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Asymptotic analysis and domain decomposition for a biharmonic problem in a thin multi-structure

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In the paper, we consider the Dirichlet boundary value problem for the biharmonic equation defined in a thin T-like shaped structure. Our goal is to construct an asymptotic expansion of its solution. We provide error estimates and also introduce and justify the asymptotic partial domain decomposition for this problem.

Keywords: Biharmonic equation; thin structure; asymptotic expansion; asymptotic partial domain decomposition.

Mathematics Subject Classification 2010: 35J40, 74K30

1. Introduction

A multi-structure is a connected domain of $\mathbb{R}^k$, usually $k = 2, 3$, composed of several components, these components might have similar or rather different geometric structure. For instance, some components might be thin in certain directions (see [8, 9, 11, 17, 21, 25] and references therein). In many applications various phenomena in multi-structures can be described in terms of boundary value problems for (system of) partial differential equations.

The commonly used method of studying thin structures and multi-structures with thin components is the so-called dimension reduction. From the mathematical point of view the idea of this method is to consider the thickness as a parameter, to pass to the limit in the studied problem as this parameter tends to zero and to obtain the limit model defined in a domain of lower dimension. In particular, the structural mechanics deals with such limit models for the elasticity system. In some applications the precision of approximation given by the limit model is not satisfactory. To improve the precision one can try to construct higher order terms of the asymptotic expansion of solutions with respect to the powers of a small parameter characterizing the structure thickness. It is then important to obtain estimates for the rate of convergence.

The limit behavior of solutions of elliptic boundary value problems defined in asymptotically thin domains depends essentially on the choice of boundary conditions. Solutions of Neumann boundary value problem in thin domains usually converge to a solution of the limit boundary value problem defined in a domain of lower dimension which does not depend on a small parameter. On the contrary, solutions to the Dirichlet problem often show more complicated behavior. For a wide class of problems an appropriate tool for studying the limit behavior of solutions is the method of asymptotic expansions. Dirichlet problems for elliptic equations defined in thin structures were considered in [19, 20]. The Stokes and Navier–Stokes equations with the no slip boundary conditions were studied in [5, 28, 29]. However, to our best knowledge, the problem of constructing the asymptotic expansion for a solution of biharmonic equation in a thin multi-structure has not been addressed in the existing literature. Boundary value problems for biharmonic operator play an important role in the two-dimensional plate models (for instance, see [10, 16, 31] and references therein). The theory of clamped elastic shells and plates of a complex geometry is widely used for designing chips (see [23]), in this model a thin narrow strip-like structure is exposed to some force having electro-magnetic nature.

The goal of this paper is to construct an asymptotic expansion of solutions to the Dirichlet boundary value problem for biharmonic equation defined in a thin multi-structure. For the sake of presentation simplicity we consider here only the case of T-like shaped domains (see Fig. 1). However our approach also applies to general rod structures (see [30, 26]).

For each $\varepsilon \in]0,1[$ we define a domain Ω_ε as follows (see Fig. 1):

$$\Omega_\varepsilon = (\,]-1,1[\times]-\varepsilon,0\,[\,) \cup \left(\left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\times[0,1[\right) \subset \mathbb{R}^2.$$

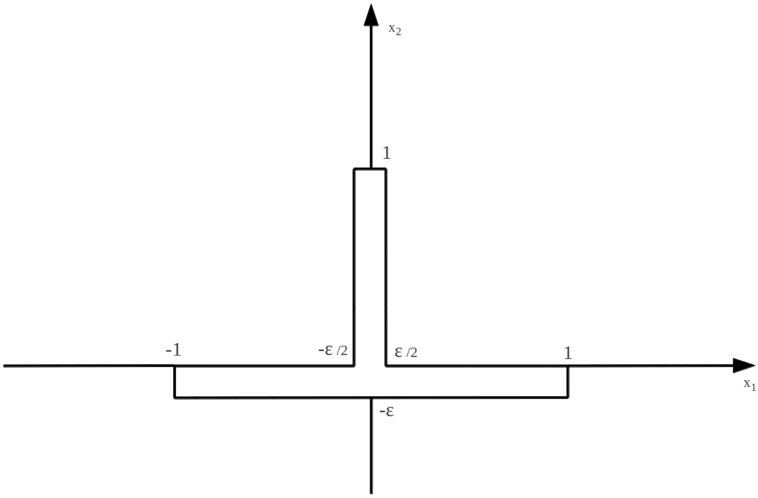

Fig. 1. Ω_ε.

Given two functions $f_1 : [-1, 1] \mapsto \mathbb{R}$ and $f_2 : [-1, 0] \mapsto \mathbb{R}$ such that, for some $N \in \mathbb{N} \cap [4, +\infty[$,

$$\begin{cases} f_1 \in C^{N+2}([-1,1]), \quad f_2 \in C^{N+2}([0,1]) : \\ \exists\, a \in]0,1[, \quad \exists\, b \in \mathbb{R} : f_1(x_1) = b, \quad \forall\, x_1 \in [-a, a], \\ \quad f_2(x_2) = b, \quad \forall\, x_2 \in [0, a], \end{cases} \tag{1.1}$$

we introduce

$$f : (x_1, x_2) \in \Omega_\varepsilon \to f(x_1, x_2) = \begin{cases} f_1(x_1) & \text{if } (x_1, x_2) \in]-1, 1[\times]-\varepsilon, 0[\,, \\ f_2(x_2) & \text{if } (x_1, x_2) \in \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\times [0, 1[\,. \end{cases}$$

In Ω_ε, we consider the following problem

$$\begin{cases} \triangle^2 u_\varepsilon = f, \quad \text{in } \Omega_\varepsilon, \\ u_\varepsilon = 0 = \dfrac{\partial u_\varepsilon}{\partial \nu}, \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{1.2}$$

where ν denotes the unit outer normal on $\partial\Omega_\varepsilon$. Define a weak solution to this problem as

$$u_\varepsilon \in H_0^2(\Omega_\varepsilon); \quad \int_{\Omega_\varepsilon} \triangle u_\varepsilon \triangle v dx = \int_{\Omega_\varepsilon} f v dx, \quad \forall\, v \in H_0^2(\Omega_\varepsilon). \tag{1.3}$$

Applying the Riesz representation theorem (or the Lax–Milgram lemma) as in [18] (referring to [32]), one can see that problem (1.3) admits a unique solution for any $f \in L^2(\Omega_\varepsilon)$ (possibly not satisfying regularity conditions (1.1)) and the weak solution satisfies *a priori* estimates

$$\|u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \leq C \|f\|_{L^2(\Omega_\varepsilon)}, \quad \forall\, \varepsilon > 0,$$

with a constant C independent of ε. This estimate follows from identity (1.3) with $v = u_\varepsilon$ and from the Cauchy–Schwarz and Poincaré–Friedrichs inequalities.

In this paper we obtain two main results.

(1) We construct and justify the complete asymptotic expansion of the solution of problem (1.2), this expansion having the form

$$\begin{aligned} u_\varepsilon^N &= \rho\left(\frac{x_1}{\varepsilon}\right) u_{1,\varepsilon}^{Nreg}(x_1,x_2) + \rho\left(\frac{x_2}{\varepsilon}\right) u_{2,\varepsilon}^{Nreg}(x_1,x_2) \\ &\quad + u_{-1,\varepsilon}^{Nbl}\left(\frac{x_1+1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\eta(x_1+1) + u_{1,\varepsilon}^{Nbl}\left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\eta(x_1-1) \\ &\quad + u_{2,\varepsilon}^{Nbl}\left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon}\right)\eta(x_2-1) + \varepsilon^4 u_0^{Nbl}\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}\right)\eta(x_1)\eta(x_2), \end{aligned} \tag{1.4}$$

with

$$\begin{cases} \rho \in C^\infty(\mathbb{R}, [0,1]), \\ \rho = 0, \quad \text{in } \left[-\dfrac{1}{2}, \dfrac{1}{2}\right], \\ \rho = 1, \quad \text{in } \mathbb{R} \Big\backslash \left]-\dfrac{2}{3}, \dfrac{2}{3}\right[\end{cases}$$

and

$$\eta = 1 - \rho.$$

The structure of the expansion in (1.4) is suggested by the following arguments. We look for a smooth asymptotic expansion of the solution of biharmonic equation $\triangle^2 w = f_1$ in $]-1,1[\times]-\varepsilon,0[$, satisfying the homogeneous Dirichlet boundary conditions on $]-1,1[\times\{-\varepsilon,0\}$, in the form

$$u_{1,\varepsilon}^{Nreg}(x_1,x_2) = \sum_{j=0}^{N} \varepsilon^j u_{1,j}\left(x_1, \frac{x_2}{\varepsilon}\right), \tag{1.5}$$

with smooth functions

$$u_{1,j} : [-1,1] \times [-1,0] \to \mathbb{R},$$

which can be explicitly computed. Namely,

$$u_{1,j}(x_1,\xi_2) = \begin{cases} 0 & \text{if } j \in \{0,1,2,3\}, \\ \mathcal{N}_j(\xi_2) f_1^{(j-4)}(x_1) & \text{if } j \in \mathbb{N} \cap [4,N], \end{cases} \quad \forall\, (x_1,\xi_2) \in [-1,1] \times [-1,0],$$

where $\mathcal{N}_j$ are the solutions of the following recurrent chain of problems:

$$\begin{cases} \mathcal{N}_j^{(4)} + 2\mathcal{N}_{j-2}^{(2)} + \mathcal{N}_{j-4} = \delta_{j4}, \quad \text{in }]-1,0[, \\ \mathcal{N}_j(-1) = 0 = \mathcal{N}_j'(-1), \\ \mathcal{N}_j(0) = 0 = \mathcal{N}_j'(0), \end{cases}$$

with $\mathcal{N}_j = 0$ for $j < 4$, and δ_{j4} denoting the Kronecker delta defined in (2.6). For instance, $\mathcal{N}_4(\xi_2) = \frac{1}{4!}(\xi_2 + 1)^2\,\xi_2^2$, $\xi_2 \in \mathbb{R}$.

Unfortunately, expansion (1.5) does not fit the homogeneous Dirichlet boundary conditions on $\{-1, 1\} \times]-\varepsilon, 0[$. Then, in order to compensate the error on $\{-1, 1\} \times]-\varepsilon, 0[$, we add boundary layer correctors of the form

$$u_{-1,\varepsilon}^{Nbl}(x_1, x_2) = \sum_{j=4}^{N} \varepsilon^j u_{-1,j}^{Nbl}\left(\frac{x_1 + 1}{\varepsilon}, \frac{x_2}{\varepsilon}\right),$$

$$u_{1,\varepsilon}^{Nbl}(x_1, x_2) = \sum_{j=4}^{N} \varepsilon^j u_{1,j}^{Nbl}\left(\frac{x_1 - 1}{\varepsilon}, \frac{x_2}{\varepsilon}\right),$$

with

$$u_{-1,j}^{Nbl} : \xi \in]0, +\infty[\times]-1, 0[\to u_{-1,j}^{Nbl}(\xi) \in \mathbb{R},$$

$$u_{1,j}^{Nbl} : \xi \in]-\infty, 0[\times]-1, 0[\to u_{1,j}^{Nbl}(\xi) \in \mathbb{R}$$

satisfying appropriate boundary value problems for the biharmonic equation in an infinite strip and having exponential decay when $|\xi| \to \infty$ (see (3.1) and (3.8)). The exponential decay of the solution to these problems is not evident. In particular, the Fourier series expansion with respect to the transversal variable fails to work. So we need to apply the general theory and to check its applicability in the case of the biharmonic operator.

Similarly, letting

$$u_{2,\varepsilon}^{Nreg}(x_1, x_2) = \sum_{j=4}^{N} \varepsilon^j f_2^{(j-4)}(x_2)\mathcal{M}_j\left(\frac{x_1}{\varepsilon}\right), \quad \forall\,(x_1, x_2) \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \times [0, 1], \tag{1.6}$$

where $\mathcal{M}_j$ are the solutions of the recurrent chain of problems

$$\begin{cases} \mathcal{M}_j^{(4)} + 2\mathcal{M}_{j-2}^{(2)} + \mathcal{M}_{j-4} = \delta_{j4}, & \text{in } \left]-\frac{1}{2}, \frac{1}{2}\right[, \\ \mathcal{M}_j\left(-\frac{1}{2}\right) = 0 = \mathcal{M}_j'\left(-\frac{1}{2}\right), \\ \mathcal{M}_j\left(\frac{1}{2}\right) = 0 = \mathcal{M}_j'\left(\frac{1}{2}\right), \end{cases}$$

with $\mathcal{M}_j = 0$ for $j < 4$ (for instance, $\mathcal{M}_4 = \frac{1}{4!}(\xi_1 - \frac{1}{2})^2(\xi_1 + \frac{1}{2})^2, \xi_1 \in \mathbb{R}$), we obtain a smooth asymptotic expansion of the solution of biharmonic equation $\triangle^2 w = f_2$ in $]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[\times]0, 1[$, satisfying homogeneous Dirichlet boundary conditions on $\{-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\} \times]0, 1[$, and $u_{2,\varepsilon}^{Nbl} = \sum_{j=4}^{N} \varepsilon^j u_{2,j}^{Nbl}(\frac{x_1}{\varepsilon}, \frac{x_2 - 1}{\varepsilon})$ is the corresponding boundary layer corrector compensating the error on $]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[\times \{1\}$ and decaying exponentially in the vicinity of the end point (see (3.9)).

Finally, $u_0^{Nbl}(\xi)$ defined by (4.6) is the boundary layer corrector exponentially decaying as $|\xi| \to +\infty$, required for matching $u_{1,\varepsilon}^{Nreg}$ with $u_{2,\varepsilon}^{Nreg}$.

Then, we prove that

$$\|u_\varepsilon - u_\varepsilon^N\|_{H^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-1}\sqrt{\varepsilon}, \tag{1.7}$$

where c_N denotes a positive constant independent of ε, but depending on N.

(2) By using estimate (1.7) and implementing the method of asymptotic partial domain decomposition of the Dirichlet boundary value problem for the biharmonic equation, we construct and justify the approximation $u_\varepsilon^{\mathrm{dec}}$. Namely, we prove that

$$\|u_\varepsilon - u_\varepsilon^{\mathrm{dec}}\|_{H^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-3}\sqrt{\varepsilon},$$

where

$$u_\varepsilon^{\mathrm{dec}}(x_1,x_2) = \begin{cases} \displaystyle\sum_{j=4}^{N} \varepsilon^j f_1^{(j-4)}(x_1)\mathcal{N}_j\left(\frac{x_2}{\varepsilon}\right) & \text{for } x_2 > 0, \\ \displaystyle\sum_{j=4}^{N} \varepsilon^j f_1^{(j-4)}(x_2)\mathcal{M}_j\left(\frac{x_1}{\varepsilon}\right) & \text{for } x_2 < 0, \end{cases}$$

if the distance from the point (x_1, x_2) to the extremities and to the "center" of the multi-structure is bigger than $\delta_\varepsilon \simeq N\varepsilon|\log\varepsilon|$. In the remaining part of Ω_ε, called $\Omega_\varepsilon^{\mathrm{dec}}$ (see Fig. 2), $u_\varepsilon^{\mathrm{dec}}$ solves biharmonic equation $\triangle^2 u_\varepsilon^{\mathrm{dec}} = f$ with appropriate Dirichlet boundary conditions (see problem (6.2)–(6.3)).

The method of asymptotic partial decomposition of domain (MAPDD) was introduced in [27] and then developed in [30]. This method reduces the 2D or 3D model of a thin structure to some model of hybrid dimension (2-1 or 3-1) conserving the dimension on a small part where the behavior of solution is singular, and reducing the dimension in the main part of the domain where the behavior of

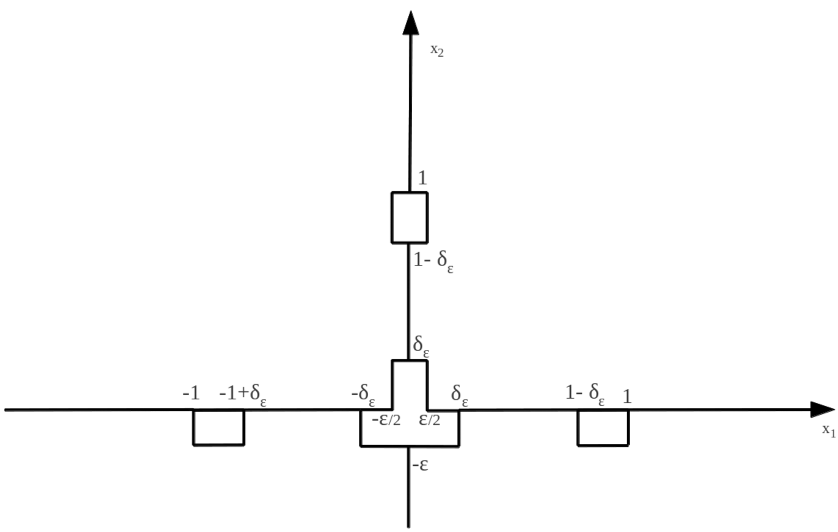

Fig. 2. $\Omega_\varepsilon^{\mathrm{dec}}$.

the solution is regular. This approach corresponds to a special dimension reduction procedure with some local zooms. Here we apply this method to problem (1.3). In this case the decomposition is complete and it makes this idea more advantageous for problem (1.3). The complete asymptotic decomposition allows us to parallelize the computations in the connected components of the set $\Omega_\varepsilon^{\text{dec}}$ (see Fig. 2).

We conclude discussing the case where the right-hand side is not C^N-smooth function.

Note that the choice of the right-hand side having structure (1.1) is standard for the analysis of thin structures. It depends on the longitudinal variable of the graph of the structure. However its shape can be generalized, and one can add some additional part depending on both variables with the support in some neighborhood of the point $(0, 0)$ or some part depending on the transversal variable. The method does not change and is still applicable to these situations.

For asymptotic expansion in periodic structures we refer to [4]. For asymptotic expansion in comb-like shaped domain, we refer to recent paper [3] and the references quoted therein. The results on dimension reduction and/or homogenization for fourth-order problems can be found in [6, 14, 15, 22]. An optimization problem for the biharmonic equation was studied in [7]. The works [12, 13] deal with dimension reduction for second-order problems in a thin T-like shaped domain, see also the papers quoted there.

2. Smooth Part of the Expansion

In this and the following two sections we weaken the regularity assumption in the first line of (1.1), assuming $f_1 \in C^N([-1,1])$ and $f_2 \in C^N([0,1])$.

This section is devoted to build (1.5) and (1.6).

We proceed with constructing (1.5). Assumption (1.5) provides that

$$\triangle^2 u_{1,\varepsilon}^{Nreg}(x_1,x_2) = \sum_{j=0}^{N} \varepsilon^j \left(\frac{\partial^4 u_{1,j}}{\partial x_1^4} + \varepsilon^{-2} 2 \frac{\partial^4 u_{1,j}}{\partial x_1^2 \partial \xi_2^2} + \varepsilon^{-4} \frac{\partial^4 u_{1,j}}{\partial \xi_2^4} \right) \left(x_1, \frac{x_2}{\varepsilon} \right),$$

$$\forall\, (x_1,x_2) \in [-1,1] \times [-\varepsilon, 0]. \tag{2.1}$$

By virtue of the change of index $j' = j+2$, the second term in the right-hand side of (2.1) can be rewritten in the following way

$$2 \sum_{j=0}^{N} \varepsilon^{j-2} \frac{\partial^4 u_{1,j}}{\partial x_1^2 \partial \xi_2^2} = 2 \sum_{j'=2}^{N+2} \varepsilon^{j'-4} \frac{\partial^4 u_{1,j'-2}}{\partial x_1^2 \partial \xi_2^2},$$

from which, setting $u_{1,j} = 0$ for j negative, it follows that

$$2 \sum_{j=0}^{N} \varepsilon^{j-2} \frac{\partial^4 u_{1,j}}{\partial x_1^2 \partial \xi_2^2} = 2 \sum_{j=0}^{N} \varepsilon^{j-4} \frac{\partial^4 u_{1,j-2}}{\partial x_1^2 \partial \xi_2^2} + 2\varepsilon^{N-3} \frac{\partial^4 u_{1,N-1}}{\partial x_1^2 \partial \xi_2^2} + 2\varepsilon^{N-2} \frac{\partial^4 u_{1,N}}{\partial x_1^2 \partial \xi_2^2}. \tag{2.2}$$

Similarly, making the change of index $j' = j+4$, the first term in the right-hand side of (2.1) can be rewritten in the following way

$$\sum_{j=0}^{N} \varepsilon^j \frac{\partial^4 u_{1,j}}{\partial x_1^4} = \sum_{j=0}^{N} \varepsilon^{j-4} \frac{\partial^4 u_{1,j-4}}{\partial x_1^4} + \varepsilon^{N-3} \frac{\partial^4 u_{1,N-3}}{\partial x_1^4} + \varepsilon^{N-2} \frac{\partial^4 u_{1,N-2}}{\partial x_1^4} + \varepsilon^{N-1} \frac{\partial^4 u_{1,N-1}}{\partial x_1^4} + \varepsilon^{N} \frac{\partial^4 u_{1,N}}{\partial x_1^4}. \tag{2.3}$$

By combining (2.1) with (2.2) and (2.3), one has that

$$\begin{aligned} \triangle^2 u_{1,\varepsilon}^{Nreg}(x_1,x_2) = {} & \sum_{j=0}^{N} \varepsilon^{j-4} \left(\frac{\partial^4 u_{1,j}}{\partial \xi_2^4} + 2 \frac{\partial^4 u_{1,j-2}}{\partial x_1^2 \partial \xi_2^2} + \frac{\partial^4 u_{1,j-4}}{\partial x_1^4} \right) \left(x_1, \frac{x_2}{\varepsilon} \right) \\ & + \left[\varepsilon^{N-3} \left(2 \frac{\partial^4 u_{1,N-1}}{\partial x_1^2 \partial \xi_2^2} + \frac{\partial^4 u_{1,N-3}}{\partial x_1^4} \right) \right. \\ & + \varepsilon^{N-2} \left(2 \frac{\partial^4 u_{1,N}}{\partial x_1^2 \partial \xi_2^2} + \frac{\partial^4 u_{1,N-2}}{\partial x_1^4} \right) \\ & \left. + \varepsilon^{N-1} \frac{\partial^4 u_{1,N-1}}{\partial x_1^4} + \varepsilon^{N} \frac{\partial^4 u_{1,N}}{\partial x_1^4} \right] \left(x_1, \frac{x_2}{\varepsilon} \right), \\ & \forall (x_1,x_2) \in [-1,1] \times [-\varepsilon, 0], \end{aligned} \tag{2.4}$$

where $u_{1,j}$ is assumed null, for j negative.

Now, let us choose $u_{1,j}, j \in \{1, \dots, N\}$, such that

$$\begin{cases} \dfrac{\partial^4 u_{1,j}}{\partial \xi_2^4} + 2 \dfrac{\partial^4 u_{1,j-2}}{\partial x_1^2 \partial \xi_2^2} + \dfrac{\partial^4 u_{1,j-4}}{\partial x_1^4} = \delta_{j4} f_1, & \text{in }]-1,1[\times]-1,0[, \\ u_{1,j}(x_1,-1) = 0 = \dfrac{\partial u_{1,j}}{\partial \xi_2}(x_1,-1), & \forall\, x_1 \in]-1,1[, \\ u_{1,j}(x_1,0) = 0 = \dfrac{\partial u_{1,j}}{\partial \xi_2}(x_1,0), & \forall\, x_1 \in]-1,1[, \end{cases} \tag{2.5}$$

where

$$\delta_{j4} = \begin{cases} 1 & \text{if } j = 4, \\ 0 & \text{if } j \neq 4. \end{cases} \tag{2.6}$$

By applying a recursive method, it is easy to see that

$$u_{1,j}(x_1,\xi_2) = \begin{cases} 0 & \text{if } j \in \{0,1,2,3\}, \\ \mathcal{N}_j(\xi_2) f_1^{(j-4)}(x_1) & \text{if } j \in \mathbb{N} \cap [4,N], \end{cases} \quad \forall\, (x_1,\xi_2) \in [-1,1] \times [-1,0], \tag{2.7}$$

where

$$\mathcal{N}_j = \begin{cases} 0 & \text{if } j \text{ is odd,} \\ \text{a specific polynomial function of degree } j & \text{if } j \text{ is even.} \end{cases} \tag{2.8}$$

Precisely, $\mathcal{N}_j$ are solutions of the recurrent chain of problems

$$\begin{cases} \mathcal{N}_j^{(4)} + 2\mathcal{N}_{j-2}^{(2)} + \mathcal{N}_{j-4} = \delta_{j4}, \quad \text{in }]-1,0[, \\ \mathcal{N}_j(-1) = 0 = \mathcal{N}_j'(-1), \\ \mathcal{N}_j(0) = 0 = \mathcal{N}_j'(0), \end{cases}$$

with $\mathcal{N}_j = 0$ for $j < 4$. For instance,

$$\mathcal{N}_4(\xi_2) = \frac{1}{4!}(\xi_2+1)^2\xi_2^2, \quad \xi_2 \in \mathbb{R}.$$

Then, combining (1.5) with (2.7) and (2.8), it results in

$$u_{1,\varepsilon}^{Nreg}(x_1,x_2) = \sum_{j=4}^{N} \varepsilon^j f_1^{(j-4)}(x_1)\mathcal{N}_j\left(\frac{x_2}{\varepsilon}\right), \quad \forall\,(x_1,x_2) \in [-1,1]\times[-\varepsilon,0]. \tag{2.9}$$

Moreover, setting

$$\begin{aligned} r_{1,\varepsilon}^N(x_1,x_2) = \Bigg[&\varepsilon^{N-3}\left(2\frac{\partial^4 u_{1,N-1}}{\partial x_1^2\partial\xi_2^2} + \frac{\partial^4 u_{1,N-3}}{\partial x_1^4}\right) + \varepsilon^{N-2}\left(2\frac{\partial^4 u_{1,N}}{\partial x_1^2\partial\xi_2^2} + \frac{\partial^4 u_{1,N-2}}{\partial x_1^4}\right) \\ &+ \varepsilon^{N-1}\frac{\partial^4 u_{1,N-1}}{\partial x_1^4} + \varepsilon^N\frac{\partial^4 u_{1,N}}{\partial x_1^4}\Bigg]\left(x_1,\frac{x_2}{\varepsilon}\right), \\ &\forall\,(x_1,x_2)\in[-1,1]\times[-\varepsilon,0], \end{aligned} \tag{2.10}$$

from (2.4) and (2.5) it follows that

$$\begin{cases} \triangle^2 u_{1,\varepsilon}^{Nreg}(x_1,x_2) = f_1(x_1) + r_{1,\varepsilon}^N(x_1,x_2), \quad \forall\,(x_1,x_2)\in]-1,1[\times]-\varepsilon,0[, \\ u_{1,\varepsilon}^{Nreg}(x_1,-\varepsilon) = 0 = \dfrac{\partial u_{1,\varepsilon}^{Nreg}}{\partial x_2}(x_1,-\varepsilon), \quad \forall\,x_1\in]-1,1[, \\ u_{1,\varepsilon}^{Nreg}(x_1,0) = 0 = \dfrac{\partial u_{1,\varepsilon}^{Nreg}}{\partial x_2}(x_1,0), \quad \forall\,x_1\in]-1,1[. \end{cases} \tag{2.11}$$

Similarly, setting

$$u_{2,\varepsilon}^{Nreg}(x_1,x_2) = \sum_{j=4}^{N}\varepsilon^j f_2^{(j-4)}(x_2)\mathcal{M}_j\left(\frac{x_1}{\varepsilon}\right), \quad \forall\,(x_1,x_2)\in\left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]\times[0,1],$$

it results in

$$\begin{cases} \triangle^2 u_{2,\varepsilon}^{Nreg}(x_1,x_2) = f_2(x_2) + r_{2,\varepsilon}^N(x_1,x_2), & \forall\,(x_1,x_2) \in \left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\times]0,1[, \\ u_{2,\varepsilon}^{Nreg}\left(\pm\frac{\varepsilon}{2},x_2\right) = 0 = \frac{\partial u_{2,\varepsilon}^{Nreg}}{\partial x_2}\left(\pm\frac{\varepsilon}{2},x_2\right), & \forall\, x_2 \in]0,1[, \end{cases}$$

where

$$\begin{aligned} r_{2,\varepsilon}^N(x_1,x_2) = \Bigg[&\varepsilon^{N-3}\left(2\frac{\partial^4 u_{2,N-1}}{\partial\xi_1^2\partial x_2^2} + \frac{\partial^4 u_{2,N-3}}{\partial x_2^4}\right) \\ &+ \varepsilon^{N-2}\left(2\frac{\partial^4 u_{2,N}}{\partial\xi_1^2\partial x_2^2} + \frac{\partial^4 u_{2,N-2}}{\partial x_2^4}\right) \\ &+ \varepsilon^{N-1}\frac{\partial^4 u_{2,N-1}}{\partial x_2^4} + \varepsilon^N\frac{\partial^4 u_{2,N}}{\partial x_2^4}\Bigg]\left(\frac{x_1}{\varepsilon},x_2\right), \\ &\forall\,(x_1,x_2) \in \left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right] \times [0,1], \end{aligned} \tag{2.12}$$

with

$$\begin{aligned} &u_{2,j}(\xi_1,x_2) \\ &\quad= \begin{cases} 0 & \text{if } j \in \{0,1,2,3\}, \\ \mathcal{M}_j(\xi_1) f_2^{(j-4)}(x_2) & \text{if } j \in \mathbb{N}\cap[4,N], \end{cases} \quad \forall\,(\xi_1,x_2) \in \left[-\frac{1}{2},\frac{1}{2}\right] \times [0,1] \end{aligned}$$

and

$$\mathcal{M}_j = \begin{cases} 0 & \text{if } j \text{ is odd}, \\ \text{a specific polynomial function of degree } j & \text{if } j \text{ is even}. \end{cases} \tag{2.13}$$

Precisely, $\mathcal{M}_j$ are solutions of the recurrent chain of problems

$$\begin{cases} \mathcal{M}_j^{(4)} + 2\mathcal{M}_{j-2}^{(2)} + \mathcal{M}_{j-4} = \delta_{j4}, \quad \text{in } \left]-\frac{1}{2},\frac{1}{2}\right[, \\ \mathcal{M}_j\left(-\frac{1}{2}\right) = 0 = \mathcal{M}_j'\left(-\frac{1}{2}\right), \\ \mathcal{M}_j\left(\frac{1}{2}\right) = 0 = \mathcal{M}_j'\left(\frac{1}{2}\right), \end{cases}$$

with $\mathcal{M}_j = 0$ for $j < 4$. For instance, $\mathcal{M}_4 = \frac{1}{4!}(\xi_1 - \frac{1}{2})^2(\xi_1 + \frac{1}{2})^2,\ \xi_1 \in \mathbb{R}$.

3. Boundary Layers Near the Extremities

Unfortunately, $u_{1,\varepsilon}^{Nreg}$ and $u_{2,\varepsilon}^{Nreg}$ do not fit the Dirichlet boundary conditions on $\{-1,1\} \times]-\varepsilon,0[$ and on $]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[\times\{1\}$, respectively. To overcome this difficulty, we introduce some boundary layers near the extremities of the multi-structure.

For $j = 4, \dots, N$, the following problem

$$\begin{cases} \triangle^2 u^{Nbl}_{-1,j} = 0, \quad \text{in }]0,+\infty[\times]-1,0[, \\ u^{Nbl}_{-1,j} = 0 = \dfrac{\partial u^{Nbl}_{-1,j}}{\partial \xi_2}, \quad \text{on }]0,+\infty[\times\{-1,0\}, \\ u^{Nbl}_{-1,j}(0,\xi_2) = -u_{1,j}(-1,\xi_2), \quad \text{on }]-1,0[, \\ -\dfrac{\partial u^{Nbl}_{-1,j}}{\partial \xi_1}(0,\xi_2) = \dfrac{\partial u_{1,j-1}}{\partial x_1}(-1,\xi_2), \quad \text{on }]-1,0[, \\ \lim\limits_{|\xi|\to+\infty} u^{Nbl}_{-1,j}(\xi) = 0 \end{cases} \tag{3.1}$$

admits a unique weak solution $u^{Nbl}_{-1,j} \in H^2(]0,+\infty[\times]-1,0[)$. Moreover, this solution pointwise-exponentially tends to zero with all its derivatives, as $|\xi|$ diverges. This result can be derived from [24, Theorem 6] and well-known estimates by Agmon, Douglis and Nirenberg [1, 2]. In order to use the results from [24] we should show that

(a) for each $j \geq 4$ problem (3.1) has a solution $u^{Nbl}_{-1,j} \in H^2(]0,+\infty[\times]-1,0[)$.
(b) the operator pencil $A(\lambda) = e^{-\lambda\xi_1}\Delta^2 e^{\lambda\xi_1}$ defined on $S =]0,1[\times]-1,0[$ with periodic boundary conditions on $\{0,1\}\times]-1,0[$ and Dirichlet boundary condition

$$u = 0, \quad \frac{\partial}{\partial \xi_2}(e^{\lambda\xi_1}u) = 0 \quad \text{on }]0,1[\times\{-1,0\},$$

does not have eigenvalues on the segment $[0, 2\pi i]$.

In order to justify (a) we first consider a sequence of cylinders $Q_L :=]0,L[\times]-1,0[, L = 1,2,\dots$, and spectral problems

$$\Delta^2 v = \lambda v \quad \text{in } Q_L, \quad v = 0 = \frac{\partial}{\partial \nu} v \quad \text{on } \partial Q_L.$$

For each L this problem has a discrete spectrum $0 < \lambda_{1,L} \leq \lambda_{2,L} \leq \cdots \leq \lambda_{k,L} \to +\infty$.

Lemma 3.1. *The estimate holds*

$$\lambda_{1,L} \geq \pi^4. \tag{3.2}$$

Moreover, for any $v \in H^2_0(Q_L)$

$$\|v\|_{L^2(Q_L)} \leq \pi^{-2}\|\Delta v\|_{L^2(Q_L)} \tag{3.3}$$

and there is a constant $C > 0$ *that does not depend on* L, *such that*

$$\|v\|_{H^2_0(Q_L)} \leq C\|\Delta v\|_{L^2(Q_L)}. \tag{3.4}$$

Proof. We use the variational representation of $\lambda_{1,L}$ which reads

$$\lambda_{1,L} = \min\left\{\int_{Q_L} |\Delta v(\xi)|^2\, d\xi : \int_{Q_L} v^2\, d\xi = 1\right\},$$

where the minimum is taken over $v \in H_0^2(Q_L)$. Developing a test function v in Fourier series

$$v = \sum \alpha_{k_1,k_2} \sin(k_1\pi\xi_1)\sin(k_2\pi\xi_2/L)$$

with positive integer k_1 and k_2, and using the fact that

$$\Delta v = -\sum \alpha_{k_1,k_2}(\pi^2 k_1^2 + \pi^2 k_2^2/L^2)\sin(k_1\pi\xi_1)\sin(k_2\pi\xi_2/L),$$

one can easily derive the desired inequality (3.2) from the variational representation of $\lambda_{1,L}$.

As a consequence of (3.2) we obtain that for any $v \in H_0^2(Q_L)$ inequality (3.3) holds true.

Combining this with the standard elliptic estimates, we obtain (3.4) with a constant C that does not depend on L. □

Lemma 3.2. *For each $j \geq 4$ problem* (3.1) *has a unique solution in* $H^2(]0,\infty[\times]-1,0[)$.

Proof. In a cylinder $Q_L :=]0,L[\times]-1,0[, L = 1,2,\dots$ we consider the following problem

$$\begin{cases} \Delta^2 u_{-1,j}^{Nbl,L} = 0, \quad \text{in } Q_L, \\ u_{-1,j}^{Nbl,L} = 0 = \dfrac{\partial u_{-1,j}^{Nbl,L}}{\partial\xi_2}, \quad \text{on }]0,L[\times\{-1,0\}, \\ u_{-1,j}^{Nbl,L}(0,\xi_2) = -u_{1,j}(-1,\xi_2), \ \text{ on }]-1,0[, \\ u_{-1,j}^{Nbl,L}(L,\xi_2) = 0, \ \text{ on }]-1,0[, \\ -\dfrac{\partial u_{-1,j}^{Nbl,L}}{\partial\xi_1}(0,\xi_2) = \dfrac{\partial u_{1,j-1}}{\partial x_1}(-1,\xi_2), \ \text{ on }]-1,0[\,, \\ \dfrac{\partial u_{-1,j}^{Nbl,L}}{\partial\xi_1}(L,\xi_2) = 0, \quad \text{on }]-1,0[. \end{cases} \tag{3.5}$$

Considering the regularity of the functions $u_{-1,j}$, one can construct functions $U_{-1,j}$ such that $U_{-1,j} \in H^2(Q_3)$, $\operatorname{supp}(U_{-1,j}) \subset [0,2[\times[-1,0]$,

$$U_{-1,j}(0,\xi_2) = u_{-1,j}(-1,\xi_2), \quad -\frac{\partial U_{-1,j}}{\partial\xi_1}(0,\xi_2) = \frac{\partial u_{1,j-1}}{\partial x_1}(-1,\xi_2), \ \text{ on }]-1,0[,$$

$$U_{-1,j} = 0 = \frac{\partial U_{-1,j}}{\partial\xi_2}, \ \text{ on }]0,3[\times\{-1,0\}, \quad \|\Delta^2 U_{-1,j}\|_{L^2(Q_3)} \leq C_j.$$

Then the function $u_{-1,j}^{Nbl,L} - U_{-1,j}$ satisfies in Q_L the equation

$$\Delta^2(u_{-1,j}^{Nbl,L} - U_{-1,j}) = -\Delta^2 U_{-1,j}$$

with homogeneous Dirichlet boundary conditions. Multiplying this equation by $(u_{-1,j}^{Nbl,L} - U_{-1,j})$, integrating by parts and using (3.4), we get

$$\|u_{-1,j}^{Nbl,L} - U_{-1,j}\|_{H_0^2(Q_L)} \le C_j;$$

here the constant C_j does not depend on L. Therefore,

$$\|u_{-1,j}^{Nbl,L}\|_{H^2(Q_L)} \le C_j.$$

If we extend the functions $u_{-1,j}^{Nbl,L}$ to Q_∞ by setting $u_{-1,j}^{Nbl,L} = 0$ for $x \in [L,\infty[\times]-1,0[$, then the extended functions satisfies the estimate

$$\|u_{-1,j}^{Nbl,L}\|_{H^2(Q_\infty)} \le C_j.$$

Passing to the weak limit, as $L \to +\infty$, we obtain the desired solution of problem (3.1). The last limit relation in (3.1) follows from the Friedrichs inequality and local elliptic estimates.

The uniqueness readily follows from Lemma 3.1. □

Applying local elliptic estimates one can easily show that $u_{-1,j}^{Nbl} \in H^4(]1,+\infty[\times]-1,0[)$. Indeed, it readily follows from local elliptic estimates that $u_{-1,j}^{Nbl} \in H^4_{\rm loc}(]1,+\infty\,[\times]-1,0[)$ and that $\|u_{-1,j}^{Nbl}\|^2_{H^4([L_1,L_1+1])} \le C\|u_{-1,j}^{Nbl}\|^2_{L^2([L_1-1,L_1+2])}$ for all $L_1 \ge 1$. It then remains to sum up the latter inequalities over $L_1 = 1, 2, \dots$.

It remains to justify item (b).

Lemma 3.3. *The operator pencil $A(\lambda)$ does not have eigenvalues on the segment $[0, 2\pi i]$.*

Proof. Assume that $\lambda_0 \in [0, 2\pi i]$ is an eigenvalue of $A(\lambda)$. Denote by V the corresponding eigenfunction. Multiplying the equation $e^{-\lambda_0\xi_1}\Delta^2 e^{\lambda_0\xi_1}V = 0$ by $\overline{V}$, integrating the resulting relation over Q_1 and integrating by parts, we obtain

$$\begin{aligned}0 &= \int_{Q_1}(A(\lambda_0)V)\overline{V}\,d\xi\\ &= \int_{Q_1}\left(\left(\frac{\partial}{\partial\xi_1}+\lambda_0\right)^2 V + \frac{\partial^2}{\partial^2\xi_2}V\right)\overline{\left(\left(\frac{\partial}{\partial\xi_1}+\lambda_0\right)^2 V + \frac{\partial^2}{\partial^2\xi_2}V\right)}d\xi.\end{aligned}$$

This yields

$$\left(\frac{\partial}{\partial\xi_1}+\lambda_0\right)^2 V + \frac{\partial^2}{\partial^2\xi_2}V = 0.$$

With the standing boundary conditions the last operator is positive definite and thus $V = 0$. Indeed, multiplying the last equation by $\overline{V}$, integrating by parts over Q_1 and taking into account the fact that λ belongs to the imaginary axis, we obtain

$$\int_{Q_1}\left|\left(\frac{\partial}{\partial\xi_1}+\lambda_0\right)V\right|^2 + \left|\frac{\partial}{\partial\xi_2}V\right|^2 d\xi = 0.$$

This completes the proof. □

According to [24, Theorem 6], there exists $\beta > 0$ such that $\|e^{\beta\xi_1}u^{Nbl}_{-1,j}\|_{H^4(]0,+\infty[\times[-1,0])} < \infty$. From the standard Schauder estimates it then follows that this solution is smooth in $[1,+\infty]\times[-1,0]$ and satisfies the estimates

$$|u^{Nbl}_{-1,j}(\xi_1,\xi_2)| \le Ce^{-\beta\xi_1}$$

as desired. The existence of exponentially decaying boundary layer functions in the vicinity of another two bases can be justified in the same way.

The function

$$u^{Nbl}_{-1,\varepsilon} : \xi = (\xi_1,\xi_2) \in]0,+\infty[\times]-1,0[\to \sum_{j=4}^{N} \varepsilon^j u^{Nbl}_{-1,j}(\xi)$$

exponentially tends to zero with all its derivatives, as $|\xi|$ diverges, and satisfies the following problem

$$\begin{cases} \triangle^2\left(u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right) = 0, \quad \text{in }]-1,+\infty[\times]-\varepsilon,0[, \\[2ex] u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right) = 0 = \dfrac{\partial\left(u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right)}{\partial x_2}, \quad \text{on }]-1,+\infty[\times\{-\varepsilon,0\}, \\[2ex] u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right) = -u^{Nreg}_{1,\varepsilon}(x_1,x_2), \quad \text{on } \{-1\}\times]-\varepsilon,0[, \\[2ex] -\dfrac{\partial\left(u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right)}{\partial x_1} = \dfrac{\partial u^{Nreg}_{1,\varepsilon}}{\partial x_1}(x_1,x_2) - \varepsilon^N f_1^{(N-3)}(x_1)\mathcal{N}_N\left(\dfrac{x_2}{\varepsilon}\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \{-1\}\times]-\varepsilon,0[. \end{cases}$$

Note that $\mathcal{N}_N(\frac{x_2}{\varepsilon}) = 0$ for odd N. Consequently, choosing N odd, it results in

$$\begin{cases} \triangle^2\left(\eta(x_1+1)u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right) \\ \qquad = \triangle^2\left(\left(\eta(x_1+1)-1\right)u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right), \quad \text{in }]-1,1[\times]-\varepsilon,0[, \\[2ex] \eta(x_1+1)u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right) = 0 = \dfrac{\partial\left(\eta(x_1+1)u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right)}{\partial\nu}, \\ \qquad\qquad\qquad \text{on } (]-1,1[\times\{-\varepsilon,0\})\cup(\{1\}\times]-\varepsilon,0[), \\[2ex] \eta(x_1+1)u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right) = -u^{Nreg}_{1,\varepsilon}(x_1,x_2), \quad \text{on } \{-1\}\times]-\varepsilon,0[, \\[2ex] \dfrac{\partial\left(\eta(x_1+1)u^{Nbl}_{-1,\varepsilon}\left(\dfrac{x_1+1}{\varepsilon},\dfrac{x_2}{\varepsilon}\right)\right)}{\partial\nu} = -\dfrac{\partial u^{Nreg}_{1,\varepsilon}}{\partial\nu}(x_1,x_2), \quad \text{on } \{-1\}\times]-\varepsilon,0[, \end{cases} \tag{3.6}$$

where ν denotes the unit outer normal on $\partial(]-1,1[\times]-\varepsilon,0[)$. We point out that, by virtue of the exponential decay of $u^{Nbl}_{-1,\varepsilon}$, there exist two positive constants c_1 and c_2, independent of ε, such that

$$\begin{aligned} &\left|\triangle^2\left((\eta(x_1+1)-1)\,u^{Nbl}_{-1,\varepsilon}\left(\frac{x_1+1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right)\right| \\ &\quad < \frac{c_1}{\varepsilon^4}\exp\left(-\frac{c_2}{\varepsilon}\right), \quad \forall\,(x_1,x_2)\in]-1,1[\times]-\varepsilon,0[. \end{aligned} \tag{3.7}$$

Similarly, for $j=4,\dots,N$, the following problems

$$\begin{cases} \triangle^2 u^{Nbl}_{1,j} = 0, \quad \text{in }]-\infty,0[\times]-1,0[, \\ u^{Nbl}_{1,j} = 0 = \dfrac{\partial u^{Nbl}_{1,j}}{\partial \xi_2}, \quad \text{on }]-\infty,0[\times\{-1,0\}, \\ u^{Nbl}_{1,j}(0,\xi_2) = -u_{1,j}(1,\xi_2), \quad \text{on }]-1,0[, \\ \dfrac{\partial u^{Nbl}_{1,j}}{\partial \xi_1}(0,\xi_2) = -\dfrac{\partial u_{1,j-1}}{\partial x_1}(1,\xi_2), \quad \text{on }]-1,0[, \\ \lim\limits_{|\xi|\to+\infty} u^{Nbl}_{1,j}(\xi) = 0, \end{cases} \tag{3.8}$$

$$\begin{cases} \triangle^2 u^{Nbl}_{2,j} = 0, \quad \text{in } \left]-\dfrac{1}{2},\dfrac{1}{2}\right[\times]-\infty,0[, \\ u^{Nbl}_{2,j} = 0 = \dfrac{\partial u^{Nbl}_{2,j}}{\partial \xi_1}, \quad \text{on } \left\{-\dfrac{1}{2},\dfrac{1}{2}\right\}\times]-\infty,0[, \\ u^{Nbl}_{2,j}(\xi_1,0) = -u_{2,j}(\xi_1,1), \quad \text{on } \left]-\dfrac{1}{2},\dfrac{1}{2}\right[, \\ \dfrac{\partial u^{Nbl}_{2,j}}{\partial \xi_2}(\xi_1,0) = -\dfrac{\partial u_{2,j-1}}{\partial x_2}(\xi_1,1), \quad \text{on } \left]-\dfrac{1}{2},\dfrac{1}{2}\right[, \\ \lim\limits_{|\xi|\to+\infty} u^{Nbl}_{2,j}(\xi) = 0, \end{cases} \tag{3.9}$$

admit a unique weak solution $u^{Nbl}_{1,j} \in H^2(]-\infty,0[\times]-1,0[)$ and $u^{Nbl}_{2,j} \in H^2(]-\frac{1}{2},\frac{1}{2}[\times]-\infty,0[)$, respectively. Moreover, they pointwise-exponentially tend to zero with all their derivatives, as $|\xi|$ diverges. Then, setting

$$u^{Nbl}_{1,\varepsilon} : \xi = (\xi_1,\xi_2) \in]-\infty,0[\times]-1,0[\to \sum_{j=4}^{N} \varepsilon^j u^{Nbl}_{1,j}(\xi)$$

and

$$u^{Nbl}_{2,\varepsilon} : \xi = (\xi_1,\xi_2) \in \left]-\frac{1}{2},\frac{1}{2}\right[\times]-\infty,0[\to \sum_{j=4}^{N} \varepsilon^j u^{Nbl}_{2,j}(\xi),$$

for N odd it results in

$$\begin{cases} \triangle^2 \left(\eta(x_1-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \right) \\ \qquad = \triangle^2 \left((\eta(x_1-1)-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \right), \quad \text{in }]-1,1[\times]-\varepsilon,0[, \\ \eta(x_1-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) = 0 = \frac{\partial \left(\eta(x_1-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \right)}{\partial \nu}, \\ \qquad \text{on } (]-1,1[\times\{-\varepsilon,0\}) \cup (\{-1\}\times]-\varepsilon,0[), \\ \eta(x_1-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) = -u_{1,\varepsilon}^{Nreg}(x_1,x_2), \quad \text{on } \{1\}\times]-\varepsilon,0[, \\ \frac{\partial \left(\eta(x_1-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \right)}{\partial \nu} = -\frac{\partial u_{1,\varepsilon}^{Nreg}}{\partial \nu}(x_1,x_2), \quad \text{on } \{1\}\times]-\varepsilon,0[; \end{cases} \tag{3.10}$$

and

$$\begin{cases} \triangle^2 \left(\eta(x_2-1) u_{2,\varepsilon}^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) \right) \\ \qquad = \triangle^2 \left((\eta(x_2-1)-1) u_{2,\varepsilon}^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) \right), \quad \text{in } \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\times]0,1[, \\ \eta(x_2-1) u_{2,\varepsilon}^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) \\ \qquad = 0 = \frac{\partial \left(\eta(x_2-1) u_{2,\varepsilon}^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) \right)}{\partial \nu}, \quad \text{on } \left\{\pm\frac{\varepsilon}{2}\right\}\times]0,1[, \\ \eta(x_2-1) u_{2,\varepsilon}^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) = -u_{2,\varepsilon}^{Nreg}(x_1,x_2), \quad \text{on } \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\times\{1\}, \\ \frac{\partial \left(\eta(x_2-1) u_{2,\varepsilon}^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2-1}{\varepsilon} \right) \right)}{\partial \nu} = -\frac{\partial u_{2,\varepsilon}^{Nreg}}{\partial \nu}(x_1,x_2), \quad \text{on } \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\times\{1\}, \end{cases} \tag{3.11}$$

where, as in (3.7), one has that

$$\left| \triangle^2 \left((\eta(x_1-1)-1) u_{1,\varepsilon}^{Nbl} \left(\frac{x_1-1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \right) \right| < \frac{c_1}{\varepsilon^4} \exp\left(-\frac{c_2}{\varepsilon}\right), \quad \forall\, (x_1,x_2) \in]-1,1[\times]-\varepsilon,0[\tag{3.12}$$

and

$$\left|\triangle^2\left((\eta(x_2-1)-1)u_{2,\varepsilon}^{Nbl}\left(\frac{x_1}{\varepsilon},\frac{x_2-1}{\varepsilon}\right)\right)\right| < \frac{c_1}{\varepsilon^4}\exp\left(-\frac{c_2}{\varepsilon}\right), \quad \forall\,(x_1,x_2)\in\left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\times]0,1[. \tag{3.13}$$

In (3.11), ν denotes the unit outer normal on $\partial(]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[\times]0,1[)$.

4. Boundary Layer Near the Junction

Now, for gluing $u_{1,\varepsilon}^{Nreg}$ with $u_{2,\varepsilon}^{Nreg}$, we multiply the first function by $\rho(\frac{x_1}{\varepsilon})$ and the second one by $\rho(\frac{x_2}{\varepsilon})$. Consequently, we have to compute $\triangle^2(\rho(\frac{x_1}{\varepsilon})u_{1,\varepsilon}^{Nreg})$ and $\triangle^2(\rho(\frac{x_2}{\varepsilon})u_{2,\varepsilon}^{Nreg})$. To this aim, we recall that if $p=p(x_1)$ and $q=q(x_1,x_2)$ are two sufficiently smooth functions, it results in

$$\triangle^2(pq)=p\triangle^2 q+4p^{(1)}\triangle\frac{\partial q}{\partial x_1}+p^{(2)}\left(6\frac{\partial^2 q}{\partial x_1^2}+2\frac{\partial^2 q}{\partial x_2^2}\right)+4p^{(3)}\frac{\partial q}{\partial x_1}+p^{(4)}q.$$

Then, by virtue of (2.11), we have

$$\begin{aligned}\triangle^2\left(\rho\left(\frac{x_1}{\varepsilon}\right)u_{1,\varepsilon}^{Nreg}(x_1,x_2)\right)&=\rho\left(\frac{x_1}{\varepsilon}\right)\triangle^2(u_{1,\varepsilon}^{Nreg}(x_1,x_2))+s_{1,\varepsilon}^N(x_1,x_2)\\&=f_1(x_1)-\eta\left(\frac{x_1}{\varepsilon}\right)f_1(x_1)+\rho\left(\frac{x_1}{\varepsilon}\right)r_{1,\varepsilon}^N(x_1,x_2)\\&\quad+s_{1,\varepsilon}^N(x_1,x_2),\quad\forall(x_1,x_2)\in]-1,1[\times]-\varepsilon,0[,\end{aligned}$$

where

$$\begin{aligned}s_{1,\varepsilon}^N(x_1,x_2)&=\varepsilon^{-1}\psi_{1,\varepsilon}^N\left(x_1,\frac{x_2}{\varepsilon}\right)\rho^{(1)}\left(\frac{x_1}{\varepsilon}\right)+\varepsilon^{-2}\psi_{2,\varepsilon}^N\left(x_1,\frac{x_2}{\varepsilon}\right)\rho^{(2)}\left(\frac{x_1}{\varepsilon}\right)\\&\quad+\varepsilon^{-3}\psi_{3,\varepsilon}^N\left(x_1,\frac{x_2}{\varepsilon}\right)\rho^{(3)}\left(\frac{x_1}{\varepsilon}\right)+\varepsilon^{-4}\psi_{4,\varepsilon}^N\left(x_1,\frac{x_2}{\varepsilon}\right)\rho^{(4)}\left(\frac{x_1}{\varepsilon}\right),\\&\quad\forall(x_1,x_2)\in]-1,1[\times]-\varepsilon,0[,\end{aligned}$$

with

$$\begin{cases}\psi_{1,\varepsilon}^N(x_1,\xi_2)=4\displaystyle\sum_{j=4}^N(\varepsilon^j f^{(j-1)}(x_1)\mathcal{N}_j(\xi_2)+\varepsilon^{j-2}f^{(j-3)}(x_1)\mathcal{N}_j^{(2)}(\xi_2)),\\ \psi_{2,\varepsilon}^N(x_1,\xi_2)=\displaystyle\sum_{j=4}^N(6\varepsilon^j f^{(j-2)}(x_1)\mathcal{N}_j(\xi_2)+2\varepsilon^{j-2}f^{(j-4)}(x_1)\mathcal{N}_j^{(2)}(\xi_2)),\\ \psi_{3,\varepsilon}^N(x_1,\xi_2)=4\displaystyle\sum_{j=4}^N(\varepsilon^j f^{(j-3)}(x_1)\mathcal{N}_j(\xi_2)),\\ \psi_{4,\varepsilon}^N(x_1,\xi_2)=4\displaystyle\sum_{j=4}^N(\varepsilon^j f^{(j-4)}(x_1)\mathcal{N}_j(\xi_2)),\\ \qquad\qquad\forall(x_1,\xi_2)\in]-1,1[\times]-1,0[.\end{cases}$$

Consequently, by virtue of the assumption on f in the second line of (1.1) and the properties of ρ, if $\varepsilon < \frac{3}{2}a$, it results in

$$s^N_{1,\varepsilon}(x_1,x_2) = b\left(2\mathcal{N}_4^{(2)}\left(\frac{x_2}{\varepsilon}\right)\rho^{(2)}\left(\frac{x_1}{\varepsilon}\right) + 4\mathcal{N}_4\left(\frac{x_2}{\varepsilon}\right)\rho^{(4)}\left(\frac{x_1}{\varepsilon}\right)\right),$$
$$\forall (x_1,x_2) \in]-1,1[\times]-\varepsilon,0[\qquad (4.1)$$

and

$$\begin{cases} \triangle^2\left(\rho\left(\frac{x_1}{\varepsilon}\right)u^{Nreg}_{1,\varepsilon}(x_1,x_2)\right) \\ \qquad = f_1(x_1) + \rho\left(\frac{x_1}{\varepsilon}\right)r^N_{1,\varepsilon}(x_1,x_2) - \eta\left(\frac{x_1}{\varepsilon}\right)b + s^N_{1,\varepsilon}(x_1,x_2), \\ \qquad\qquad \forall\,(x_1,x_2) \in]-1,1[\times]-\varepsilon,0[\,, \\ \rho\left(\frac{x_1}{\varepsilon}\right)u^{Nreg}_{1,\varepsilon}(x_1,-\varepsilon) = 0 = \rho\left(\frac{x_1}{\varepsilon}\right)\frac{\partial u^{Nreg}_{1,\varepsilon}}{\partial x_2}(x_1,-\varepsilon), \quad \forall\, x_1 \in]-1,1[\,, \\ \rho\left(\frac{x_1}{\varepsilon}\right)u^{Nreg}_{1,\varepsilon}(x_1,0) = 0 = \rho\left(\frac{x_1}{\varepsilon}\right)\frac{\partial u^{Nreg}_{1,\varepsilon}}{\partial x_2}(x_1,0)\,, \quad \forall\, x_1 \in]-1,1[\,. \end{cases} \qquad (4.2)$$

Similarly, if $\varepsilon < \frac{3}{2}a$, it results in

$$\begin{cases} \triangle^2\left(\rho\left(\frac{x_2}{\varepsilon}\right)u^{Nreg}_{2,\varepsilon}(x_1,x_2)\right) \\ \qquad = f_2(x_2) + \rho\left(\frac{x_2}{\varepsilon}\right)r^N_{2,\varepsilon}(x_1,x_2) - \eta\left(\frac{x_2}{\varepsilon}\right)b + s^N_{2,\varepsilon}(x_1,x_2), \\ \qquad\qquad \forall (x_1,x_2) \in \left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\times]0,1[\,, \\ \rho\left(\frac{x_2}{\varepsilon}\right)u^{Nreg}_{2,\varepsilon}\left(\pm\frac{\varepsilon}{2},x_2\right) = 0 = \rho\left(\frac{x_2}{\varepsilon}\right)\frac{\partial u^{Nreg}_{2,\varepsilon}}{\partial x_2}\left(\pm\frac{\varepsilon}{2},x_2\right), \quad \forall\, x_2 \in]0,1[\,, \end{cases} \qquad (4.3)$$

where

$$s^N_{2,\varepsilon}(x_1,x_2) = b\left(2\mathcal{M}_4^{(2)}\left(\frac{x_1}{\varepsilon}\right)\rho^{(2)}\left(\frac{x_2}{\varepsilon}\right) + 4\mathcal{M}_4\left(\frac{x_1}{\varepsilon}\right)\rho^{(4)}\left(\frac{x_2}{\varepsilon}\right)\right),$$
$$\forall\,(x_1,x_2) \in \left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\times]0,1[\,. \qquad (4.4)$$

Now, let

$$h : (\xi_1,\xi_2) \in D = (\mathbb{R}\times]-1,0[) \cup \left(\left]-\frac{1}{2},\frac{1}{2}\right[\times[0,+\infty[\right) \to h(\xi_1,\xi_2)$$

$$= \begin{cases} b(-\eta(\xi_1) + 2\mathcal{N}_4^{(2)}(\xi_2)\rho^{(2)}(\xi_1) + 4\mathcal{N}_4(\xi_2)\rho^{(4)}(\xi_1)), \\ \qquad\qquad\qquad\qquad \text{if } (\xi_1, \xi_2) \in \mathbb{R} \times]-1, 0[\,, \\ b(-\eta(\xi_2) + 2\mathcal{M}_4^{(2)}(\xi_1)\rho^{(2)}(\xi_2) + 4\mathcal{M}_4(\xi_1)\rho^{(4)}(\xi_2)), \\ \qquad\qquad\qquad\qquad \text{if } (\xi_1, \xi_2) \in \left]-\frac{1}{2}, \frac{1}{2}\right[\times [0, +\infty[\,. \end{cases} \tag{4.5}$$

Note that $h \in C^\infty(D)$ and $\mathrm{supp}(h) \subseteq ([-\frac{2}{3}, \frac{2}{3}] \times [-1, 0]) \cup ([-\frac{1}{2}, \frac{1}{2}] \times [0, \frac{2}{3}])$. After a minor modification of the arguments of Lemmas 3.1–3.2 one can show that the following problem

$$\begin{cases} \triangle^2 u_0^{Nbl} = -h, \quad \text{in } D, \\ u_0^{Nbl} = 0 = \dfrac{\partial u_0^{Nbl}}{\partial \nu}, \quad \text{on } \partial D, \\ \lim\limits_{|\xi| \to +\infty} u_0^{Nbl}(\xi) = 0, \end{cases} \tag{4.6}$$

admits a unique weak solution in $H^2(D)$. Moreover, due to Lemma 3.3 and [24, Theorem 6] this solution exponentially tends to zero with all its derivatives, as $|\xi|$ diverges. Consequently, $u_0^{Nbl}(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon})$ solves the following problem

$$\begin{cases} \varepsilon^4 \triangle^2 \left(u_0^{Nbl} \left(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon} \right) \right) \\ \qquad = -h \left(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon} \right), \quad \text{in } D_\varepsilon = (\mathbb{R} \times]-\varepsilon, 0[) \cup \left(\left]-\dfrac{\varepsilon}{2}, \dfrac{\varepsilon}{2}\right[\times [0, +\infty[\right), \\ u_0^{Nbl} \left(\dfrac{x_1}{\varepsilon}, \dfrac{x_2}{\varepsilon} \right) = 0 = \dfrac{\partial u_0^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right)}{\partial \nu}, \quad \text{on } \partial D_\varepsilon. \end{cases} \tag{4.7}$$

By combining (4.2), (4.1), (4.3) and (4.4) with (4.7) and (4.5), one obtains that

$$v_\varepsilon = \rho \left(\frac{x_1}{\varepsilon} \right) u_{1,\varepsilon}^{Nreg}(x_1, x_2) + \rho \left(\frac{x_2}{\varepsilon} \right) u_{2,\varepsilon}^{Nreg}(x_1, x_2) + \varepsilon^4 u_0^{Nbl} \left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon} \right) \in H^2(\Omega_\varepsilon)$$

and

$$\begin{cases} \triangle^2 v_\varepsilon = f(x_1, x_2) + \rho \left(\dfrac{x_1}{\varepsilon} \right) r_{1,\varepsilon}^N(x_1, x_2) + \rho \left(\dfrac{x_2}{\varepsilon} \right) r_{2,\varepsilon}^N(x_1, x_2), \quad \text{in } \Omega_\varepsilon, \\ v_\varepsilon = 0 = \dfrac{\partial v_\varepsilon}{\partial \nu}, \quad \text{on } \Gamma_\varepsilon, \end{cases} \tag{4.8}$$

where $\Gamma_\varepsilon = \partial\Omega_\varepsilon \backslash ((\{\pm 1\} \times]-\varepsilon, 0[) \cup (]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[\times \{1\}))$, and $\rho(\frac{x_1}{\varepsilon}) u_{1,\varepsilon}^{Nreg}(x_1, x_2)$ and $\rho(\frac{x_1}{\varepsilon}) r_{1,\varepsilon}^N(x_1, x_2)$ are extended by zero on $[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}] \times [0, 1]$, while $\rho(\frac{x_2}{\varepsilon}) u_{2,\varepsilon}^{Nreg}(x_1, x_2)$

and $\rho(\frac{x_2}{\varepsilon})r^N_{2,\varepsilon}(x_1,x_2)$ are extended by zero on $[-1,1]\times[-\varepsilon,0]$. Then, from (4.8) it follows that

$$w_\varepsilon = \rho\left(\frac{x_1}{\varepsilon}\right) u^{Nreg}_{1,\varepsilon}(x_1,x_2) + \rho\left(\frac{x_2}{\varepsilon}\right) u^{Nreg}_{2,\varepsilon}(x_1,x_2)$$
$$+\varepsilon^4\eta(x_1)\eta(x_2)u^{Nbl}_0\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\in H^2(\Omega_\varepsilon)$$

and

$$\begin{cases} \triangle^2 w_\varepsilon = f(x_1,x_2) + \rho\left(\frac{x_1}{\varepsilon}\right) r^N_{1,\varepsilon}(x_1,x_2) + \rho\left(\frac{x_2}{\varepsilon}\right) r^N_{2,\varepsilon}(x_1,x_2) \\ \qquad + \varepsilon^4\triangle^2\left((\eta(x_1)\eta(x_2)-1)u^{Nbl}_0\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right), \quad \text{in } \Omega_\varepsilon, \\ w_\varepsilon = 0 = \dfrac{\partial w_\varepsilon}{\partial \nu}, \quad \text{on } \Gamma_\varepsilon, \\ w_\varepsilon = u^{Nreg}_{1,\varepsilon}, \quad \dfrac{\partial w_\varepsilon}{\partial\nu} = \dfrac{\partial u^{Nreg}_{1,\varepsilon}}{\partial\nu}, \quad \text{on } \{-1,1\}\times]-\varepsilon,0[, \\ w_\varepsilon = u^{Nreg}_{2,\varepsilon}, \quad \dfrac{\partial w_\varepsilon}{\partial\nu} = \dfrac{\partial u^{Nreg}_{2,\varepsilon}}{\partial\nu}, \quad \text{on } \left]-\dfrac{\varepsilon}{2},\dfrac{\varepsilon}{2}\right[\times\{1\}. \end{cases} \tag{4.9}$$

We point out that, by virtue of the exponential decay of u^{Nbl}_0, there exist two positive constants c_1 and c_2, independent of ε, such that

$$\left|\triangle^2\left((\eta(x_1)\eta(x_2)-1)u^{Nbl}_0\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right)\right|$$
$$< \frac{c_1}{\varepsilon^4}\exp\left(-\frac{c_2}{\varepsilon}\right), \quad \forall\,(x_1,x_2)\in\Omega_\varepsilon. \tag{4.10}$$

5. Error Estimate for the Asymptotic Expansion

Now, let u^N_ε be the function defined in Ω_ε by (1.4) with N odd and

$$\varepsilon < \min\left\{1, \frac{3}{2}a\right\}. \tag{5.1}$$

From (4.9), (3.6), (3.10) and (3.11) it follows that

$$\begin{cases} \triangle^2 u^N_\varepsilon = f + r^N_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ u^N_\varepsilon = 0 = \dfrac{\partial u^N_\varepsilon}{\partial\nu}, \quad \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.2}$$

that is

$$\begin{cases} u^N_\varepsilon \in H^2_0(\Omega_\varepsilon), \\ \displaystyle\int_{\Omega_\varepsilon} \triangle u^N_\varepsilon \triangle v\,dx = \int_{\Omega_\varepsilon}(f + r^N_\varepsilon)v\,dx, \quad \forall\, v\in H^2_0(\Omega_\varepsilon), \end{cases} \tag{5.3}$$

where

$$r_\varepsilon^N = \rho\left(\frac{x_1}{\varepsilon}\right) r_{1,\varepsilon}^N(x_1,x_2) + \rho\left(\frac{x_2}{\varepsilon}\right) r_{2,\varepsilon}^N(x_1,x_2) + \varepsilon^4 \triangle^2\left((\eta(x_1)\eta(x_2)-1)u_0^{Nbl}\left(\frac{x_1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right) + \triangle^2\left((\eta(x_1+1)-1)u_{-1,\varepsilon}^{Nbl}\left(\frac{x_1+1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right) + \triangle^2\left((\eta(x_1-1)-1)u_{1,\varepsilon}^{Nbl}\left(\frac{x_1-1}{\varepsilon},\frac{x_2}{\varepsilon}\right)\right) + \triangle^2\left((\eta(x_2-1)-1)u_{2,\varepsilon}^{Nbl}\left(\frac{x_1}{\varepsilon},\frac{x_2-1}{\varepsilon}\right)\right).$$

Then, by virtue of (2.10), (2.12), (3.7), (3.12), (3.13) and (4.10), it results in $r_\varepsilon^N \in L^\infty(\Omega_\varepsilon)$ and

$$\|r_\varepsilon^N\|_{L^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-3}\sqrt{\varepsilon}, \tag{5.4}$$

where, by now on, c_N denotes a positive constant independent of ε (but depending on N). By applying *a priori* estimates for the solution of (1.3), comparing (1.3) with (5.3) and (5.4), and using the Poincaré inequality in $]-1,1[^2$ (in such way the Poincaré constant is independent of ε), one obtains that

$$\|u_\varepsilon^N - u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-3}\sqrt{\varepsilon}. \tag{5.5}$$

In particular, rewriting (5.5) with N replaced by $N+2$ and assuming $f_1 \in C^{N+2}([-1,1])$, $f_2 \in C^{N+2}([0,1])$, one has that

$$\|u_\varepsilon^{N+2} - u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-1}\sqrt{\varepsilon}. \tag{5.6}$$

On the other hand, by construction it results in

$$\|u_\varepsilon^{N+2} - u_\varepsilon^N\|_{H^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-1}\sqrt{\varepsilon}. \tag{5.7}$$

Then, comparing (5.6) with (5.7), one obtains that

$$\|u_\varepsilon^N - u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \le c_N \varepsilon^{N-1}\sqrt{\varepsilon}, \tag{5.8}$$

if assumption (1.1) holds true.

6. Implementation of the Method of Asymptotic Partial Domain Decomposition

Now, let us set $\delta_\varepsilon = \frac{N+3}{c_2}\varepsilon|\log\varepsilon|$,

$$\eta_{\delta_\varepsilon} : t \in \mathbb{R} \to \eta\left(\frac{t\delta_\varepsilon}{\varepsilon}\right),$$

and in formula (1.4) let us replace $\eta(x_1), \eta(x_2), \eta(x_1 \pm 1), \eta(x_2 - 1)$ by $\eta_{\delta_\varepsilon}(x_1), \eta_{\delta_\varepsilon}(x_2), \eta_{\delta_\varepsilon}(x_1 \pm 1), \eta_{\delta_\varepsilon}(x_2 - 1)$, respectively. Then, assumption (5.1) does not change, while the left-hand side in estimates (3.7), (3.12), (3.13) and (4.10) is replaced by

$$\frac{c_1}{\varepsilon^4}\exp\left(-\frac{c_2\delta_\varepsilon}{\varepsilon}\right) = c_1\varepsilon^{N-1}.$$

Then, if u_ε^N denotes also the function defined in Ω_ε by (1.4) with N odd and η replaced by $\eta_{\delta_\varepsilon}$, if $f_1 \in C^{N+2}([-1,1])$ and $f_2 \in C^{N+2}([0,1])$ one has that, as in (5.8),

$$\|u_\varepsilon^N - u_\varepsilon\|_{H^2(\Omega_\varepsilon)} \le c_N\varepsilon^{N-1}\sqrt{\varepsilon}. \tag{6.1}$$

Now, let

$$\begin{aligned} H_\varepsilon^{\rm dec} = \Bigg\{ & v \in H_0^2(\Omega_\varepsilon) : v(x_1,x_2) = \sum_{j=4}^{N} \varepsilon^j f_1^{(j-4)}(x_1)\mathcal{N}_j\left(\frac{x_2}{\varepsilon}\right), \\ & \forall\,(x_1,x_2) \in ([-1+\delta_\varepsilon, -\delta_\varepsilon] \cup [\delta_\varepsilon, 1-\delta_\varepsilon]) \times [-\varepsilon, 0], \\ & v(x_1,x_2) = \sum_{j=4}^{N} \varepsilon^j f_2^{(j-4)}(x_2)\mathcal{M}_j\left(\frac{x_1}{\varepsilon}\right), \\ & \forall\,(x_1,x_2) \in \left[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right] \times [\delta_\varepsilon, 1-\delta_\varepsilon] \Bigg\}. \end{aligned} \tag{6.2}$$

Note that $H_\varepsilon^{\rm dec}$ is a closed subset of $H_0^2(\Omega_\varepsilon)$. Let (see Fig. 2)

$$\begin{cases} \Omega_{\varepsilon,0}^{\rm dec} = (]-\delta_\varepsilon, \delta_\varepsilon[\times]-\varepsilon, 0[) \cup \left(\left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\times\ [0, \delta_\varepsilon[\right), \\ \Omega_{\varepsilon,-1}^{\rm dec} =]-1, -1+\delta_\varepsilon, [\times]-\varepsilon, 0[, \\ \Omega_{\varepsilon,1}^{\rm dec} =]1-\delta_\varepsilon, 1[\times]-\varepsilon, 0[, \\ \Omega_{\varepsilon,2}^{\rm dec} = \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\times]1-\delta_\varepsilon, 1[, \\ \Omega_\varepsilon^{\rm dec} = \Omega_{\varepsilon,0}^{\rm dec} \cup \Omega_{\varepsilon,-1}^{\rm dec} \cup \Omega_{\varepsilon,1}^{\rm dec} \cup \Omega_{\varepsilon,2}^{\rm dec}. \end{cases}$$

Then, the following problem

$$\begin{cases} u_\varepsilon^{\rm dec} \in H_\varepsilon^{\rm dec}, \\ \triangle^2 u_\varepsilon^{\rm dec} = f, \quad \text{in } \Omega_\varepsilon^{\rm dec}, \\ u_\varepsilon^{\rm dec} = 0 = \dfrac{\partial u_\varepsilon^{\rm dec}}{\partial\nu}, \quad \text{on } \partial\Omega_\varepsilon \cap \partial\Omega_\varepsilon^{\rm dec} \end{cases} \tag{6.3}$$

admits a unique weak solution. In fact, problem (6.3) can be reduced to four independent Dirichlet biharmonic problems in $\Omega_{\varepsilon,0}^{\rm dec}$, $\Omega_{\varepsilon,-1}^{\rm dec}$, $\Omega_{\varepsilon,1}^{\rm dec}$ and $\Omega_{\varepsilon,2}^{\rm dec}$. For instance, in $\Omega_{\varepsilon,0}^{\mathrm{dec}}$ we get

$$\begin{cases} u_\varepsilon^{\mathrm{dec}} \in H^2(\Omega_{\varepsilon,0}^{\mathrm{dec}}), \\ \triangle^2 u_\varepsilon^{\mathrm{dec}} = f, \quad \text{in } \Omega_{\varepsilon,0}^{\mathrm{dec}}, \\ u_\varepsilon^{\mathrm{dec}} = 0 = \dfrac{\partial u_\varepsilon^{\mathrm{dec}}}{\partial \nu}, \quad \text{on } \partial\Omega_\varepsilon \cap \partial\Omega_{\varepsilon,0}^{\mathrm{dec}}, \\ u_\varepsilon^{\mathrm{dec}} = \varepsilon^4 b \mathcal{N}_4\left(\dfrac{x_2}{\varepsilon}\right), \quad \dfrac{\partial u_\varepsilon^{\mathrm{dec}}}{\partial \nu} = 0, \quad \text{on } \{-\delta_\varepsilon, \delta_\varepsilon\} \times]-\varepsilon, 0[, \\ u_\varepsilon^{\mathrm{dec}} = \varepsilon^4 b \mathcal{M}_4\left(\dfrac{x_2}{\varepsilon}\right), \quad \dfrac{\partial u_\varepsilon^{\mathrm{dec}}}{\partial \nu} = 0, \quad \text{on } \left]-\dfrac{\varepsilon}{2}, \dfrac{\varepsilon}{2}\right[\times \{\delta_\varepsilon\}. \end{cases}$$

Clearly, this problem admits a unique weak solution in $H^2(\Omega_{\varepsilon,0}^{\mathrm{dec}})$.

On the other side, it results in

$$u_\varepsilon^N - u_\varepsilon^{\mathrm{dec}} = 0, \quad \text{in } \Omega_\varepsilon \backslash \Omega_\varepsilon^{\mathrm{dec}},$$

and

$$\begin{cases} \triangle^2 (u_\varepsilon^N - u_\varepsilon^{\mathrm{dec}}) = \begin{cases} r_{1,\varepsilon}^N & \text{if } x_2 < 0, \\ r_{2,\varepsilon}^N & \text{if } x_2 > 0, \end{cases} \quad \text{in } \Omega_\varepsilon^{\mathrm{dec}}, \\ u_\varepsilon^N - u_\varepsilon^{\mathrm{dec}} = 0 = \dfrac{\partial (u_\varepsilon^N - u_\varepsilon^{\mathrm{dec}})}{\partial \nu} \quad \text{on } \partial\Omega_\varepsilon^{\mathrm{dec}}. \end{cases}$$

Consequently, if $f_1 \in C^{N+2}([-1,1])$ and $f_2 \in C^{N+2}([0,1])$, from (2.10) and (2.12) it follows that

$$\|u_\varepsilon^N - u_\varepsilon^{\mathrm{dec}}\|_{H^2(\Omega_\varepsilon)} \leq c_N \varepsilon^{N-3}\sqrt{\varepsilon}. \tag{6.4}$$

Finally, (6.1) and (6.4) provide that

$$\|u_\varepsilon - u_\varepsilon^{\mathrm{dec}}\|_{H^2(\Omega_\varepsilon)} \leq c_N \varepsilon^{N-3}\sqrt{\varepsilon}.$$

7. Remarks on the Smoothness Hypothesis for the Right-Hand Side

In this section we discuss the case where the right-hand side is not C^N-smooth function.

If $g_1 \in L^2(]0,1[)$ and $g_2 \in L^2(]-1,1[)$, for any fixed $\delta > 0$ there exist two functions $g_{1,\delta} \in C_0^{N+2}(]-1,1[)$ and $g_{2,\delta} \in C_0^{N+2}(]0,1[)$, satisfying (1.1) with $b = 0$, such that

$$\|g_1 - g_{1,\delta}\|_{L^2(]0,1[)} < \delta, \quad \|g_2 - g_{2,\delta}\|_{L^2(]-1,1[)} < \delta.$$

Let $u_{\delta,\varepsilon}$ be the solution of problem (1.2) with

$$g_\delta : (x_1,x_2) \in \Omega_\varepsilon \to g(x_1,x_2) = \begin{cases} g_{1,\delta}(x_1) & \text{if } (x_1,x_2) \in]-1,1[\times]-\varepsilon,0[, \\ g_{2,\delta}(x_2) & \text{if } (x_1,x_2) \in]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}[\times[0,1[. \end{cases}$$

Then, from Sec. 2 it follows that

$$u^{Nreg}_{\delta,\varepsilon}(x_1,x_2) = \begin{cases} \displaystyle\sum_{j=4}^{N} \varepsilon^j g^{(j-4)}_{1,\delta}(x_1)\mathcal{N}_j\left(\frac{x_2}{\varepsilon}\right), & \text{if } (x_1,x_2) \in [-1,1]\times[-\varepsilon,0], \\ \displaystyle\sum_{j=4}^{N} \varepsilon^j g^{(j-4)}_{2,\delta}(x_2)\mathcal{M}_j\left(\frac{x_1}{\varepsilon}\right), & \text{if } (x_1,x_2) \in \left[-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right]\times[0,1] \end{cases}$$

belongs to $C_0^{N+2}(\Omega_\varepsilon)$ and satisfies

$$\|u^{Nreg}_{\delta,\varepsilon} - u_{\delta,\varepsilon}\|_{H^2(\Omega_\varepsilon)} \le c_{N,\delta}\varepsilon^{N-1}\sqrt{\varepsilon},$$

where $c_{N,\delta}$ is a positive constant independent of ε, but depending on N and δ.

On the other hand, the solution u_ε of problem (1.2) with

$$g : (x_1,x_2) \in \Omega_\varepsilon \to g(x_1,x_2) = \begin{cases} g_1(x_1), & \text{if } (x_1,x_2) \in]-1,1[\times]-\varepsilon,0[, \\ g_2(x_2), & \text{if } (x_1,x_2) \in \left]-\frac{\varepsilon}{2},\frac{\varepsilon}{2}\right[\times[0,1[, \end{cases}$$

in the right-hand side satisfies the estimate

$$\|u_\varepsilon - u_{\delta,\varepsilon}\|_{H^2(\Omega_\varepsilon)} \le c\sqrt{\varepsilon}\delta,$$

where c is a positive constant independent of N,δ and ε. So, applying the triangle inequality we get

$$\|u_\varepsilon - u^{Nreg}_{\delta,\varepsilon}\|_{H^2(\Omega_\varepsilon)} \le c\sqrt{\varepsilon}\delta + c_{N,\delta}\varepsilon^{N-1}\sqrt{\varepsilon}.$$

If $g_{1,\delta}$ and $g_{2,\delta}$ have the first $N+2$ derivatives bounded by some constant independent of δ (but this estimate is not evident), then it results in also $c_{N,\delta} = c_N$ is independent of δ. In this case, the last estimate reduce to

$$\|u_\varepsilon - u^{Nreg}_{\delta,\varepsilon}\|_{H^2(\Omega_\varepsilon)} \le c\sqrt{\varepsilon}\delta + c_N\varepsilon^{N-1}\sqrt{\varepsilon}.$$

On the other hand, in applications, functions are mostly piecewise smooth. Let us see what happens if, for instance, f_1 is C^N-smooth everywhere, except for the point $x_1 = \frac{1}{2}$. In this case, the construction of u^N_ε should be modified by an additional boundary layer $u^{Nbl}_{\frac{1}{2},\varepsilon}$ corresponding to the point $x_1 = \frac{1}{2}$. Namely, the smooth part of the expansion is no more smooth in $x_1 = \frac{1}{2}$. It may have a gap as well as its derivatives with respect to x_1:

$$[u^{Nreg}_{1,\varepsilon}]_{x_1=\frac{1}{2}} = \sum_{j=4}^{N} \varepsilon^j [f_1^{(j-4)}]_{x_1=\frac{1}{2}}\mathcal{N}_j\left(\frac{x_2}{\varepsilon}\right), \quad \forall\, x_2 \in [-\varepsilon,0],$$

where $[v]_{x_1=\frac{1}{2}}$ denotes the jump of v in $x_1 = \frac{1}{2}$. Similarly, one has that

$$\left[\frac{\partial^{(i)} u_{1,\varepsilon}^{Nreg}}{\partial x_1^{(i)}}\right]_{x_1=\frac{1}{2}} = \sum_{j=4}^{N} \varepsilon^j [f_1^{(j+i-4)}]_{x_1=\frac{1}{2}} \mathcal{N}_j\left(\frac{x_2}{\varepsilon}\right), \quad \forall\, x_2 \in [-\varepsilon, 0], \ \ \forall\, i \in \{1,2,3\}.$$

These gaps should be compensated by boundary layer $u_{\frac{1}{2},\varepsilon}^{Nbl}(\frac{x_1-\frac{1}{2}}{\varepsilon}, \frac{x_2}{\varepsilon})$ such that $u_{\frac{1}{2},\varepsilon}^{Nbl}$ satisfies the following problem

$$\begin{cases} \triangle^2 u_{\frac{1}{2},\varepsilon}^{Nbl} = 0, \quad \text{in } (]-\infty,0[\cup]0,+\infty[) \times]-1,0[, \\ u_{\frac{1}{2},\varepsilon}^{Nbl}(\xi_1,-1) = 0 = \dfrac{\partial u_{\frac{1}{2},\varepsilon}^{Nbl}}{\partial \nu}(\xi_1,-1), \quad \xi_1 \in]-\infty,0[\cup]0,+\infty[, \\ u_{\frac{1}{2},\varepsilon}^{Nbl}(\xi_1,0) = 0 = \dfrac{\partial u_{\frac{1}{2},\varepsilon}^{Nbl}}{\partial \nu}(\xi_1,0), \quad \xi_1 \in]-\infty,0[\cup]0,+\infty[, \\ \left[\dfrac{\partial^{(i)} u_{\frac{1}{2},\varepsilon}^{Nbl}}{\partial \xi_1^{(i)}}\right]_{\xi_1=0} = -\displaystyle\sum_{j=4}^{N} \varepsilon^j [f_1^{(j+i-4)}]_{x_1=\frac{1}{2}} \mathcal{N}_j(\xi_2), \\ \qquad\qquad \forall\, \xi_2 \in [-1,0], \quad \forall\, i \in \{0,1,2,3\}. \end{cases}$$

Clearly, this boundary layer can be found as a sum $\sum_{j=4}^{N} \varepsilon^j u_{\frac{1}{2},j}^{Nbl}$, where $u_{\frac{1}{2},j}^{Nbl}$ are independent of ε. Note that this boundary layer corrector is of order ε^4, i.e. the same order as the exact solution u_ε.

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References

[1] S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, *Comm. Pure Appl. Math.* **12** (1959) 623–727.

[2] ———, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II, *Comm. Pure Appl. Math.* **17** (1964) 35–92.

[3] Y. Amirat, O. Bodart, U. De Maio and A. Gaudiello, Effective boundary condition for Stokes flow over a very rough surface, *J. Differential Equations* **254** (2013) 3395–3430.

[4] A. Bensoussan, J. L. Lions and G. Papanicolaou, *Asymptotic Analysis for Periodic Structures* (North-Holland, Amsterdam, 1978).

[5] F. Blanc, O. Gipouloux, G. Panasenko and A. M. Zine, Asymptotic analysis and partial asymptotic decomposition of the domain for Stokes equation in tube structure, *Math. Models Methods Appl. Sci.* **9**(9) (1999) 1351–1378.

[6] D. Blanchard, A. Gaudiello and T. A. Mel'nik, Boundary homogenization and reduction of dimension in a Kirchhoff–Love plate, *SIAM J. Math. Anal.* **39**(6) (2008) 1764–1787.

[7] G. Buttazzo and S. A. Nazarov, An optimization problem for the biharmonic equation with Sobolev conditions, *J. Math. Sci.* (*N.Y.*) **176**(6) (2011) 786–796.

[8] J. Casado-Díaz and M. Luna-Laynez, A multiscale method to the homogenization of elastic thin reticulated structures. Homogenization, 2001 (Naples), *GAKUTO Internat. Ser. Math. Sci. Appl.* **18** (2003) 155–168.

[9] P. G. Ciarlet, *Plates and Junctions in Elastic Multi-Structures. An Asymptotic Analysis*, Research in Applied Mathematics, Vol. 24 (Masson, 1990).

[10] ______, *Mathematical Elasticity. Volume II*: *Theory of Plates*, Studies in Mathematics and its Applications, Vol. 27 (Elsevier Sciences B.V., 1997).

[11] D. Cioranescu and J. Saint Jean Paulin, *Homogenization of Reticulated Structures*, Applied Mathematical Sciences, Vol. 139 (Springer, 1999).

[12] A. Gaudiello, B. Gustafsson, C. Lefter and J. Mossino, Asymptotic analysis of a class of minimization problems in a thin multidomain, *Calc. Var. Partial Differ. Eqs.* **15**(2) (2002) 181–201.

[13] A. Gaudiello and A. Sili, Asymptotic analysis of the eigenvalues of an elliptic problem in an anisotropic thin multidomain, *Proc. Roy. Soc. Edinburgh Sect. A* **141**(4) (2011) 739–754.

[14] A. Gaudiello and E. Zappale, Junction in a thin multi-domain for a fourth order problem, *Math. Models Methods Appl. Sci.* **16**(12) (2006) 1–32.

[15] ______, A model of joined beams as limit of a 2D plate, *J. Elast.* **103** (2011) 205233.

[16] F. Gazzola, H.-C. Grunau and G. Sweers, *Polyharmonic Boundary Value Problems*, Lectures Notes in Mathematics, Vol. 1991 (Springer, 2010).

[17] V. A. Kozlov, V. G. Ma'zya and A. B. Movchan, *Asymptotic Analysis of Fields in a Multi-Structure*, Oxford Mathematical Monographs (Oxford University Press, 1999).

[18] O. A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics* (Springer, 1985).

[19] A. E. Lapshin and G. P. Panasenko, Asymptotic analysis of the solution of Dirichlet's problem for Poisson's equation posed in periodic lattice-like domain, *Vestnik Moskov. Univ. Ser. Math. Mech.* **5** (1995) 43–50.

[20] ______, Asymptotic expansion of the solution of Dirichlet's problem for Poisson's equation posed in non-periodic lattice-like domain, *Trudy Seminara I.G. Petrovskogo* **19** (1996) 99–108.

[21] H. Le Dret, *Problèmes variationnels dans les multi-domaines*: *Modélisation des jonctions et applications*, Research in Applied Mathematics, Vol. 19 (Masson, 1991).

[22] H. Le Dret and N. Meunier, Modeling heterogeneous martensitic wires, *Math. Models Methods Appl. Sci.* **15**(3) (2005) 375–406.

[23] S. E. Lyshevski, *MEMS and NEMS*: *Systems, Devices, and Structures* (CRC Press, 2002).

[24] S. A. Nazarov, Elliptic boundary value problems with periodic coefficients in a cylinder, *Math. USSR Izvestija* **18**(3) (1982) 89–98.

[25] G. P. Panasenko, Homogenization processes in lattice structures, *Math. Sb.* **122**(2) (1983) 220–231.

[26] ———, Asymptotic solutions of the elasticity theory system of equations for lattice and skeletal structures, *Math. Sb.* **183**(1) (1992) 89–113 (in Russian); English translation: *Russian Acad. Sci. Sbornik Math.* **75**(1) (1993) 85–110.

[27] ———, Method of asymptotic partial decomposition of domain, *Math. Models Methods Appl. Sci.* **8**(1) (1998) 139–156.

[28] ———, Asymptotic expansion of the solution of Navier–Stokes equation in a tube structure, *C. R. Acad. Sci. Paris, Sér. IIb* **326** (1998) 867–872.

[29] ———, Asymptotic expansion of the solution of Navier–Stokes equation in tube structure and partial asymptotic decomposition of the domain, *Applicable Anal.* **76**(3–4) (2000) 363–381.

[30] ———, *Multi-Scale Modelling for Structures and Composites* (Springer, 2005).

[31] G. Sweers, A survey on boundary conditions for the biharmonic, *Complex Var. Elliptic Eqs.* **54**(2) (2009) 79–93.

[32] M. I. Visik, On the strongly elliptic systems of differential equations, *Math. USSR Sbornik* **29**(3) (1951) 615–676.