It follows from (2.13) that $W$ has the representation $W(x) = \min\{d_{\overline{H}}(x,\xi) + W(\xi);\ \xi \in \mathcal{A}_{\overline{H}}\}$.

We say that a point $\xi \in \mathcal{A}_{\overline{H}}$ is *significant* if

$$W(x) = d_{\overline{H}}(x,\xi) + W(\xi) \quad \text{in a neighborhood of } \xi.$$

Otherwise we call $\xi$ *negligible*. For every negligible point $\xi \in \mathcal{A}_{\overline{H}}$ there are sequences $x^n \to \xi$ and $\xi^n \in \mathcal{A}_{\overline{H}} \backslash \{\xi\}$ such that $d_{\overline{H}}(x^n,\xi) + W(\xi) > d_{\overline{H}}(x^n,\xi^n) + W(\xi^n)$. Since the Aubry set consists of a finite number of points, the sequence $\xi^n$ converges (possibly along a subsequence) to $\xi' \in \mathcal{A}_{\overline{H}}$, $\xi' \neq \xi$. Passing to the limit $n \to \infty$ and using the continuity of the distance function, we get $d_{\overline{H}}(\xi,\xi') = W(\xi) - W(\xi')$ (we always have $d_{\overline{H}}(\xi,\xi') \geq W(\xi) - W(\xi')$). Now we introduce a (partial) order relation $\preceq$ on $\mathcal{A}_{\overline{H}}$ by setting

$$\xi' \preceq \xi \iff d_{\overline{H}}(\xi,\xi') = W(\xi) - W(\xi'). \tag{7.1}$$

This relation is clearly reflexive, its transitivity is a consequence of the triangle inequality $d_{\overline{H}}(\xi,\xi'') \leq d_{\overline{H}}(\xi,\xi') + d_{\overline{H}}(\xi',\xi'')$ while the antisymmetry follows from the inequality $S_{\overline{H}}(\xi,\xi') > 0$ held for all $\xi,\xi' \in \mathcal{A}_{\overline{H}}$ with $\xi \neq \xi'$. Then we see that every minimal element $\xi \in \mathcal{A}_{\overline{H}}$ is a significant point. Since $\mathcal{A}_{\overline{H}}$ is finite there exists a minimal element, i.e. there is at least one significant point $\xi \in \mathcal{A}_{\overline{H}}$.

**Theorem 15.** *Let assumptions of Theorem 4 be satisfied, and let $\xi$ be a significant point of $\mathcal{A}_{\overline{H}}$, associated to a converging (sub)sequence $W_\varepsilon \to W$. Then $\lim_{\varepsilon\to 0} \lambda_\varepsilon/\varepsilon = \sigma(\xi)$ and functions $u_\varepsilon(\xi + \sqrt{\varepsilon}z)/u_\varepsilon(\xi)$ converge weakly in $H^1(K)$ for every compact set $K \subset \mathbb{R}^N$ to the limit function $w(z)$ which is the unique positive eigenfunction of*

$$Q^{ij}\frac{\partial^2 w}{\partial z_i \partial z_j} + z_i B^{ji}\frac{\partial w}{\partial z_j} = \sigma w \quad \text{in } \mathbb{R}^N \tag{7.2}$$

*corresponding to the eigenvalue $\sigma = \sigma(\xi)$, normalized by $w(\xi) = 1$ and satisfying the additional condition*

$$w(z)e^{\mu|\Pi_s z|^2 - \nu|\Pi_u z|^2} \text{ is bounded on } \mathbb{R}^N \text{ for some } \mu > 0 \text{ and every } \nu > 0, \tag{7.3}$$

*where $\Pi_s$ and $\Pi_u$ denote spectral projectors on the invariant subspaces of the matrix $B$ corresponding to the eigenvalues with positive and negative real parts (stable and unstable subspaces of the system $\dot{z}_i = -B^{ij}z_j$). Here coefficients $Q^{ij}$ and $B^{ij}$ are as in Theorem 4.*

*Proof.* From now on we will assume that $u_\varepsilon$ is normalized by $u_\varepsilon(\xi) = 1$, unless otherwise is specified; the $W$ will also refer to the limit of scaled logarithmic transformations of $u_\varepsilon$ normalized in this way. Thanks to the upper and lower bounds for the eigenvalue $\lambda_\varepsilon$ the ratio $\lambda_\varepsilon/\varepsilon$ converges, as $\varepsilon \to 0$, along a subsequence, to a finite limit, denoted by $\sigma_0$. Then we argue exactly as in the proof of the lower bound for $\lambda_\varepsilon$. We consider rescaled eigenfunctions

$w_\varepsilon(z) = u_\varepsilon(\xi + \sqrt{\varepsilon}z)$ that are solutions of

$$\frac{\partial}{\partial z_i}\Big(q^{ij}_{\xi,\frac{\xi}{\varepsilon}}(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon})\frac{\partial w_\varepsilon}{\partial z_j}\Big) + \Big(\frac{\overline{b}^j(\sqrt{\varepsilon}z+\xi)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon}\tilde{h}^j_\varepsilon(z)\Big)\frac{\partial w_\varepsilon}{\partial z_j} = \frac{\lambda_\varepsilon}{\varepsilon}\theta^*_{\xi,\frac{\xi}{\varepsilon}}\big(\sqrt{\varepsilon}z, z/\sqrt{\varepsilon}\big)w_\varepsilon \quad \text{in } \frac{\Omega-\xi}{\sqrt{\varepsilon}}.$$

Up to extracting a further subsequence, $w_\varepsilon$ converge in $C(K)$ and weakly in $H^1(K)$, for every compact $K$, to a positive solution $w_0$ of (7.2) with $\sigma = \sigma_0$. Eigenvalue problem (7.2) has, in general, many solutions even in the class of positive eigenfunctions $w(z)$. We will show that the above defined eigenfunction $w_0$ also satisfies (7.3). Under condition (7.3) the following uniqueness result holds.

**Lemma 16.** *Spectral problem (7.2) has a unique eigenpair $(\sigma, w)$ with a positive eigenfunction $w$ satisfying (7.3) and normalized by $w(0) = 1$. Furthermore, $w(z) = e^{-\Gamma^{ij}z_iz_j}$, where $\Gamma$ is the maximal positive semi-definite solution of (6.8), and $\sigma = -2\mathrm{tr}(\Gamma Q)$.*

*Proof.* First observe that $w(z) = e^{-\Gamma^{ij}z_iz_j}$ satisfies (7.3). This follows from the relation $\Gamma = \Pi_s^*\Gamma\Pi_s \geq \gamma\Pi_s^*\Pi_s$ with $\gamma > 0$, see Proposition 25 in Appendix C. It is also clear that $w(z)$ does solve (7.2) with $\sigma = -2\mathrm{tr}(\Gamma Q)$.

To justify the uniqueness of $\sigma$ and $w(z)$ we make use of a transformation $\tilde{w}(z) = e^{\phi(z)}w(z)$, with a quadratic function $\phi(z)$ to be constructed later on, which leads to the equation of the form

$$Q^{ij}\frac{\partial^2\tilde{w}}{\partial z_i\partial z_j} + z_i\widetilde{B}^{ji}\frac{\partial\tilde{w}}{\partial z_j} + \widetilde{C}(z)\tilde{w} = \sigma\tilde{w} \quad \text{in } \mathbb{R}^N. \tag{7.4}$$

We will choose $\phi(z)$ so that $\widetilde{C}(z) \to -\infty$, $\tilde{w}(z) \to 0$ as $|z| \to \infty$. Then, by [41], there is a unique $\sigma$ such that (7.4) has a positive solution $\tilde{w}(z)$ vanishing as $|z| \to \infty$ ($\tilde{w}(z)$ is also unique up to multiplication by a positive constant).

We proceed with constructing $\phi(z)$. By setting $\phi = rA_s^{ij}z_iz_j - rA_u^{ij}z_iz_j$, with symmetric matrices $A_s$ and $A_u$, we get in (7.4)

$$\widetilde{B}^{ji} = B^{ji} + 4rQ^{jl}(A_u^{li} - A_s^{li})$$

and

$$\widetilde{C}(z) = 4r^2(A_u^{il} - A_s^{il})Q^{lm}(A_u^{mj} - A_s^{mj})z_iz_j + r\Big(\big(B^{li}(A_u^{lj} - A_s^{lj}) + (A_u^{il} - A_s^{il})B^{jl}\big)z_iz_j + 2\mathrm{tr}\big(Q(A_u - A_s)\big)\Big).$$

Define $A_s$ and $A_u$ as particular solutions of the Lyapunov matrix equations

$$A_sB + B^*A_s = \Pi_s^*\Pi_s, \quad A_uB + B^*A_u = -\Pi_u^*\Pi_u, \tag{7.5}$$

which are given by

$$A_s = \int_{-\infty}^0 e^{B^*t}\Pi_s^*\Pi_se^{Bt}\,\mathrm{d}t, \quad A_u = \int_0^\infty e^{B^*t}\Pi_u^*\Pi_ue^{Bt}\,\mathrm{d}t, \tag{7.6}$$

and choose sufficiently small $r_0 > 0$ in such a way that the matrix

$$4r(A_u-A_s)Q(A_u-A_s)+(A_u-A_s)B+B^*(A_u-A_s) = 4r(A_u-A_s)Q(A_u-A_s)-\Pi_s^*\Pi_s-\Pi_u^*\Pi_u$$

is negative definite for $0 < r < r_0$. Then $\widetilde{C}(z) \to -\infty$ as $|z| \to \infty$. It remains to see that if $w(z)$ satisfies (7.3) then choosing small enough $r > 0$ we have $\tilde{w}(z) \to 0$ as $|z| \to \infty$. Here we have used the fact that the inequalities $A_s \leq \gamma_1 \Pi_s^* \Pi_s$ and $A_u \geq \gamma_2 \Pi_u^* \Pi_u$ hold for some $\gamma_1, \gamma_2 > 0$. □

So far we know that $\lambda_\varepsilon/\varepsilon \to \sigma_0$ and $w_\varepsilon(z) = u_\varepsilon(\xi + \sqrt{\varepsilon} z)$ converge uniformly on compacts to a positive solution $w_0$ of (7.2) with $\sigma = \sigma_0$. In order to apply Lemma 16 we need only to show (7.3). To this end we first construct a quadratic function $\Phi_\mu^\nu(x)$ satisfying

$$\overline{H}(\nabla \Phi_\mu^\nu(x), x) \leq -\delta |x - \xi|^2 \quad \text{in a neighborhood } U(\xi) \text{ of } \xi \tag{7.7}$$

for some $\delta > 0$.

**Lemma 17.** *Let us set $\phi_s(x) := A_s^{ij} x_i x_j$ and $\phi_u(x) := A_u^{ij} x_i x_j$, where $A_s$ and $A_u$ are solutions of the Lyapunov matrix equation (7.5) given by (7.6). Then the function*

$$\Phi_\mu^\nu(x) := \mu \phi_s(x - \xi) - \nu \phi_u(x - \xi) \tag{7.8}$$

*satisfies (7.7) for some $\delta > 0$, provided that $0 < \mu, \nu < r$ and $r > 0$ is sufficiently small.*

*Proof.* We have, as $x \to \xi$,

$$\begin{aligned}
\overline{H}(\nabla \Phi_\mu^\nu(x), x) &\leq \overline{H}(0, x) + \frac{\partial \overline{H}}{\partial p_j}(0, x) \frac{\partial \Phi_\mu^\nu}{\partial x_j}(x) + C |\nabla \Phi_\mu^\nu(x)|^2 \\
&= -(x_i - \xi_i) \frac{\partial \bar{b}^j}{\partial x_i}(\xi) \frac{\partial \Phi_\mu^\nu}{\partial x_j}(x) + C |\nabla \Phi_\mu^\nu(x)|^2 + \bar{o}(|x - \xi|^2) \\
&\leq -2(x_i - \xi_i) B^{ji} (\mu A_s^{lj} - \nu A_u^{lj})(x_l - \xi_l) \\
&\quad + C_1 (\mu^2 |\Pi_s^*(x - \xi)|^2 + \nu^2 |\Pi_u^*(x - \xi)|^2) + \bar{o}(|x - \xi|^2).
\end{aligned} \tag{7.9}$$

Note that $(\mu A_s - \nu A_u) B + B^* (\mu A_s - \nu A_u) = \mu \Pi_s^* \Pi_s + \nu \Pi_u^* \Pi_u$, therefore the first term in the right hand side of (7.9) can be written as $-\mu |\Pi_s(x - \xi)|^2 - \nu |\Pi_u(x - \xi)|^2$. Thus (7.7) does hold if $0 < \mu < 1/C_1$ and $0 < \nu < 1/C_1$. □

Next we prove

**Lemma 18.** *If $\Phi_\mu^\nu(x)$ and $\mu, \nu$ are as in Lemma 17, then $W(x) > \Phi_\mu^\nu(x)$ in $\overline{U'(\xi)} \setminus \{\xi\}$, where $U'(\xi) \subset U(\xi)$ is a neighborhood of $\xi$.*

*Proof.* Since $\xi$ is a significant point, we have $W(x) = d_{\overline{H}}(x, \xi)$ in some neighborhood $U(\xi)$ of $\xi$. Due to the representation formula (2.11), there exists a sequence of positive numbers $\{t^n > 0\}_{n=1}^\infty$ and absolutely continuous curves $\eta^n : [0, t^n] \to \overline{\Omega}$ satisfying the initial and the terminal conditions $\eta^n(0) = \xi$ and $\eta^n(t^n) = x$ such that

$$d_{\overline{H}}(x, \xi) = \lim_{n \to \infty} \int_0^{t^n} \overline{L}(\dot{\eta}^n, \eta^n) \, d\tau.$$

We claim that there is a neighborhood $U'(\xi) \subset U(\xi)$ such that for all sufficiently large $n$ and any $x \in U'(\xi)$ we have $\{\eta^n(\tau) ; \tau \in [0, t^n]\} \subset U(\xi)$. Indeed, if we assume that such a neighborhood does not exist then there are sequences of points $x^n \to \xi$ and curves $\eta^n(t)$ such that

- $\eta^n$ connects $\xi$ to $x^n$, that is $\eta^n(0) = \xi$, $\eta^n(t^n) = x^n$;
- $\eta^n(\tau^n) \in \partial U(\xi)$ for some $\tau^n \in (0, t^n)$;
- $\lim_{n\to\infty} \int_0^{t^n} \overline{L}(\dot\eta^n, \eta^n)\, \mathrm{d}\tau = 0$.

Letting $y^n := \eta^n(\tau^n) \in \partial U(\xi)$ and considering the continuity of the distance function, we obtain $\lim_{n\to\infty} S_{\overline{H}}(y^n, \xi) = 0$, where $S_{\overline{H}}(y^n, \xi) = d_{\overline{H}}(y^n, \xi) + d_{\overline{H}}(\xi, y^n)$ is the symmetrized distance. After extracting a subsequence $y^n \to y \in \partial U(\xi)$ we obtain $S_{\overline{H}}(y, \xi) = 0$. Therefore $y \in \mathcal{A}_{\overline{H}}$. Repeating this reasoning we conclude that there is a point of Aubry set on the boundary of any open neighborhood of $\xi$. Therefore, $\xi$ cannot be an isolated point of $\mathcal{A}_{\overline{H}}$. This contradicts (2.19).

Now using (7.7) we get, for every $x \in U'(\xi)$

$$\Phi^\nu_\mu(x) = \int_0^{t^n} \nabla\Phi^\nu_\mu(\eta^n)\cdot\dot\eta^n\, \mathrm{d}\tau = \int_0^{t^n} \big(\nabla\Phi^\nu_\mu(\eta^n)\cdot\dot\eta^n - \overline{H}(\nabla\Phi^\nu_\mu(\eta^n), \eta^n)\big)\, \mathrm{d}\tau$$
$$+ \int_0^{t^n} \overline{H}(\nabla\Phi^\nu_\mu(\eta^n), \eta^n)\, \mathrm{d}\tau \le \int_0^{t^n} \overline{L}(\dot\eta^n, \eta^n)\, \mathrm{d}\tau,$$

when $n$ is sufficiently large. It follows that $\Phi^\nu_\mu \le W$ in $U'(\xi)$. On the other hand if $\Phi^\nu_\mu = W$ at a point $x_0 \in U'(\xi)$ then $x_0$ is a local minimum of $W - \Phi^\nu_\mu$ and $\overline{H}(\nabla\Phi^\nu_\mu(x_0), x_0) \ge 0$ since $W$ is a viscosity solution of $\overline{H}(\nabla W(x), x) = 0$ in $\Omega$. Therefore, $x_0 = \xi$ by (7.7), i.e. $\Phi^\nu_\mu < W$ in $U'(\xi)\setminus\{\xi\}$ and choosing, if necessary, a smaller neighborhood $U'(\xi)$, we obtain the desired statement. □

The following step is crucial in establishing (7.3). We want to construct a test function $\Psi_\varepsilon(x)$ of the form $\Psi_\varepsilon(x) = \Phi^\nu_\mu(x) + \varepsilon\tilde\theta_\varepsilon(x, x/\varepsilon)$ such that

$$-\varepsilon a^{ij}(x, x/\varepsilon)\frac{\partial^2\Psi_\varepsilon}{\partial x_i\partial x_j} + H\big(\nabla\Psi_\varepsilon(x), x, x/\varepsilon\big) \le \overline{H}(\nabla\Phi^\nu_\mu(x), x) + C\varepsilon \quad \text{in } U'(\xi). \tag{7.10}$$

To this end we first assume that the solution $\vartheta(p, x, y)$ of (2.4), normalized by $\int_Y \vartheta(p, x, y)\mathrm{d}y = 1$, is sufficiently smooth, and set $\tilde\theta_\varepsilon(x, y) = \theta\big(\nabla\Phi^\nu_\mu(x), x, y\big)$, where $\theta(p, x, y) = \log\vartheta(p, x, y)$. Then, since

$$-a^{ij}(x, y)\frac{\partial^2\theta(p, x, y)}{\partial y_i\partial y_j} + H\big(p + \nabla_y\theta(p, x, y), x, y\big) = \overline{H}(p, x),$$

(7.10) is straightforward. Note that in this case $\tilde\theta_\varepsilon(x, y)$ does not depend on $\varepsilon$. In the general case, thanks to $C^1$-regularity of the coefficients $a^{ij}(x, y)$ and $b^j(x, y)$, all the first and second order partial derivatives of $\vartheta(p, x, y)$ exist and continuous on $\mathbb{R}^N \times \overline{\Omega} \times \mathbb{R}^N$, except (possibly) $\partial^2\vartheta(p, x, y)/\partial x_i\partial x_j$. To obtain sufficient regularity of $\tilde\theta_\varepsilon(x, y)$ we set

$$\tilde\theta_\varepsilon(x, y) = \int \varphi_\varepsilon(x - x')\theta\big(\nabla\Phi^\nu_\mu(x), x', y\big)\mathrm{d}x',$$

where $\varphi_\varepsilon(x) = \varepsilon^{-N}\varphi(x/\varepsilon)$, with $\varphi(x)$ being a $C_0^\infty(\mathbb{R}^N)$ nonnegative function, $\int\varphi(x)\, \mathrm{d}x = 1$. Then we have

$$a^{ij}(x, x/\varepsilon)\Big(\frac{\partial^2\theta}{\partial y_i\partial y_j}\big(\nabla\Phi^\nu_\mu(x), x, x/\varepsilon\big) - \frac{\partial^2}{\partial x_i\partial x_j}\big(\theta_\varepsilon(x, x/\varepsilon)\big)\Big) \le \frac{C}{\varepsilon}$$

and $\big|\nabla_y\theta\big(\nabla\Phi^\nu_\mu(x), x, x/\varepsilon\big) - \nabla\big(\theta_\varepsilon(x, x/\varepsilon)\big)\big| \le C$. This yields (7.10).

It follows from (7.10) and (7.7) that

$$-\varepsilon a^{ij}(x,x/\varepsilon)\frac{\partial^2\Psi_\varepsilon}{\partial x_i\partial x_j}+H\big(\nabla\Psi_\varepsilon(x),x,x/\varepsilon\big)\le-\delta|x-\xi|^2+C\varepsilon\quad\text{in } U'(\xi).\tag{7.11}$$

Consider now the function $W_\varepsilon-\Psi_\varepsilon$. By Lemma 18 we have $W_\varepsilon>\Psi_\varepsilon$ on $\partial U'(\xi)$ for sufficiently small $\varepsilon$, therefore either $W_\varepsilon\ge\Psi_\varepsilon$ in $U'(\xi)$ or $W_\varepsilon-\Psi_\varepsilon$ attains its negative minimum in $U'(\xi)$ at a point $x_\varepsilon$. In the latter case we have $\nabla W_\varepsilon(x_\varepsilon)=\nabla\Psi_\varepsilon(x_\varepsilon)$ and

$$a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon)\frac{\partial^2W_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon)\ge a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon)\frac{\partial^2\Psi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon).$$

Therefore,

$$\begin{aligned}\lambda_\varepsilon&=-\varepsilon a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon)\frac{\partial^2W_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon)+H\big(\nabla W_\varepsilon(x_\varepsilon),x_\varepsilon,x_\varepsilon/\varepsilon\big)\\&\le-\varepsilon a^{ij}(x_\varepsilon,x_\varepsilon/\varepsilon)\frac{\partial^2\Psi_\varepsilon}{\partial x_i\partial x_j}(x_\varepsilon)+H\big(\nabla\Psi_\varepsilon(x_\varepsilon),x_\varepsilon,x_\varepsilon/\varepsilon\big)\le-\delta|x_\varepsilon-\xi|^2+C\varepsilon.\end{aligned}$$

Thus either $W_\varepsilon>\Psi_\varepsilon$ in $U'(\xi)$ or $W_\varepsilon\ge\Psi_\varepsilon+W_\varepsilon(x_\varepsilon)-\Psi_\varepsilon(x_\varepsilon)$ in $U'(\xi)$ and $x_\varepsilon$ satisfies $|x_\varepsilon-\xi|\le C\sqrt{\varepsilon}$. Both cases lead to the bound $W_\varepsilon(x)\ge\Phi^\nu_\mu(x)+W_\varepsilon(\tilde x_\varepsilon)-\beta\varepsilon$, where $\tilde x_\varepsilon$ is either $\xi$ or $x_\varepsilon$ (recall that $u_\varepsilon$ is normalized by $u_\varepsilon(\xi)=1$, i.e. $W_\varepsilon(\xi)=0$). Then, setting $z=(x-\xi)/\sqrt{\varepsilon}$ and recalling the definition of $\Phi^\nu_\mu$ in (7.8), we get

$$w_\varepsilon(z)\le Cw_\varepsilon(z_\varepsilon)e^{-\mu\phi_s(z)+\nu\phi_u(z)}\quad\text{in }(U(\xi)-\xi)/\sqrt{\varepsilon},$$

where $z_\varepsilon=(\tilde x_\varepsilon-\xi)/\sqrt{\varepsilon}$, and hence $|z_\varepsilon|\le C$. Observe that since $z_\varepsilon$ stay in a fixed compact as $\varepsilon\to0$, then $w_\varepsilon(z_\varepsilon)\le C$ and in the limit $\varepsilon\to0$ we therefore obtain

$$w(z)\le Ce^{-\mu\phi_s(z)+\nu\phi_u(z)}\quad\text{in }\mathbb{R}^N.$$

It remains to note that $\phi_s(z)\ge\gamma_3|\Pi_sz|^2$ and $\phi_u(z)\le\gamma_4|\Pi_uz|^2$ for some $\gamma_3,\gamma_4>0$. Hence $w(z)$ does satisfy (7.3), and applying Lemma 16 in conjunction with claim (iii) of Proposition 25 (see Appendix C) we complete the proof of Theorem 15. □

*Proof of Theorem 4.* Theorem 13 and Theorem 15 along with the fact that the set of significant points is nonempty yield formula (2.20). Moreover they imply the uniqueness of the limiting additive eigenfunction $W(x)$, provided that the maximum in (2.20) is attained at exactly one point $\xi=\bar\xi$ of the Aubry set. Indeed, we know that, up to extracting a subsequence, functions $W_\varepsilon$ converge uniformly (on compacts in $\Omega$) to an additive eigenfunction $W(x)$; here $W_\varepsilon=-\varepsilon\log u_\varepsilon$ and $u_\varepsilon$ are referred to the eigenfunctions normalized by (2.2). By Theorem 15 the unique significant point (associated to the chosen subsequence) is $\bar\xi$. Therefore $\bar\xi$ is the only minimal element in $\mathcal{A}_{\overline{H}}$ with respect to the order relation $\preceq$ defined in (7.1); hence it is the least element of $\mathcal{A}_{\overline{H}}$, i.e. $\bar\xi\preceq\xi$ for every $\xi\in\mathcal{A}_{\overline{H}}$. This means that $W(\xi)=W(\bar\xi)+d_{\overline{H}}(\xi,\bar\xi)$ for all $\xi\in\mathcal{A}_{\overline{H}}$, and consequently $W(x)=d_{\overline{H}}(x,\bar\xi)+W(\bar\xi)$. Thus, taking into account Corollary 12, we have $W(\bar\xi)=0$, and claim (i) of Theorem 4 is now completely proved. Finally, claim (ii) of Theorem 4 is addressed in Theorem 15. □

## 8. Case of zero potential. Other scalings

The statement of Theorem 4 remains valid in the case of $\varepsilon^\alpha$-scaling in (1.1) with $\alpha > 1$. This section focuses on this scaling. As in Theorem 4 we suppose that $c(x, y) = 0$. In this case the effective drift is still given by formula (2.16), however the function $\theta^*$ is now defined as a $Y$-periodic solution of the equation

$$\frac{\partial^2}{\partial y_i \partial y_j}\big(a^{ij}(x, y)\theta^*(x, y)\big) = 0, \qquad \int_Y \theta^* \, dy = 1.$$

It should be noted that the discontinuous dependence of the effective drift on the parameter $\alpha \ge 1$ at $\alpha = 1$ might lead to a significant shift of the concentration set of the eigenfunction $u_\varepsilon$ from the fixed points $\xi$ of the vector field $\overline{b}$, if $\alpha > 1$ is sufficiently close to 1.

In order to define more precisely the location of concentration points of $u_\varepsilon$, let us introduce the approximate effective Hamiltonian $\overline{H}_\varepsilon(p, x)$ as the (additive) eigenvalue corresponding to a $Y$-periodic eigenfunction of

$$-a^{ij}(x, y)\frac{\partial^2 \theta_\varepsilon(p, x, y)}{\partial y_i \partial y_j} + H(p + \varepsilon^{\alpha-1}\nabla_y \theta_\varepsilon(p, x, y), x, y) = \overline{H}_\varepsilon(p, x), \tag{8.1}$$

and the approximate effective drift $\overline{b}_\varepsilon(x)$ by

$$\overline{b}^j_\varepsilon(x) = -\frac{\partial \overline{H}_\varepsilon}{\partial p_j}(0, x).$$

The eigenvalue $\overline{H}_\varepsilon$ is unique and $\theta_\varepsilon$ is unique up to an additive constant, moreover $\theta_\varepsilon$ can be found as the scaled logarithmic transformation $\theta_\varepsilon = -\frac{1}{\varepsilon^{2(\alpha-1)}} \log \vartheta_\varepsilon$ of a positive $Y$-periodic eigenfunction of the linear eigenvalue problem

$$\varepsilon^{2(1-\alpha)} a^{ij}(x, y)\frac{\partial^2 \vartheta_\varepsilon}{\partial y_i \partial y_j} + \varepsilon^{1-\alpha}(b^j(x, y) - 2a^{ij}(x, y)p_i)\frac{\partial \vartheta_\varepsilon}{\partial y_j} + H(p, x, y)\vartheta_\varepsilon = \overline{H}_\varepsilon(p, x)\vartheta_\varepsilon.$$

Similarly to the case $\alpha = 1$, the drift $\overline{b}_\varepsilon(x)$ is defined by

$$\overline{b}_\varepsilon(x) = \int_Y b(x, y)\theta^*_\varepsilon(x, y) \, dy, \tag{8.2}$$

via the $Y$-periodic solution $\theta^*_\varepsilon$ of the equation

$$\frac{\partial^2}{\partial y_i \partial y_j}\big(a^{ij}(x, y)\theta^*_\varepsilon\big) - \varepsilon^{\alpha-1}\frac{\partial}{\partial y_j}\big(b^j(x, y)\theta^*_\varepsilon\big) = 0 \tag{8.3}$$

normalized by $\int_Y \theta^*_\varepsilon \, dy = 1$. From the smallness of the second term in the last equation, it follows that, under our assumptions on the coefficients, $\overline{b}_\varepsilon \to \overline{b}$ in $C^1(\overline{\Omega})$ topology, moreover $\|\overline{b}_\varepsilon - \overline{b}\|_{C^1(\overline{\Omega})} = O(\varepsilon^{\alpha-1})$. Therefore, if $\overline{b}$ has a finite number of zeros in $\Omega$, and all of them are hyperbolic fixed points of the ODE $\dot{x} = -\overline{b}(x)$, then, for sufficiently small $\varepsilon > 0$, $\overline{b}_\varepsilon$ has the same number of zeros, and the distance of these zeros from the corresponding zeros of $\overline{b}$ is at most $O(\varepsilon^{\alpha-1})$.

**Theorem 19.** *Let $\alpha > 1$, and $c(x, y) = 0$. Then, under conditions (2.19), all the statements of Theorem 4 remain valid except for claim (ii), where $\overline{\xi}$ should be replaced with the nearest to $\overline{\xi}$ zero of the vector field $\overline{b}_\varepsilon(x)$.*

*Proof.* Since the proof is quite similar to that of Theorem 4, we just outline main changes to be made in order to adapt the arguments of Sections 6 and 7 to the case $\alpha > 1$.

In order to obtain the lower bound for the eigenvalues $\lambda_\varepsilon$, one can follow the lines of Section 6. However, the arguments of Section 6 should apply to the zeros $\xi_\varepsilon$ of $\bar{b}_\varepsilon$ in place of the corresponding zeros $\xi$ of $\bar{b}$. Also $\theta^*_\varepsilon$ should be used in place of $\theta^*$. Note that although $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$, the distance between this two points might be of order $\varepsilon^{\alpha-1}$, so that in the local scale $\sqrt{\varepsilon}$ this distance might tend to infinity. Nevertheless, up to the shift from $\xi$ to $\xi_\varepsilon$ the local analysis is exactly the same as in Section 6. Let us emphasize that for $\alpha \in (1, 3/2)$ the statement of Lemma 14 remains valid only if at least one of zeros of $\bar{b}$ is an interior point of $\Omega$. Clearly, this condition is satisfied if (2.19) holds.

The argument of Section 7 can also be adapted to the case $\alpha > 1$. As in the proof of the lower bound one obtains equation (7.2) for the limit of rescaled functions $w_\varepsilon(z) = u_\varepsilon(\xi_\varepsilon + \sqrt{\varepsilon}z)$, while the construction of the functions $\Phi^\nu_\mu$ and $\Psi_\varepsilon$ is to be modified. One can linearize the drift $\bar{b}_\varepsilon$ at $\xi_\varepsilon$ and construct the quadratic function $\Phi^\nu_\mu$ (which now depends on $\varepsilon$) following Section 7 with $B^{ji}_\varepsilon = \frac{\partial \bar{b}^j}{\partial x_i}(\xi_\varepsilon)$ in place of $B^{ji}$; also, in the construction of the function $\Psi_\varepsilon$ one makes use of the eigenfunction $\theta_\varepsilon$ (cf. (8.1)) and sets $\Psi_\varepsilon(x) = \Phi^\nu_\mu(x) + \varepsilon^{2\alpha-1}\theta_\varepsilon(\nabla\Phi^\nu_\mu(x), x, x/\varepsilon^\alpha)$. The details are left to the reader. □

Finally note that the case $\alpha < 1$ remains completely open. The strategy used in the case $\alpha \geq 1$ fails to work for $\alpha < 1$. In particular, we cannot define in a natural way the effective drift because the corresponding periodic cell problem (8.3) becomes singular for $\alpha < 1$.

## 9. Example

Here we consider an example of an operator of the form (2.14) for which conditions (2.19) are fulfilled. Let $x^y(t)$ be a solution of the ODE $\dot{x}^y = -\bar{b}(x^y)$, $x^y(0) = y$. We assume that

- The vector field $\bar{b}(x)$ has exactly three zeros $\xi^1, \xi^2, \xi^3$ in $\overline{\Omega}$. All of them are interior points of $\Omega$.
- $\xi^1$ and $\xi^3$ are stable hyperbolic points, that is the eigenvalues of $\big(-\frac{\partial \bar{b}^j}{\partial x_i}(\xi^1)\big)_{i,j=\overline{1,N}}$ and $\big(-\frac{\partial \bar{b}^j}{\partial x_i}(\xi^3)\big)_{i,j=\overline{1,N}}$ have negative real parts; $\xi^2$ is a hyperbolic point and $\sigma(\xi^2) > \max\{\sigma(\xi^1), \sigma(\xi^3)\}$.
- The ODE $\dot{x} = -\bar{b}(x)$ does not have a solution with $\lim_{t\to+\infty} x(t) = \lim_{t\to-\infty} x(t) = \xi^2$.
- For every $y \in \overline{\Omega}\backslash\bigcup_{j=1}^3\{\xi^j\}$, either $\lim_{t\to-\infty} x^y(t) = \xi^2$, or $\inf\{t < 0; x^y(t) \in \overline{\Omega}\} > -\infty$.

**Proposition 20.** *Under the above assumptions the Aubry set $\mathcal{A}_{\overline{H}}$ coincides with $\bigcup_{j=1}^3\{\xi^j\}$.*

*Proof.* According to (2.18), $\varkappa_1(v - \bar{b}(x))^2 \leq L(v, x) \leq \varkappa_2(v - \bar{b}(x))^2$ with some $0 < \varkappa_1 \leq \varkappa_2$. One can easily check that the desired statement follows from variational representation (2.12) of the Aubry set and the assumptions on $\bar{b}$. □

Hence, by Theorem 4, $W(x) = d_{\overline{H}}(x, \xi^2)$ and $\lambda_\varepsilon = \varepsilon\sigma(\xi^2) + \bar{o}(\varepsilon)$.

It is interesting to trace in this example the possible structure of the set $\mathcal{Z} = \{x \in \Omega;\ W(x) = 0\}$. Observe that this set can also be defined as the set of points at which the eigenfunction $u_\varepsilon$ does not show an exponential decay, as $\varepsilon \to 0$. Its structure depends on whether there are trajectories of the equation $\dot{x} = -\bar{b}(x)$ going from $\xi^1$ or $\xi^3$ to $\xi^2$, or not.

Let $\mathcal{Z}^1$ be the set of all points $y \in \overline{\Omega}$ such that $\lim_{t\to+\infty} x^y(t) = \xi^1$ and $\lim_{t\to-\infty} x^y(t) = \xi^2$, and let $\mathcal{Z}^3$ be the set of all points $y \in \overline{\Omega}$ such that $\lim_{t\to+\infty} x^y(t) = \xi^3$ and $\lim_{t\to-\infty} x^y(t) = \xi^2$. Notice that the sets $\mathcal{Z}^1$ and $\mathcal{Z}^3$ might be empty.

**Proposition 21.** *We have* $\mathcal{Z} = \{\xi^2\} \cup \overline{\mathcal{Z}^1} \cup \overline{\mathcal{Z}^3}$.

*Proof.* The desired statement easily follows from (2.11) and the fact that $W(x) = d_{\overline{H}}(x, \xi^2)$. □

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## Appendices

### A. Uniqueness of additive eigenfunction and variational representation of the Aubry set

The following simple result provides a uniqueness criterion for problem (1.4)–(1.5).

**Proposition 22.** *Let $\lambda = \lambda_{\overline{H}}$ so that (1.4)–(1.5) has a solution $W$. Then $W$ is unique (up to an additive constant) if and only if $S_{\overline{H}-\lambda}(x,y) = 0$ for all $x,y \in \mathcal{A}_{\overline{H}-\lambda}$, where $S_{\overline{H}-\lambda}(x,y) = d_{\overline{H}-\lambda}(x,y) + d_{\overline{H}-\lambda}(y,x)$.*

*Proof.* If $S_{\overline{H}-\lambda}(x,y) = 0$ then $W(x) - W(y) = d_{\overline{H}-\lambda}(x,y)$; this follows from the fact that for any $x,y$ we have $W(x) - W(y) \leq d_{\overline{H}-\lambda}(x,y)$. In particular, if $S_{\overline{H}-\lambda}(x,y) = 0$ for all $x,y \in \mathcal{A}_{\overline{H}-\lambda}$, then taking $\xi \in \mathcal{A}_{\overline{H}-\lambda}$, we get $W(x) = d_{\overline{H}-\lambda}(x,\xi) + W(\xi)$ on $\mathcal{A}_{\overline{H}-\lambda}$. Thus, according to the representation formula (2.13), $W(x) = d_{\overline{H}-\lambda}(x,\xi) + W(\xi)$ in $\Omega$, i.e. $W$ is unique up to an additive constant.

If there are two points $\xi,\xi' \in \mathcal{A}_{\overline{H}-\lambda}$ such that $S_{\overline{H}-\lambda}(\xi,\xi') > 0$, then $W_0(x) = d_{\overline{H}-\lambda}(x,\xi)$ and $W_1(x) = d_{\overline{H}-\lambda}(x,\xi') - d_{\overline{H}-\lambda}(\xi,\xi')$ are two solutions of (1.4)–(1.5) and $0 = W_0(\xi) = W_1(\xi)$, while $W_0(\xi') - W_1(\xi') = S_{\overline{H}-\lambda}(\xi,\xi') > 0$. □

The following statement justifies the variational definition of Aubry set given by (2.12).

**Proposition 23.** *Let $\overline{H}(p,x) \in C(\mathbb{R}^N \times \overline{\Omega})$ be a convex in $p$ Hamiltonian such that $\min\{H(p,x)/|p|;\, x \in \overline{\Omega}\} \to +\infty$ as $|p| \to \infty$. Assume that $\lambda_{\overline{H}}$ is the additive eigenvalue of problem (2.6)–(2.7). Then the Aubry set $\mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$ defined by (2.10) can be equivalently given by*

$$y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}} \Longleftrightarrow \sup_{\delta>0} \inf \left\{ \int_0^t (\overline{L}(\dot{\eta},\eta) + \lambda_{\overline{H}})\, d\tau \,;\, \eta(0) = \eta(t) = y, t > \delta \right\} = 0; \qquad \text{(A.1)}$$

*the infimum here is taken over absolutely continuous curves $\eta : [0,t] \to \overline{\Omega}$.*

*Proof.* Let $y \in \mathcal{A}_{\overline{H}-\lambda_{\overline{H}}}$. Then $u(t,x) := d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$, being a solution of (2.6)–(2.7), is a stationary viscosity subsolution of $\frac{\partial u}{\partial t} + \overline{H}(\nabla u, x) - \lambda_{\overline{H}} = 0$ in $(0,+\infty) \times \Omega$ and a supersolution of this equation in $(0,+\infty) \times \overline{\Omega}$. According to [14, item (5) of Theorem X.1],

$u(t,x)$ is given by the Lax-Oleinik formula

$$u(t,x) = \inf\left\{\int_0^t (\overline{L}(\dot\eta,\eta) + \lambda_{\overline{H}})\,d\tau + d_{\overline{H}-\lambda_{\overline{H}}}(z,y);\ \eta(0)=z,\ \eta(t)=x,\ z\in\overline{\Omega}\right\}. \quad \text{(A.2)}$$

The minimization here (and below) is taken over absolutely continuous curves $\eta$ in $\overline{\Omega}$. Then, using the representation formula (2.11) for $d_{\overline{H}-\lambda_{\overline{H}}}(\eta,y)$ we have, for every $\delta > 0$,

$$\begin{aligned} 0 &= d_{\overline{H}-\lambda_{\overline{H}}}(y,y) = u(\delta,y) \\ &= \inf\left\{\int_0^\delta (\overline{L}(\dot\eta,\eta)+\lambda_{\overline{H}})\,d\tau + \int_0^{t'} (\overline{L}(\dot\eta',\eta')+\lambda_{\overline{H}})\,d\tau;\ \eta'(0)=\eta(t)=y,\ \eta'(t')\right. \\ &\quad \left.=\eta(0),\ t'>0\right\} \\ &= \inf\left\{\int_0^t (\overline{L}(\dot\eta'',\eta'')+\lambda_{\overline{H}})\,d\tau;\ \eta''(0)=\eta''(t)=y,\ t>\delta\right\}. \end{aligned}$$

Conversely, let $\{t^n\}_{n=1}^\infty$ be a sequence of positive real numbers such that $t^n \to \infty$, and

$$\Delta_n := \inf\left\{\int_0^{t^n} (\overline{L}(\dot\eta,\eta)+\lambda_{\overline{H}})\,d\tau;\ \eta(0)=\eta(t^n)=y\right\} \to 0, \quad \text{as } n\to\infty.$$

Consider $u(t,x)$ given by (A.2). In view of (2.11) we have $u(t,x) \geq \inf\{d_{\overline{H}-\lambda_{\overline{H}}}(x,z) + d_{\overline{H}-\lambda_{\overline{H}}}(z,y);\ z\in\overline{\Omega}\} \geq d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ . Since (2.6)–(2.7) has a solution $W$, after adding a constant to $W$ (if necessary) we obtain $u(t,x) \leq W(x) + C \leq C_1$ by the comparison principle (see [14, Section III]). According to [14, item (5) of Theorem X.1], the functions $v_s(t,x) = u(t+s,x)$ are uniformly Lipschitz continuous on $[0,+\infty)\times\overline{\Omega}$ for $s \geq 1$. Therefore,

$$w(x) = \liminf_{\tau\to\infty} u(\tau,x) = \lim_{r\to\infty}\inf_{s\geq r} u(s,x) = \lim_{r\to\infty}\inf_{s\geq r} u(t+s,x)$$

is a Lipschitz continuous supersolution of

$$\frac{\partial w}{\partial t} + \overline{H}(\nabla w,x) - \lambda_{\overline{H}} = 0 \quad \text{in } (0,+\infty)\times\overline{\Omega}.$$

Since $w$ does not depend on $t$, it is also a supersolution of the equation $\overline{H}(\nabla w,x) - \lambda_{\overline{H}} = 0$ in $\overline{\Omega}$. On the other hand

$$\begin{aligned} w(x) &\leq \limsup_{n\to\infty}\inf_{t>0} u(t^n+t,x) \\ &\leq \inf\left\{\int_0^t (\overline{L}(\dot\eta,\eta)+\lambda_{\overline{H}})\,d\tau;\ \eta(t)=x,\ \eta(0)=y,\ t>0\right\} + \limsup_{n\to\infty}\Delta_n = d_{\overline{H}-\lambda_{\overline{H}}}(x,y). \end{aligned}$$

Thus $d_{\overline{H}-\lambda_{\overline{H}}}(x,y) = w(x)$, and, therefore, $d_{\overline{H}-\lambda_{\overline{H}}}(x,y)$ satisfies (2.7). □

## B. Aubry set for small perturbations of a gradient field

We outline here the proof of the claim stated in Remark 5.

**Lemma 24.** *Let the vector field $b(x,y)$ is a $C^1$-small perturbation of a gradient field $\nabla P(x)$ with $C^2$ potential $P(x)$, and assume that*

- *the set $\{x\in\overline{\Omega};\ \nabla P(x)=0\}$ is formed by a finite collection of points in $\Omega$,*
- *the Hessian matrix $\left(\frac{\partial^2}{\partial x_i\partial x_j}P(x)\right)_{i,j=\overline{1,N}}$ at every such a point is nonsingular.*

*Then condition (2.19) is satisfied.*

*Proof.* Consider a vector field $b(x,y)$ which is a $C^1$-small perturbation of $\nabla P(x)$, i.e. $\|b(x,y)-\nabla P(x)\|_{C^1(\overline{\Omega}\times\overline{Y})} = \delta$, and $\delta$ is sufficiently small. Let us show that the Aubry set $\mathcal{A}_{\overline{H}}$ of the Hamiltonian $\overline{H}(p,x)$ given by (2.4) (with $c(x,y)=0$) is exactly the set of zeros of $\bar{b}(x)$ in $\Omega$, provided that $\delta$ is sufficiently small and $P\in C^2(\overline{\Omega})$ satisfies the conditions specified in Remark 5. Since the Aubry set of this Hamiltonian coincides with that of the effective Hamiltonian given by (2.4), we can assume without loss of generality that $\overline{H}(p,x)=\sum p_i^2-\bar{b}^i(x)p_i$. Let us first find the Aubry set $\mathcal{A}_{H^0}$ of the Hamiltonian

$$H^0(p,x)=\sum p_i^2-p_i\frac{\partial P(x)}{\partial x_i}.$$

We calculate the corresponding Lagrangian $L^0(v,x)=\frac{1}{4}|v+\nabla P(x)|^2$ and use criterion (2.12). Let $\xi\in\mathcal{A}_{H^0}$, then there exist a sequence of absolutely continuous curves $\eta^n:[0,t^n]\to\overline{\Omega}$, $\eta^n(0)=\eta^n(t^n)=\xi$, such that $t^n\to\infty$ and

$$\lim_{n\to\infty}\int_0^{t^n}|\dot{\eta}^n+\nabla P(\eta^n)|^2\mathrm{d}\tau=0.$$

This yields

$$0=\lim_{n\to\infty}\int_0^{t^n}(|\dot{\eta}^n|^2+2\nabla P_\delta(\eta^n)\cdot\dot{\eta}^n+|\nabla P(\eta^n)|^2)\mathrm{d}\tau=\lim_{n\to\infty}\int_0^{t^n}(|\dot{\eta}^n|^2+|\nabla P(\eta^n)|^2)\mathrm{d}\tau.$$

Therefore, $\eta^n(t)\to\xi$ uniformly on every fixed interval $[0,T]$. It follows that $\xi$ belongs to the set $K=\{x\in\Omega;\ \nabla P(x)=0\}$. Clearly, we also have $K\subset\mathcal{A}_{H^0}$. Now note that the effective drift $\bar{b}(x)$ given by (2.16), can be written as $\bar{b}(x)=\nabla P(x)+\tilde{b}_\delta(x)$ with $C^1$-small $\tilde{b}_\delta(x)$, $\|\tilde{b}_\delta\|_{C^1}=\overline{O}(\delta)$ as $\delta\to 0$. The latter bound readily follows from the regularity of $\theta^*$ and perturbation arguments. Thanks to the assumption on the critical points of $P(x)$, by the implicit function theorem, zeros of $\bar{b}(x)$ are isolated and are close to $K$ when $\delta$ is sufficiently small. Moreover, if $\omega$ is a small neighborhood of $\xi\in K$ then $\bar{b}(x)$ vanishes at exactly one point $\xi_\delta\in\omega$ and $|\xi-\xi_\delta|=O(\delta)$. Therefore, we can define a $C^2$ function $P_\delta$ such that $|\nabla P_\delta(x)|>0$ in $\overline{\Omega}\backslash K_\delta$, where $K_\delta$ is the set of zeros of $\bar{b}(x)$, and $|\bar{b}(x)-\nabla P_\delta(x)|=g_\delta(x)|\nabla P_\delta(x)|$ with $\max_{x\in\overline{\Omega}}g_\delta(x)=\overline{O}(\delta)$ as $\delta\to 0$. This yields the following bound (for small $\delta$)

$$|v+\bar{b}(x)|^2\geq\frac{1}{2}|v|^2+2\nabla P_\delta(x)\cdot v+V_\delta(x),\quad\forall v\in\mathbb{R}^N,\ x\in\overline{\Omega},$$

where $V_\delta>0$ in $\overline{\Omega}\backslash K_\delta$. Then, arguing as above we see that $\mathcal{A}_{\overline{H}}=K_\delta$. Moreover, every $\xi\in K_\delta$ is a hyperbolic fixed point of the ODE $\dot{x}=-\bar{b}(x)$, as $\delta$ is sufficiently small. □

## C. Properties of solutions of Bernoulli matrix equation

We provide here some results on Bernoulli equation (6.8), used in Sections 6 and 7. Recall that the matrix $Q$ in (6.8) is positive definite, $\Pi_s$ and $\Pi_u$ denote spectral projectors on the invariant subspaces of the matrix $B$ corresponding to eigenvalues with positive and negative real parts.

**Proposition 25.** *The maximal positive semi-definite solution $\Gamma$ of (6.8) possesses the following properties: (i) $\Gamma=\Pi_s^*\Gamma\Pi_s$, (ii) $\Gamma\geq\gamma\Pi_s^*\Pi_s$ (in the sense of quadratic forms) for some $\gamma>0$, (iii) $2\mathrm{tr}(Q\Gamma)=\mathrm{tr}(B\Pi_s)$, i.e. $2\mathrm{tr}(Q\Gamma)$ is the sum of positive real parts of eigenvalues of $B$.*

*Proof.* It follows from (6.8) that $X = \Pi_u^* \Gamma \Pi_u$ satisfies

$$4\Pi_u^* \Gamma Q \Gamma \Pi_u - X(B\Pi_u) - (B\Pi_u)^* X = 0. \tag{C.1}$$

Consider the symmetric solution of (C.1) given by

$$\widetilde{X} = \int_0^\infty Y(t)\,\mathrm{d}t, \tag{C.2}$$

where $Y(t) = -4e^{(B\Pi_u)^* t}\Pi_u^* \Gamma Q \Gamma \Pi_u e^{B\Pi_u t}$ (note that $\dot{Y}(t) = Y(t)(B\Pi_u) + (B\Pi_u)^* Y(t)$ and $Y(t) \to 0$ as $t \to +\infty$, therefore integrating we get $\widetilde{X}(B\Pi_u) + (B\Pi_u)^*\widetilde{X} = -Y(0) = 4\Pi_u^*\Gamma Q \Gamma \Pi_u$, i.e. $\widetilde{X}$ does solve (C.1)). We claim that $X = \widetilde{X}$. Otherwise $Z := X - \widetilde{X}$ is a nonzero solution of equation $Z(B\Pi_u) + (B\Pi_u)^* Z = 0$ and $Z = \Pi_u^* Z \Pi_u$. Then $Z(t) = Z$ is a stationary solution of the differential equation $\dot{Z}(t) = Z(t)(B\Pi_u) + (B\Pi_u)^* Z(t)$. The latter equation has the solution $\widetilde{Z}(t) = e^{(B\Pi_u)^* t}\Pi_u^* Z \Pi_u e^{B\Pi_u t}$ which vanishes as $t \to +\infty$ and satisfies the initial condition $\widetilde{Z}(0) = Z$. Thus $Z = 0$, i.e. $X = \widetilde{X}$. On the other hand it follows from (C.2) that $\widetilde{X} \leq 0$ while $X \geq 0$, this yields $X = \widetilde{X} = 0$. Since $\Gamma$ is positive semi-definite we also have $\Gamma \Pi_u = \Pi_u^* \Gamma = 0$ and the calculation $\Gamma = (\Pi_u + \Pi_s)^* \Gamma (\Pi_u + \Pi_s) = \Pi_s^* \Gamma \Pi_s$ completes the proof of (i). As a byproduct we also have established that $\Gamma$ is the maximal positive semi-definite solution of

$$4\Gamma Q \Gamma - \Gamma(B\Pi_s) - (B\Pi_s)^* \Gamma = 0. \tag{C.3}$$

Indeed, assuming that $\widetilde{\Gamma}$ is another positive semi-definite solution of (C.3) we get $\Pi_u^* \widetilde{\Gamma} Q \widetilde{\Gamma} \Pi_u = 0$. This yields $\widetilde{\Gamma}\Pi_u = 0$ so that $\widetilde{\Gamma} = \Pi_s^* \widetilde{\Gamma} \Pi_s$, therefore $\widetilde{\Gamma} B = \Pi_s^* \widetilde{\Gamma} (\Pi_s)^2 B = \widetilde{\Gamma}(B\Pi_s)$ and $\widetilde{\Gamma}$ thus solves (6.8).

To show (ii) and (iii) consider the maximal positive definite solution $\widetilde{\Gamma}_\delta$ of

$$4\widetilde{\Gamma}_\delta Q \widetilde{\Gamma}_\delta - \widetilde{\Gamma}_\delta (B\Pi_s + \delta I) - (B\Pi_s + \delta I)^* \widetilde{\Gamma}_\delta = 0 \tag{C.4}$$

for $\delta > 0$. The existence of the unique positive definite solution follows from the fact that $\widetilde{\Gamma}_\delta^{-1}$ is the unique solution of the Lyapunov matrix equation

$$4Q - (B\Pi_s + \delta I)\widetilde{\Gamma}_\delta^{-1} - \widetilde{\Gamma}_\delta^{-1}(B\Pi_s + \delta I)^* = 0 \tag{C.5}$$

given by

$$\widetilde{\Gamma}_\delta^{-1} = 4\int_{-\infty}^0 e^{(B\Pi_s + \delta I)t} Q e^{(B\Pi_s + \delta I)^* t}\,\mathrm{d}t.$$

It is known (see [31, Theorem 11.2.1]) that $\widetilde{\Gamma}_\delta$ converges to the (maximal positive semi-definite) solution $\Gamma$ of (C.3) as $\delta \to +0$. This allows to establish (iii) easily,

$$2\mathrm{tr}(Q\Gamma) = 2\lim_{\delta \to +0} 2\mathrm{tr}(Q\widetilde{\Gamma}_\delta) = \frac{1}{2}\lim_{\delta \to +0} \mathrm{tr}\left(\widetilde{\Gamma}_\delta^{-1}(B\Pi_s + \delta I)^* \widetilde{\Gamma}_\delta + (B\Pi_s + \delta I)\right) = \mathrm{tr}(B\Pi_s).$$

Finally, if we assume that (ii) is false, then there is $\eta \in \mathbb{R}^N$ such that $\Gamma\eta = 0$ while $\Pi_s \eta \neq 0$. On the other hand, $\Gamma\left(\lim_{\delta \to +0} \widetilde{\Gamma}_\delta^{-1} \Pi_s^* \Pi_s \eta\right) = \Pi_s^* \Pi_s \eta$, where the limit $\lim_{\delta \to +0} \widetilde{\Gamma}_\delta^{-1} \Pi_s^* \Pi_s \eta$ exists, for $e^{(B\Pi_s + \delta I)t} Q e^{(B\Pi_s + \delta I)^* t} \Pi_s^* \Pi_s \eta$ decays exponentially fast as $t \to -\infty$, uniformly in $\delta \geq 0$. According to the Fredholm alternative $\Pi_s^* \Pi_s \eta$ and $\eta$ must be orthogonal, yielding $|\Pi_s \eta| = 0$. We obtained a contradiction showing that (ii) does hold. □