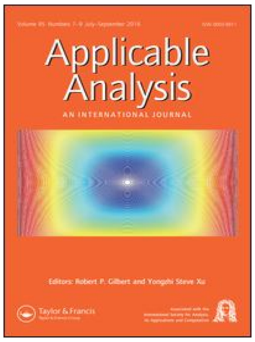

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Resolvent bounds for jump generators

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ABSTRACT
The paper deals with jump generators with a convolution kernel. Assuming that the kernel decays either exponentially or polynomially, we prove a number of lower and upper bounds for the resolvent of such operators. In particular we focus on sharp estimates of the resolvent kernel for small values of the spectral parameter. We consider two applications of these results. First we obtain pointwise estimates for principal eigenfunction of jump generators perturbed by a compactly supported potential (so-called nonlocal Schrödinger operators). Then we consider the Cauchy problem for the corresponding inhomogeneous evolution equations and study the behaviour of its solutions.

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1. Introduction

This paper deals with nonlocal operators with an integrable convolution kernel. The properties of such operators depend crucially on the behaviour of the convolution kernel at infinity. We consider different cases covering both polynomial and exponential rates of decay of its tails. Our main aim is to investigate the behaviour at infinity of the resolvent kernel of these operators. We prove a number of lower and upper bounds for the said resolvent kernel and, with the help of these results, deduce pointwise bounds for the principal eigenfunction of the operator obtained from the original convolution type operator by adding a compactly supported potential. Another application concerns the inhomogeneous Cauchy problems for convolution type operators.

The nonlocal operators considered here are generators of Markov jump processes with the jump distribution defined by the convolution kernel. The analysis of the behaviour of the processes can be performed in terms of the resolvent of its generators. On the other hand, such operators appear in the kinetic description of birth-and-death Markov dynamics of populations in spatial ecology, see e.g. [1] and the literature therein. In this setting the tail of the kernel can be thought of as the range of dispersion of newborn individuals. In many applications this kernel might have heavy tails.

The problem of existence of discrete spectrum and the principal eigenfunction for the perturbed operator has been studied in recent papers [2–5]. This operator may be considered as a nonlocal version of the Schrödinger operator in which the Laplacian is changed to a convolution type operator. It is interesting to observe that the ground state problem for such nonlocal Schrödinger operator is rather different comparing with that in the classical local case. We will come to this point later on.

To our best knowledge the only work in the existing literature where the behaviour of the resolvent kernel for nonlocal operators has been studied is [5]. In the mentioned work the authors consider the

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case of nonlocal operators with the convolution kernel decaying super exponentially. The fast decay of the kernel allows one to use the techniques of analytic functions.

In the present paper we consider both polynomially and exponentially decaying convolution kernels. However, the main focus of this work is on polynomially decaying kernels: under some natural conditions we show that the resolvent kernel has the same polynomial decay as the convolution kernel, see Theorem 2.1. In the later case the resolvent kernel decays exponentially, see Theorems 2.2, 2.3. An important part of our study is obtaining sharp upper and lower bounds for the resolvent kernel as the spectral parameter is getting sufficiently small. These bounds play then a crucial role when we study the spectral problem for nonlocal Schrödinger operators being a perturbation of the convolution operator by a bounded localized potential. If the perturbation is "small enough", then the corresponding principal eigenvalue is small, and the behaviour of the principal eigenfunction at infinity can be expressed in terms of the resolvent kernel with a small positive value of the spectral parameter, see Theorem 3.2.

Notice that, in the case of exponentially decaying convolution kernels, the exponential decay of the corresponding resolvent kernel can be deduced from Paley–Wiener theorem, see e.g. [6]. However, our approach allows us to find sharp estimates for the rate of decay and, in some cases, even the asymptotics of the resolvent kernel.

We also consider in this work the behaviour of solutions to the Cauchy problem for a nonlocal heat equations with a stationary source term. In particular, we provide lower and upper bounds for the solutions and prove the convergence to the stationary solution. Previously a number of qualitative results for nonlocal heat equations have been obtained in a number of works, see [7,8] and references therein.

The paper is organized as follows. Section 2 deals with the behaviour of the resolvent of integral operators with convolution kernels. In Section 2.1 we study the case of polynomially decaying kernels while Section 2.2 is devoted to the case of exponentially decaying tails.

These results are then used in Section 3 to study the properties of the principal eigenfunction of the perturbed operator and the large time behaviour of solutions to the Cauchy problem for the corresponding non-homogeneous evolution equation.

2. Bounds for resolvent kernel

We consider the operator

$$(L_0 u)(x) = \int_{\mathbb{R}^d} a(x-y)(u(y)-u(x))\mathrm{d}y. \tag{1}$$

Throughout this paper we assume the following properties for the dispersal kernel: $a(\,\cdot\,) \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ is a nonnegative bounded even continuous function of unit mass, i.e.

$$\int_{\mathbb{R}^d} a(x)\mathrm{d}x = 1. \tag{2}$$

Consequently

$$a(\,\cdot\,) \in L^2(\mathbb{R}^d) \quad \text{and its Fourier transform } \tilde{a}(\,\cdot\,) = \int_{\mathbb{R}^d} e^{-i(\cdot,x)} a(x)\mathrm{d}x \in L^2(\mathbb{R}^d) \cap C_b(\mathbb{R}^d). \tag{3}$$

For the further analysis it is convenient to rewrite operator L_0 in (1) as follows:

$$L_0 u(x) = \mathcal{S}_a u(x) - u(x), \quad \mathcal{S}_a u(x) = \int_{\mathbb{R}^d} a(x-y)u(y)\mathrm{d}y. \tag{4}$$

In this section we study the behaviour of the kernel of the resolvent for the operator L_0 under the condition that the function $a(x)$ decays either polynomially or exponentially. Denote by $\mathcal{R}_\lambda(x,y)$ the kernel of the resolvent $(\lambda - L_0)^{-1}$. This kernel admits the representation

$$\mathcal{R}_\lambda(x,y) = (1+\lambda)^{-1}\Big(\delta(x-y) + G_\lambda(x-y)\Big), \quad \lambda \in (0,\infty), \tag{5}$$

where $G_\lambda(x-y)$ is the kernel of the convolution operator

$$\sum_{k=1}^{\infty} \frac{\mathcal{S}_a^k}{(1+\lambda)^k}. \tag{6}$$

Denote by $a_k(x-y)$ the kernel of the operator $\mathcal{S}_a^k$, then

$$a_k(x) = a^{*k}(x) = \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} a(x-y_1)a(y_1-y_2)\dots a(y_{k-2}-y_{k-1})a(y_{k-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{k-1}, \tag{7}$$

and

$$G_\lambda(x) = \sum_{k=1}^{\infty} \frac{a_k(x)}{(1+\lambda)^k} \tag{8}$$

with $a_k(x)$ defined by (7).

2.1. Polynomial tail of dispersal kernel

In this section we deal with functions $a(x)$ that satisfy the following bounds

$$c_-(1+|x|)^{-(d+\alpha)} \le a(x) \le c_+(1+|x|)^{-(d+\alpha)}, \tag{9}$$

with an arbitrary $\alpha > 0$.

Theorem 2.1: *There exist constants $0 < \tilde{c}_-(\lambda) \le \tilde{c}_+(\lambda)$, such that*

$$\tilde{c}_-(d+|x|)^{-(d+\alpha)} \le G_\lambda(x) \le \tilde{c}_+(1+|x|)^{-(d+\alpha)}, \tag{10}$$

where $G_\lambda(x)$ is defined by (6).

Furthermore, $\tilde{c}_+(\lambda) = O(\lambda^{-(2+d+\alpha)})$ as λ goes to 0, and

$$G_\lambda(x) \ge \frac{C_0}{\lambda}(1+|x|)^{-(d+\alpha)}$$

for large enough $|x|$ with a constant $C_0 > 0$.

***Proof*:** I. The upper bound. In $\mathbb{R}^{(k-1)d}$ introduce the sets

$$A_1 = \left\{y \in \mathbb{R}^{(k-1)d} \,:\, |x-y_1| \ge \tfrac{|x|}{k}\right\},\ A_2 = \left\{y \in \mathbb{R}^{(k-1)d} \,:\, |y_1-y_2| \ge \tfrac{|x|}{k}\right\},\dots,$$
$$A_k = \left\{y \in \mathbb{R}^{(k-1)d} \,:\, |y_{k-1}| \ge \tfrac{|x|}{k}\right\}.$$

One can easily check that $\mathbb{R}^{(k-1)d} \subset \bigcup_{j=1}^k A_j$ and, therefore,

$$a_k \le \sum_{j=1}^{k} \int_{A_j} a(x-y_1)a(y_1-y_2)\dots a(y_{k-2}-y_{k-1})a(y_{k-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{k-1}.$$

Thus considering (9) we obtain

$$\begin{aligned}
&\int_{A_1} a(x-y_1)a(y_1-y_2)\dots a(y_{k-2}-y_{k-1})a(y_{k-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{k-1}\\
&\quad\le \frac{c_+}{(1+|k^{-1}x|)^{d+\alpha}}\int_{A_1} a(y_1-y_2)\dots a(y_{k-2}-y_{k-1})a(y_{k-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{k-1}\\
&\quad\le \frac{c_+}{(1+|k^{-1}x|)^{d+\alpha}}\int_{\mathbb{R}^{k-1}} a(y_1-y_2)\dots a(y_{k-2}-y_{k-1})a(y_{k-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{k-1}\\
&\quad= \frac{c_+}{(1+|k^{-1}x|)^{d+\alpha}}.
\end{aligned}$$

Similarly,

$$\int_{A_j} a(x-y_1)a(y_1-y_2)\dots a(y_{k-2}-y_{k-1})a(y_{k-1})\,\mathrm{d}y_1\dots\mathrm{d}y_{k-1}\le \frac{c_+}{(1+|k^{-1}x|)^{d+\alpha}}$$

for $j=1,2,\dots,k-1$. Finally,

$$a_k(x)\le \frac{c_+k}{(1+|k^{-1}x|)^{d+\alpha}} \tag{11}$$

Denote $\Lambda^2=(1+\lambda)$, then

$$\Lambda^{-2k}a_k(x)\le \frac{c_+k\Lambda^{-k}}{(\Lambda^{k/(d+\alpha)}+|\Lambda^{k/(d+\alpha)}k^{-1}x|)^{d+\alpha}}\le \frac{c_+k\Lambda^{-k}}{(1+|\gamma x|)^{d+\alpha}}$$

with $\gamma=\min\limits_k \Lambda^{k/(d+\alpha)}k^{-1}$. Notice that $\gamma=\gamma(\lambda)>0$, and $\gamma=O(\lambda)$ for small $\lambda>0$. Summing up these inequalities in k we conclude that the kernel $G_\lambda(x)$ is bounded from above by

$$\frac{\tilde c_+}{(1+|x|)^{d+\alpha}}.$$

This completes the proof of the upper bound in (10).

We proceed with the asymptotics of $\tilde c_+(\lambda)$ as $\lambda\to 0$. Using (11) we get the following upper bound for the sum (8):

$$\sum_{k=1}^{\infty}\frac{c_+k^{1+d+\alpha}(1+\lambda)^{-k}}{(k+|x|)^{d+\alpha}}\le \frac{c_+}{(1+|x|)^{d+\alpha}}\sum_{k=1}^{\infty}k^{1+d+\alpha}(1+\lambda)^{-k}.$$

Consequently,

$$\tilde c_+(\lambda)=c_+\sum_{k=1}^{\infty}\frac{k^{1+d+\alpha}}{(1+\lambda)^k}=O(\lambda^{-(2+d+\alpha)})\quad \text{as}\ \ \lambda\to 0.$$

In order to justify the last relation we first restrict the summation to a finite range of k. Namely,

$$\sum_{k=1}^{\infty}\frac{k^{1+d+\alpha}}{(1+\lambda)^k}\ge \sum_{k=1}^{[\log 2/\lambda]}\frac{k^{1+d+\alpha}}{(1+\lambda)^k}\ge \frac12\sum_{k=1}^{[\log 2/\lambda]}k^{1+d+\alpha}\ge C\lambda^{-(2+d+\alpha)}$$

To obtain an upper bound we estimate the contribution of the terms related to $k \in J_m = \{k \in \mathbb{Z}^+ : m\lambda^{-1}\log 2 \le k \le (m+1)\lambda^{-1}\log 2\}$, $m \in \mathbb{Z}^+$:

$$\sum_{k\in J_m} \frac{k^{1+d+\alpha}}{(1+\lambda)^k} \le \sum_{k=J_m} \frac{k^{1+d+\alpha}}{2^m} \le \frac{(m+1)^{2+d+\alpha}}{2^m}\lambda^{-(2+d+\alpha)}.$$

Summing up in m we arrive at the desired bound

$$\sum_{k=0}^{\infty} \frac{k^{1+d+\alpha}}{(1+\lambda)^k} \le C\lambda^{-(2+d+\alpha)}.$$

II. The lower bound. The lower bound is quite straightforward, if we take in (5) the first term of the sum. To trace the dependence of the lower bound on λ for small values of $\lambda > 0$ we should take into account the higher order terms in (8).

From (9) it follows that

$$\int\limits_{|y|\ge r} a(y)\,\mathrm{d}y \le Cr^{-\alpha}$$

with a constant $C = C(c_+, \alpha, d)$ that only depends on c_+, α and d. Then

$$\int\limits_{|y|\le r} a(y)\,\mathrm{d}y \ge 1 - Cr^{-\alpha}. \tag{12}$$

□

Lemma 2.1: *For all $\lambda \in (0,1)$ and for all*

$$|x| \ge \left(\frac{1}{\lambda}\right)^{\frac{\alpha+1}{\alpha}} m \quad \text{with} \;\; m = (2C)^{\frac{\alpha+1}{\alpha}}, \tag{13}$$

where C is the same constant as in (12), we have

$$G_\lambda(x) \ge \frac{C_0}{\lambda}(1+|x|)^{-(d+\alpha)},$$

where C_0 is a positive constant that does not depend on λ.

***Proof*:** After change of variables we get

$$\begin{aligned} a_k(x) &= \int\limits_{\mathbb{R}^d} \dots \int\limits_{\mathbb{R}^d} a(z_1)a(z_2)\dots a(z_{k-1})a(x - z_1 - z_2 - \dots - z_{k-1})\,\mathrm{d}z_1\dots\mathrm{d}z_{k-1} \\ &\ge \int\limits_{B_r} a(z_1)a(z_2)\dots a(z_{k-1})a(x - z_1 - z_2 - \dots - z_{k-1})\,\mathrm{d}z_1\dots\mathrm{d}z_{k-1}, \end{aligned}$$

where $B_r = \{z \in \mathbb{R}^{(k-1)d} : |z_j| \le r\}$. For $z \in B_r$ the following inequality is fulfilled:

$$a(x - z_1 - z_2 - \dots - z_{k-1}) \ge c_-(1 + |x| + kr)^{-(d+\alpha)}.$$

Moreover,

$$\int\limits_{B_r} a(z_1)a(z_2)\dots a(z_{k-1})\,\mathrm{d}z \ge (1 - Cr^{-\alpha})^{k-1}$$

with a constant C. This yields

$$a_k(x) \geq c_-(1 - Cr^{-\alpha})^{k-1}(1 + |x| + kr)^{-(d+\alpha)}.$$

Considering the properties of the function $(1 - x - e^{-2x})$ in the vicinity of zero, one observes that under our choice of the constant m in (13) for all $\xi \geq m$ the inequality holds

$$1 - C\xi^{-\frac{\alpha}{\alpha+1}} \geq e^{-2C\xi^{-\frac{\alpha}{\alpha+1}}}. \tag{14}$$

Then we consider all positive integer k that satisfy the estimate

$$k \leq k_0(|x|) = \left[|x|^{\frac{\alpha}{\alpha+1}}\right] + 1,$$

and set

$$r = r(|x|) = |x|^{\frac{1}{\alpha+1}}.$$

Then using (14) we have that

$$1 - Cr^{-\alpha} \geq e^{-2Cr^{-\alpha}} \qquad \text{for all} \quad |x| \geq m,$$

and for all $k \leq k_0$ we have the uniform lower bound (for all x, meeting (13)):

$$(1 - Cr^{-\alpha})^{k-1} \geq e^{-2Ck_0 r^{-\alpha}} \geq e^{-2C} = \tilde{C}_0.$$

In addition,

$$(1 + |x| + kr)^{-(d+\alpha)} \geq (1 + 2|x|)^{-(d+\alpha)}.$$

Finally, we have

$$a_k(x) \geq C_0(1 + |x|)^{-(d+\alpha)}, \quad C_0 = 2^{-(d+\alpha)}\tilde{C}_0 c_-, \quad \alpha > 0. \tag{15}$$

Summing up in k and recalling inequality (13), we obtain

$$\begin{aligned} G_\lambda(x) &\geq C_0(1 + |x|)^{-(d+\alpha)} \sum_{k=1}^{k_0(|x|)} (1 + \lambda)^{-k} \\ &\geq C_0(1 + |x|)^{-(d+\alpha)} \frac{1 - (1 + \lambda)^{-|x|^{\frac{\alpha}{\alpha+1}}}}{\lambda} \geq \frac{C_0}{2\lambda}(1 + |x|)^{-(d+\alpha)}. \end{aligned}$$

We used in the last inequality that the function $g(u) = (1 + u)^{-\frac{1}{u}}$ is increasing as $u \in (0, 1)$, and $g(0) = e^{-1}$, $g(1) = \frac{1}{2}$. This completes the proof of Lemma 2.1 and of Theorem 2.1. □

In the next section we study the behaviour of the function $G_\lambda(x)$ in the case of exponentially decaying convolution kernel $a(x)$.

2.2. Exponential tail of dispersal kernel

In this section we consider the exponentially decaying dispersal kernels. Namely, we assume that in addition to (2) - (3) the function $a(x)$ satisfies the following upper bound:

$$a(x) \;\leq\; c\; \exp\{-\delta|x|\} \tag{16}$$

with a positive constant δ. We prove here that the function $G_\lambda(x)$ defined by (6) decays also exponentially and find the rate of decay of the function $G_\lambda(x)$ for small enough λ.

Theorem 2.2: *There exist positive constants $k = k(\lambda)$, $m = m(\lambda)$, such that the function $G_\lambda(x)$ satisfies the following upper bound:*

$$G_\lambda(x) \le k(\lambda)e^{-m(\lambda)|x|}. \tag{17}$$

Moreover, $k(\lambda) \to \infty$ and $m(\lambda) = O(\lambda)$, as $\lambda \to 0$.

***Proof*:** Since the function $G_\lambda(x)$ has the following representation in terms of the Fourier transform $\tilde{a}(p)$:

$$G_\lambda(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)}\mathrm{d}p, \tag{18}$$

then for a fixed $\lambda > 0$ inequality (17) is a consequence of the Paley–Wiener theorem, see e.g. [6, Theorem IX.14]. However, our goal is to obtain sharp estimates on $k(\lambda)$, $m(\lambda)$ for small values of λ. To this end we use probabilistic arguments based on the following statement.

Lemma 2.2: *Let X_i be i.i.d. random variables taking values in $\mathbb{R}^d$ with the distribution density $a(x)$ satisfying the upper bound (16), and denote $S_n = X_1 + \ldots + X_n$. Then for any unit vector $\theta \in \mathbb{R}^d$ there exists a constant $c_1 = c_1(c, \delta)$ such that for all $n \ge 1$ the following estimates hold:*

$$P\{\theta \cdot S_n > r\} \le \begin{cases} e^{-\frac{r^2}{4c_1 n}}, & 0 < r \le \delta c_1 n \\ e^{-\frac{\delta r}{4}}, & r > \delta c_1 n \end{cases} \tag{19}$$

where δ is the same as in (16).

***Proof*:** Estimate (16) on the density $a(x)$ is isotropic, hence we can take $\theta = e_1$ and consider an 1-d random variable $\xi = \theta \cdot X_1 = X_1^{(1)}$. The distribution $a_\xi(x)$ of ξ satisfies the estimate similar to (16) with the same δ and some constant $\tilde{c} = \tilde{c}(c, \delta)$:

$$a_\xi(x) \le \tilde{c}e^{-\delta|x|}, \quad x \in \mathbb{R}^d.$$

Let $k \in (0, \frac{\delta}{2}]$ be a constant, then using the Taylor decomposition for the exponent $e^{k\xi}$ and estimates on the moments of ξ we get

$$\mathbb{E}e^{k\,\xi} \le e^{c_1 k^2} \quad \text{for } 0 < k \le \frac{\delta}{2} \tag{20}$$

with a constant $c_1 = c_1(\tilde{c}, \delta)$.

Let us take a unit vector $\theta \in \mathbb{R}^d$ and fix $n \in \mathbb{N}$. Then using the Markov inequality, the independence of random variables X_j and (20) we get for the 1-d random variable $\theta \cdot S_n$:

$$\begin{aligned} P\{\theta \cdot S_n > r\} &= P\{k\theta \cdot S_n > kr\} = P\{e^{k\theta \cdot S_n} > e^{kr}\} \\ &\le \min_{0<k\le\frac{\delta}{2}} \frac{(\mathbb{E}e^{k\theta\cdot X_1})^n}{e^{kr}} \le \min_{0<k\le\frac{\delta}{2}} e^{c_1k^2n - kr} = \exp\Big\{\min_{0<k\le\frac{\delta}{2}} f(k)\Big\} \end{aligned}$$

with $f(k) = c_1k^2n - kr$. We consider two cases. If

$$k_0 = \operatorname{argmin} f(k) = \frac{r}{2c_1 n} \le \frac{\delta}{2},$$

then $\min_{0<k\le\frac{\delta}{2}} f(k) = f(k_0) = -\frac{r^2}{4c_1 n}$, and

$$P\{\theta\cdot S_n > r\} \le e^{-\frac{r^2}{4c_1 n}}.$$

If $r > \delta c_1 n$, then $k_0 > \frac{\delta}{2}$. Consequently, $\min_{0<k\le\frac{\delta}{2}} f(k) = f(\frac{\delta}{2})$ and

$$P\{\theta\cdot S_n > r\} \le e^{\frac{c_1\delta^2 n}{4} - \frac{\delta r}{2}} = e^{(\frac{c_1\delta^2 n}{4} - \frac{\delta r}{4}) - \frac{\delta r}{4}} \le e^{-\frac{\delta r}{4}}.$$

□

For a d-dimensional random vector η and arbitrary $\varepsilon > 0$ one can find a finite collection of unit vectors $\theta_1,\dots,\theta_N,\ N = N(\varepsilon, d)$ such that

$$\{|\eta| > r\} \subset \bigcup_{i=1}^{N}\{\theta_i\cdot\eta > (1-\varepsilon)r\}$$

Then

$$P\{|\eta| > r\} \le N(\varepsilon, d)\, P\{\theta_1\cdot\eta > (1-\varepsilon)r\}. \tag{21}$$

Together with the result of Lemma 2.2 it gives

$$P\{|S_n| > r\} \le \begin{cases} c_2\, e^{-\frac{r^2}{4c_1 n}}, & 0 < r \le d_1 n \\ c_3\, e^{-\frac{\delta r}{4}}, & r > d_1 n \end{cases} \tag{22}$$

with constants $c_2,\ c_3, d_1$ which do not depend on r and n.

Next we show that the density $a_n(x)$ satisfies the estimate similar to (22). Indeed, denote by $F_n(y)$ the distribution function of S_n, then from (16) it follows

$$\begin{aligned} a_{n+1}(x) &= \int_{\mathbb{R}^d} a(x-y)\, \mathrm{d}F_n(y) \le c \int_{\mathbb{R}^d} e^{-\delta|x-y|}\, \mathrm{d}F_n(y) \\ &= c \int_{|y-x|\le\frac{1}{2}|x|} e^{-\delta|x-y|}\, \mathrm{d}F_n(y) + c \int_{|y-x|>\frac{1}{2}|x|} e^{-\delta|x-y|}\, \mathrm{d}F_n(y) \le cP\Big\{|S_n| \ge \frac{1}{2}|x|\Big\} + ce^{-\frac{1}{2}\delta|x|}. \end{aligned}$$

Together with (22) this yields for all $n \ge 1$:

$$a_n(x) \le \begin{cases} \tilde{c}_1 \max\left\{e^{-l_1\frac{|x|^2}{n}},\ e^{-\frac{1}{2}\delta|x|}\right\}, & |x| \le Bn \\ \tilde{c}_2\, e^{-\frac{\delta}{8}|x|}, & |x| > Bn \end{cases} \tag{23}$$

with constants $\tilde{c}_1, \tilde{c}_2, l_1, B$ which do not depend on $|x|$ and n. Let us estimate now $G_\lambda(x)$ from above using (8) and (23):

$$G_\lambda(x) = \sum_{n=1}^{\left[\frac{|x|}{B}\right]} \frac{a_n(x)}{(1+\lambda)^n} + \sum_{n>\left[\frac{|x|}{B}\right]} \frac{a_n(x)}{(1+\lambda)^n}$$
$$\leq \frac{\tilde{c}_2}{\lambda} e^{-\frac{\delta}{8}|x|} + \tilde{c}_1 \sum_{n\geq\left[\frac{|x|}{B}\right]} \frac{1}{(1+\lambda)^n} \leq \frac{\tilde{c}_3}{\lambda}\left(e^{-\frac{\delta}{8}|x|} + e^{-l_3(\lambda)|x|}\right) \leq k(\lambda) e^{-m(\lambda)|x|}$$

with $k(\lambda) = \frac{2\tilde{c}_3}{\lambda}$, and $l_3(\lambda) = \frac{1}{B}\ln(1+\lambda)$, $m(\lambda) = \min\{\frac{\delta}{8}, l_3(\lambda)\}$. Consequently, if λ is small enough, then $m(\lambda) = l_3(\lambda) = \frac{\lambda}{B}(1+o(1))$.

Theorem is proved. □

Remark 2.1: If we assume that there exist two constants c_1, c_2, such that

$$c_1 e^{-\delta|x|} \leq a(x) \leq c_2 e^{-\delta|x|}, \quad x \in \mathbb{R}^d,$$

then

$$k_1(\lambda) e^{-\delta|x|} \leq G_\lambda(x) \leq k_2(\lambda) e^{-m(\lambda)|x|}$$

with positive constants $k_1(\lambda), k_2(\lambda), m(\lambda)$.

In the one-dimensional case, the constants $m(\lambda)$ in Theorem 2.2 can be found more precisely. We assume that the function $a(x)$ meets the following asymptotics at infinity:

$$\lim_{|x|\to\infty} \frac{\ln a(x)}{|x|} = -c \tag{24}$$

with a constant $c > 0$. Then the Fourier transform $\tilde{a}(p)$ is an analytic function in the strip $|Im\, p| < c$. Assume additionally some smoothness of the function $a(x)$ guaranteeing that

$$\tilde{a}_\kappa(v) \equiv \tilde{a}(i\kappa + v) = \int a(x) e^{\kappa x} e^{-ivx} \mathrm{d}x \in L^1(\mathbb{R}) \quad \text{for any } 0 \leq \kappa < c. \tag{25}$$

In particular, (25) is valid if $a(x) \in C^2(\mathbb{R})$ and $\left(a(x)e^{\kappa x}\right)', \left(a(x)e^{\kappa x}\right)'' \in L^1(\mathbb{R})$ for any $0 \leq \kappa < c$.

We are interested in the asymptotics at infinity of $G_\lambda(x)$ introduced in (8). Let us consider solutions $\hat{p} = \hat{p}(\lambda)$ of the equation

$$\tilde{a}(\hat{p}) = 1 + \lambda. \tag{26}$$

It is easy to see, that if there exist solutions of (26), then there exists at least a pure imaginary solution $\hat{p} = iq$, and $\beta > q$ for any other solutions $\hat{p} = \alpha + i\beta$ of (26). Indeed, considering the properties of $a(x)$ we conclude that $\tilde{a}(iq)$ is a continuous positive increasing function of q, and $\tilde{a}(iq) > |\tilde{a}(iq+\alpha)|$.

Theorem 2.3: *Assume that $a(x)$ satisfies conditions (2), (3), (24), (25), and let $\lambda > 0$ be a positive constant. There exist constants $C(\lambda), C_-(\lambda), C_+(\lambda)$ such that*

(1) *if there exists a pure imaginary solution $\hat{p}(\lambda) = iq(\lambda)$ of Equation (26) with $q(\lambda) < c$, then $G_\lambda(x)$ has the asymptotics*

$$G_\lambda(x) = e^{-q(\lambda)|x|}(C(\lambda) + o(1)), \quad |x| \to \infty; \tag{27}$$

(2) *otherwise we have the following two-sided bound:*

$$C_-(\lambda) e^{-(c+\varepsilon)|x|} \leq G_\lambda(x) \leq C_+(\lambda) e^{-(c-\varepsilon)|x|} \tag{28}$$

with any $\varepsilon > 0$.

Remark 2.2: Notice that if

$$\lim_{q\to c-} \int_{\mathbb{R}} a(x)e^{qx}\mathrm{d}x = +\infty, \tag{29}$$

then the Equation (26) has a pure imaginary solution $\hat{p}(\lambda) = iq(\lambda)$ for any $\lambda > 0$. In this case the asymptotics of $G_\lambda(x)$ is given by (27).

If the limit (29) is finite, then depending on the value of λ both (27) and (28) can realize.

Proof: The proof relies on the analyticity of $\tilde{a}(p)$ in an appropriate strip. We use the representation (18) for $G_\lambda(x)$ in terms of $\tilde{a}(p)$. Let $x > 0$, and construct a rectangular closed contour containing a segment of a real line $[-K, K]$ a parallel segment $[K + i\kappa, -K + i\kappa]$ and two segments $I_1(K)$, $I_2(K)$ parallel to the imaginary axis: $I_1(K) = [K, K + i\kappa]$, $I_2(K) = [-K + i\kappa, -K]$, where $0 \le \kappa < c$. In the first case when $q < c$ we can take $\kappa : q < \kappa < c$ with no other solutions of (26) in the strip $0 < \mathcal{I}\mathrm{m}\, p < \kappa$. Then we have:

$$\int_{-K}^{K} \frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)}\mathrm{d}p + \int_{I_1(K)\cup I_2(K)} \frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)}\mathrm{d}p$$
$$+ \int_{K}^{-K} \frac{\tilde{a}(i\kappa + u)}{\lambda + 1 - \tilde{a}(i\kappa + u)} e^{i(i\kappa+u)x}\mathrm{d}u \;=\; 2\pi i\, res_q \left(\frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)} \right),$$

and

$$G_\lambda(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)}\mathrm{d}p = 2\pi i\, res_q \left(\frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)} \right)$$
$$+ \int_{\mathbb{R}} \frac{\tilde{a}(i\kappa + u)}{\lambda + 1 - \tilde{a}(i\kappa + u)} e^{-\kappa x + iux}\mathrm{d}u - \lim_{K\to\infty} \int_{I_1(K)\cup I_2(K)} \frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)}\mathrm{d}p. \tag{30}$$

Let us prove first that the limit in (30) is equal to 0. Since the function $a(x)e^{ux} \le a(x)e^{\kappa x} \le Be^{-\delta|x|}$ with some $B > 0$ and $\delta > 0$ uniformly in $u \in [0, \kappa]$, then

$$\int_{\mathbb{R}} a(x)e^{ux}\mathrm{d}x < \infty \quad \text{for all } \; u \in [0, \kappa].$$

Consequently,

$$\tilde{a}(K + iu) = \int a(x)e^{-i(K+iu)x}\mathrm{d}x = \int a(x)e^{ux}e^{-iKx}\mathrm{d}x \to 0 \quad \text{as } \; K \to \infty$$

uniformly in $u \in [0, \kappa]$. Analogously, the function $\tilde{\Phi}(p) = \frac{\tilde{a}(p)}{\lambda+1-\tilde{a}(p)}$ is uniformly bounded on $I_1(K) \cup I_2(K)$ and $|\tilde{\Phi}(K + iu)| \to 0$ as $K \to \infty$ uniformly over $u \in [0, \kappa]$. Since $|e^{i(\pm K+iu,x)}| = e^{-ux} \le 1$, then we obtain that

$$\int_{I_1(K)\cup I_2(K)} \frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)}\mathrm{d}p \to 0 \quad \text{as } \; K \to \infty.$$

Let us consider the upper segment $[-K + i\kappa, K + i\kappa]$ of the contour. As above we have that the function $\tilde{a}(i\kappa + u)$ is uniformly bounded for all $u \in [-K, K]$, and since $a(x)e^{\kappa x} \in L^1(\mathbb{R})$ we get

$$\tilde{a}(i\kappa + u) = \int a(x)e^{\kappa x}e^{-iux}\mathrm{d}x \to 0 \quad \text{as } \; |u| \to \infty.$$

Thus $\tilde{\Phi}(i\kappa + u)$ is uniformly bounded for all $u \in [-K, K]$, and (25) implies that

$$\int_{\mathbb{R}} |\tilde{\Phi}(i\kappa + u)| \mathrm{d}u < \infty.$$

Consequently, we have for the integral in (30):

$$\int_{\mathbb{R}} \frac{\tilde{a}(i\kappa + u)}{\lambda + 1 - \tilde{a}(i\kappa + u)} e^{-\kappa x + iux} \mathrm{d}u = O(e^{-\kappa x}), \tag{31}$$

and the main contribution to the asymptotics for (18) comes from the residue at iq, and we get

$$G_\lambda(x) \sim 2\pi i \operatorname*{res}_{iq} \left(\frac{\tilde{a}(p)}{\lambda + 1 - \tilde{a}(p)} e^{i(p,x)} \right) = e^{-q(\lambda)x}(C(\lambda) + o(1)), \quad x > 0.$$

In the second case of the Theorem the above sum over the closed contour is equal to 0, and the main contribution to the asymptotics of $G_\lambda(x)$ comes from the term (31) with any $\kappa < c$. Consequently, in this case we can conclude the following upper bound (as $x > 0$):

$$G_\lambda(x) \le C_+(\lambda) e^{-(c-\varepsilon)x} \quad \text{for any } \varepsilon > 0.$$

The lower estimate on $G_\lambda(x)$ immediately follows from representation (8) if we take in (8) the first term:

$$G_\lambda(x) > \frac{1}{2} a(x) \ge C_-(\lambda) e^{-(c+\varepsilon)x} \quad \text{with any } \varepsilon > 0.$$

The case when $x < 0$ can be considered in the same way using a rectangular closed contour in the negative imaginary semi-plane. □

Remark 2.3: Since $q(\lambda)$ is the solution of Equation (26), then $q(\lambda) = O(\sqrt{\lambda})$ for small values of λ. Thus in the one-dimensional case the asymptotics (27) from Theorem 2.3 improves the general bound (17).

3. Applications

In this section we present some applications of the results obtained in the previous section.

3.1. Asymptotic of the principal eigenfunction ψ_λ

We consider the operator

$$Lu(x) = L_0 u(x) \; + \; V(x)u(x), \quad u(x) \in L^2(\mathbb{R}^d), \tag{32}$$

with L_0 defined by (1). For the potential V, we assume that

$$0 \le V \le 1, \quad V(x) \in C_0(\mathbb{R}^d). \tag{33}$$

The operator L is a bounded self-adjoint operator in $L^2(\mathbb{R}^d)$. The equation for the principal eigenfunction (the ground state) ψ_λ of the operator L

$$(L_0 + V - \lambda)\psi_\lambda \; = \; 0, \quad \lambda > 0, \tag{34}$$

can be rewritten in the following way

$$(1 + \lambda)\psi_\lambda(x) - \mathcal{S}_a \psi_\lambda(x) = V(x)\psi_\lambda(x) =: F(x).$$

where $\mathcal{S}_a$ stands for the convolution operator. After proper rearrangements this yields

$$\psi_\lambda(x) = (1+\lambda)^{-1}\Big(F(x) + \sum_{k=1}^{\infty} \frac{(\mathcal{S}_a^k F)(x)}{(1+\lambda)^k}\Big)$$
$$= (1+\lambda)^{-1}\Big(F(x) + \int G_\lambda(x-y)F(y)\mathrm{d}y\Big), \tag{35}$$

where $G_\lambda(x)$ is defined by (6).

We remind below of the spectral properties of operator L and of the conditions ensuring the existence of the principle eigenfunction $\psi_\lambda(x)$. Notice that the operator L has these properties for any kernel $a(x)$ that meets conditions (2), independently on the behaviour of the tail of $a(x)$. The following statements have been proved in [4].

Theorem 3.1 [4]:

- *The operator L has only discrete spectrum in the half-plane $\mathcal{D} = \{\lambda \in \mathbb{C} \mid Re\lambda > 0\}$.*
- *Assume that $V(x) \equiv 1$ on some open set in $\mathbb{R}^d$. Then the ground state of L exists.*
- *For any $\delta > 0$ there is $\varepsilon > 0$ such that for any potential $V(x)$ that satisfies the inequality $V(x) \geq 1-\varepsilon$ on a ball of radius δ, the ground state of L exists.*
- *If $V(x) \geq \beta$, $\beta \in (0,1)$, in a large enough ball (depending on β) then the ground state exists.*
- *Let $d = 1, 2$, and assume that $\int_{\mathbb{R}^d} |x|^2 a(x)\mathrm{d}x < \infty$. Then for any $V(x) \not\equiv 0$ the ground state of L exists.*

We proceed with studying the asymptotic behaviour of the function $\psi_\lambda(x)$ as $|x| \to \infty$.

Theorem 3.2: *Let $V \in C_0(\mathbb{R}^d)$, and assume that $V(x) \not\equiv 0$. If*

$$c_-(1+|x|)^{-(d+\alpha)} \leq a(x) \leq c_+(1+|x|)^{-(d+\alpha)}, \tag{36}$$

then the principal eigenfunction satisfies the estimates

$$\tilde{c}_-(\lambda)(1+|x|)^{-(d+\alpha)} \leq \psi_\lambda(x) \leq \tilde{c}_+(\lambda)(1+|x|)^{-(d+\alpha)} \tag{37}$$

with $0 < \tilde{c}_-(\lambda) \leq \tilde{c}_+(\lambda)$.

If bound (16) holds, then for $\psi_\lambda(x)$ the following estimate holds:

$$\psi_\lambda(x) \leq K(\lambda)e^{-m(\lambda)|x|}. \tag{38}$$

Let $d = 1$, and assume that condition (24) is fulfilled. Then

- *if there exists a solution $\hat{p}(\lambda) = iq(\lambda)$ of Equation (26) with $q(\lambda) < c$, then*

$$\psi_\lambda(x) = e^{-q(\lambda)|x|}(C(\lambda) + o(1)), \quad |x| \to \infty, \tag{39}$$

- *if the solution of (26) does not exists for $q < c$, then*

$$C_-(\lambda)e^{-(c+\varepsilon)|x|} \leq \psi_\lambda(x) \leq C_+(\lambda)e^{-(c-\varepsilon)|x|} \tag{40}$$

  with any $\varepsilon > 0$.

***Proof*:** Since function $F(x) = V(x)\psi_\lambda(x)$ has a compact support and is positive, then relations (37), (39) and (40) follow from representation (35) and estimates (10), (27)-(28) on the function $G_\lambda(x)$. □

3.2. Application to the solution of the Cauchy problem

Let us consider the following Cauchy problem

$$\frac{\partial u}{\partial t} = L_0 u(x,t) - mu(x,t) + f(x), \quad u(x,0) = 0, \tag{41}$$

where $m > 0$ is a constant, $f(x) \geq 0$, $f \not\equiv 0$, $f \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d)$. Equation (41) describes the evolution of the population density in a spacial contact model with mortality rate $1 + m$ and space inhomogeneous source term $f(x)$. In the absence of the source term, the density of the population goes to zero as $t \to \infty$ (see [9]). The question of interest is how a flow coming from the source into the population may change the asymptotics of the density.

Since $S(t) = e^{t(L_0 - m)}$ is a contraction semigroup both in $L^2(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$ spaces, the solution $u(x,t)$ of problem (41) converges to the corresponding stationary solution $\hat{u}(x)$:

$$\|u(x,t) - \hat{u}(x)\| \to 0, \quad t \to \infty.$$

The convergence takes place both in $L^2(\mathbb{R}^d)$ and in $C_b(\mathbb{R}^d)$ norms. Concerning the stationary solution $\hat{u}(x)$ let us observe that according to (35) it can be expressed as

$$\hat{u}(x) = (m - L_0)^{-1} f(x) = \frac{1}{1+m}\Big(f(x) + \int G_m(x-y) f(y) \mathrm{d}y\Big). \tag{42}$$

It turns out that the behaviour of $u(x,t)$ and $\hat{u}(x)$ depends on both the tail of the convolution kernels and of the source term.

As above we consider separately the cases of polynomial and exponential tails of a and f. Assume first that the function $a(x)$ satisfies inequalities (9) and that the function $f(x)$ satisfies analogous inequalities with some α_1. Then, due to (10), $\hat{u}(x)$ admits the following bounds

$$c_-(1 + |x|)^{-(d+\tilde{\alpha})} \leq \hat{u}(x) \leq c_+(1 + |x|)^{-(d+\tilde{\alpha})}, \tag{43}$$

with $\tilde{\alpha} = \min(\alpha_1, \alpha)$. Moreover, $u(x,t)$ at any time $t > 0$ admits the inequality

$$0 \leq u(x,t) \leq \hat{u}(x). \tag{44}$$

To prove this inequality we define $v(x,t) = u(x,t) - \hat{u}(x)$. Then $v(x,0) = -\hat{u}(x) < 0$. Since $e^{t(L_0-m)}$ is the positivity improving semigroup, then $v(x,t) < 0$ for all $t > 0$. Therefore, $u(x,t) < \hat{u}(x)$ for all $t > 0$. The lower bound in (44) follows from Duhamel's formula.

In the case of exponentially decaying a and f both $\hat{u}$ and u decay exponentially. This can be justified in the same way as above.

If $f(x) \in C_0(\mathbb{R}^d)$, then as a direct consequence of (42), the stationary solution $\hat{u}(x)$ satisfies the same tail estimates as the dispersal kernel.

In conclusion we provide some comments on possible interpretation of the above results.

- Consider the contact model in continuum (see [9]) describing an infection spreading process in a society with the recovering intensity $1 + m$. For $m > 0$ this intensity is sufficient to make the density of infected population degenerate. Suppose that the source of the infection is localized, for instance $f(x) = \lambda \mathbf{1}_B(x)$, $\lambda > 0$, with a bounded (small) set B where infected individual appear (from outside). Then we can estimate the density of infected population on a distance from the infection source B which essentially depends on the infection spreading rate. Even very small region B may produce a drastic effect!
- The same process we have in the information spreading in the society. Even if you will have very strong real information delivering rate ($m >> 1$) the influence of a constant mass-media

flow of wrong information may be determining for the opinion formation, especially due to long range spreading possibilities of mass-media.

- Free Kawasaki dynamics of continuous particle system (see [10]) can be used for modelling a pollution spreading process. Equation (41) with $f = 0$ and some u_0 describes the evolution of the pollution density in the presence of a self cleaning ability of the environment with intensity $m > 0$. In the general case the function $f \geq 0$ represents the density of the pollution source. The solution $u(x, t)$ is the density of the pollution after time t, and $\hat{u}(x)$ is the large time limit of this density. The estimates (43) reflect the fact that even for a source localized in a small area, for instance for $f = \lambda \mathbf{1}_B(x)$ with a small ball B, the strong pollution spreading can be observed at long distances from the source.

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