FOLD BIFURCATIONS AND LINEAR STABILITY ANALYSIS IN SYSTEMS WITH PREISACH HYSTERESIS

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We propose an algorithm of linear stability analysis for periodic solutions of operator-differential equations involving the time derivative of the output of the Preisach operator. The results are tested numerically by examples where a periodic solution undergoes the fold bifurcation.

Keywords: Operator-differential equation; Preisach operator; Stability analysis; Fold bifurcation; Hysteresis

1. Introduction

We consider scalar operator-differential equations that contain the time derivative of the output of the Preisach operator P[1]. Such equations appear, for example, as models of the water flow through unsaturated soil exhibiting soil-moisture hysteresis in terrestrial hydrology (and, more generally, as models of other liquid flows through porous media). In particular, models

$$ax'(t) + (Px)'(t) = f(t, x(t)), \qquad a > 0, \tag{1}$$

where P describes the hysteresis relation between the matric potential and the water content in the soil matrix[a], have been proposed and studied in [2]. The function f in the balance equation (1) can have different form depending on the type and nature of flows present in the system[3].

[a]Systems involving equations of the type (1) are also used for modelling electronic circuits with ferromagnetic hysteresis in inductance elements, see e.g. [4].

The Cauchy problem for equation (1) has been studied in [5]; numerical algorithms for its solution have been proposed and implemented in [6]. The Preisach operator introduces a special type of memory in the system: it remembers certain local extrema of the solution from the past (the so-called main extrema or shock values). This memory manifests itself, for example, through the jumps of the derivative of the solution at the moments when a past shock value is reached again.

In [7], an algorithm for linear stability analysis of periodic solutions of equation (1) has been proposed. The results refer to the class of the periodic solutions that have exactly one maximum and one minimum on a period. In this letter we test the proposed algorithm numerically by applying it to systems where a periodic solution undergoes the fold bifurcation, i.e. a stable and an unstable solutions collide and annihilate at the bifurcation point. Then we propose and discuss a modification of the algorithm, which extends it to a more general class of periodic solutions with an arbitrary number of local extrema on the period. To explain this modification it suffices to consider the solutions with four local extrema. The main result is formulated and illustrated by a numerical example for this case.

We note that the variation of initial data of equation (1) includes the variation of the initial state of the Preisach nonlinearity in an infinite dimensional metric space without a good linear structure. However, we show that for a generic class of admissible perturbations of the initial data stability of a solution is determined by a finite dimensional linear system.

2. Preisach nonlinearity

We use the following class of the Preisach models (operators); a more general definition and the phenomenology can be found, for example, in [1]. The output of the model is a scalar continuous function defined by the formula

$$y(t) = P[\eta_0]x(t) := \iint_{\omega(t)} \mu(\alpha,\beta)\, d\alpha\, d\beta \tag{2}$$

where a nonnegative integrable function $\mu : \Pi \to \mathbb{R}$, called the Preisach measure density, is defined on the strip $\Pi = \{0 \le \beta - \alpha \le d\}$ of the (α,β) plane; the domain $\omega(t) \subset \Pi$ of integration changes in time; $\eta_0 = \eta_0(\xi)$ is the initial state. For each t, the domain $\omega(t)$ has the form $\omega(t) = \{(\alpha,\beta) \colon \alpha + \beta \le 2x(t) + \eta(t;\beta-\alpha)\}$, where $x(t)$ is a scalar continuous input of the model and the function $\eta(t;\cdot) : [0,d] \to \mathbb{R}$, called the state of the model at the moment t, satisfies $\eta(\cdot;0) = 0$, $|\eta(\cdot;\xi_1) - \eta(\cdot;\xi_2)| \le |\xi_1 - \xi_2|$. The evolution of the state is determined by simple rules, see [1]. We use the standard distance in the state space, $\rho(\eta_0^1,\eta_0^2)=\max_{\xi\in[0,d]}|\eta_0^1(\xi)-\eta_0^2(\xi)|$.

3. Properties of solutions of equation (1)

A solution $x=x(t)$ of equation (1) with the Preisach operator $P=P[\eta_0]$ defined by (2) satisfies the equation everywhere; the function $x+aPx$ in the left hand side is continuously differentiable, although x' and $(Px)'$ can have jumps. Solutions depend continuously on the initial data $t_0, x(t_0)=x_0$ and η_0. Algorithms of numerical solution of equation (1) and linearisation of the evolution operator is based on the following two properties of solutions [5,?].

(i) Each solution of (1) decreases in the area of the (t,x)-plane where $f(t,x)<0$, increases in the area where $f(t,x)>0$ and has extrema on the lines $f(t,x)=0$.

(ii) Suppose that a solution $x(t)$ of is monotone on a segment $[t_1,t_2]$ and $\eta_1=\eta_1(\xi)$ is the state of the Preisach operator at the moment t_1. Set

$$J_\beta(y,x)=\int_x^y \mu(x,\beta)d\beta,\qquad J_\alpha(y,x)=\int_y^x \mu(\alpha,x)d\alpha$$

and denote by $\alpha_{\eta_1}(\beta),\beta_{\eta_1}(\alpha)$ the left-continuous monotone functions, which are uniquely defined by the relations (where one excludes ξ)

$$\alpha=x(t_1)+(-\xi+\eta_1(\xi))/2,\qquad \beta=x(t_1)+(\xi+\eta_1(\xi))/2.$$

If $x(t)$ decreases, then almost everywhere on $[t_1,t_2]$

$$x'=f(t,x)/(a+J_\beta(\beta_{\eta_1}(x),x));\tag{3}$$

if $x(t)$ increases, then almost everywhere

$$x'=f(t,x)/(a+J_\alpha(\alpha_{\eta_1}(x),x)).\tag{4}$$

These properties imply that a solution $x=x(t)$ of (1) satisfies ordinary differential equations (3), (4) on consecutive intervals $[t_k,t_{k+1}]$ where t_k are the moments when $x(t)$ crosses the lines $f(t,x)=0$. We note that the jumps of the functions $\alpha_{\eta_1},\beta_{\eta_1}$ generate the jumps of the derivative of x.

If $x_*=x_*(t)$ is a periodic solution of Eq. (1) and $\eta_*(t_0)=\eta_0^*$ is a corresponding initial state of the Preisach operator, then x_* is a periodic solution for each initial state η_0 from a certain class $\Xi=\Xi(x_*)$. We denote by $\Xi_\kappa(x_*)$ the subclass of the class $\Xi(x_*)$, which consists of the states η_0

satisfying the relation $|\eta_0(\xi)-\eta_0(r_*)| \le \kappa(\xi-r_*)$ for $\xi > r_*$ with a $\kappa < 1$ and $r_* := \max_{t>\tau} |x_*(t)-x_*(\tau)|$. Following [7], we say that a periodic solution x_* of (1) is asymptotically stable with respect to admissible perturbations of initial data if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x(t) - x_*(t)| < \varepsilon$ for all $t > t_0$ and $|x(t) - x_*(t)| \to 0$ whenever $|x(t_0) - x^*(t_0)| < \delta$ and the distance from the initial state $\eta(t_0)$ of the Preisach operator for the perturbed solution x from the set $\Xi_\kappa(x_*)$ is less than δ.

4. Stability of a simple periodic solution

Formulas (3), (4) can be used as a basis for linearisation of equation (1) and the linear stability analysis. From now on, we consider T-periodic in t smooth functions f. Assume that (1) has a T-periodic solution $x_*(t)$, which has one minimum and one maximum on a period: the minimum x_*^m is reached at a point t_1^* and the maximum x_*^M is reached at a point t_2^* with $t_0 < t_1^* < t_2^* < t_0 + T$. Also, assume that the right and left second derivatives of $x_*(t)$ are positive at the point t_1^* and negative at t_2^*. Set

$$\hat{f}(t,x,\alpha_0) := f(t,x)/(a + J_\alpha(\alpha_0,x)), \quad \widetilde{f}(t,x,\beta_0) := f(t,x)/(a + J_\beta(\beta_0,x)),$$

$a_1(t) = \hat{f}_x(t,x_*,x_*^m)$, $b_1(t) = \hat{f}_{\alpha_0}(t,x_*,x_*^m)$, $a_2(t) = \widetilde{f}_x(t,x_*,x_*^M)$, $b_2(t) = \widetilde{f}_{\beta_0}(t,x_*,x_*^M)$ and consider the solutions u, v, z_1, z_2 of the problems

$$u'(t) = a_2(t)u(t),\ u(t_0) = 1; \qquad v'(t) = a_2(t)v(t) + b_2(t),\ v(t_0) = 0;$$
$$z_1'(t) = a_1(t)z_1(t) + b_1(t),\ z_1(t_1^*) = 1;\ z_2'(t) = a_2(t)z_2(t) + b_2(t),\ z_2(t_2^*) = 1.$$

As shown in [7], the solution x_* of (1) is asymptotically stable with respect to admissible perturbations of initial data if the number $\gamma = u(t_1^*)z_1(t_2^*)z_2(t_0+T) + v(t_1^*)z_1(t_2^*)$ satisfies $|\gamma| < 1$.

5. Example: fold bifurcation of periodic solution

In construction of the example we follow [8] where also the physics of different terms and the role of positive and negative feedbacks were discussed.

Consider the example of equation (1) of the form

$$(x + Px)' = (1 - \sin t)e^{-(x+1)^2} + \lambda - e^{-0.0625}(0.5x + 1.625) \qquad (5)$$

with the scalar parameter λ. Assume that the Preisach measure density is $\mu(\alpha,\beta) \equiv 0.125$ in the triangle $-4 < \beta < \alpha < 0$ and is zero outside this triangle. Figure 1(a) shows periodic solutions of equation (5) for $\lambda = 0.2$: all of them have one maximum and one minimum on the period $[0, 2\pi]$.

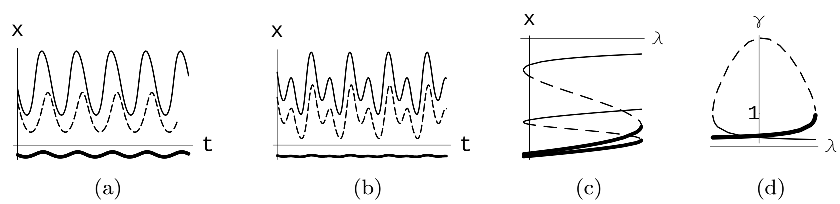

Figure 1. Periodic solutions with different number of extrema on a period: (a) 2 extrema; (b) 4 extrema. (c) Minimum and maximum values of periodic solutions with 2 extrema and (d) corresponding γ, depending on λ. The solid lines show stable solutions; the dashed lines are unstable solutions.

Numerical simulation shows that for any λ from the interval $0.155 < \lambda < 0.255$ equation (5) has three periodic solutions: two asymptotically stable and one unstable. The S-shaped curves on Figure 1(c) show the maximum and the minimum value of the three periodic solutions as functions of λ. Figure 1(d) presents the value of γ evaluated for these solutions: $0 < \gamma < 1$ for both the stable solutions and $\gamma > 1$ for the unstable solution. At the fold bifurcation points γ approaches the value 1 from below and above for the colliding pair.

6. Periodic solutions with multiple local extrema

Now we adapt the above algorithm of linear stability analysis to solutions with multiple minima on a period. To illustrate the required modification it suffices to consider a T-periodic solution $x_*(t)$ of (1) which has four local extrema: a local minimum x_1^* at a point t_1^*, a local maximum x_2^* at a point t_2^*, a global minimum x_4^* at a point t_4^* and a global maximum x_5^* at a point t_5^* with $t_0 < t_1^* < t_2^* < t_4^* < t_5^* < t_0 + T$. Assume that the right and left second derivatives of $x_*(t)$ are positive at the points t_1^*, t_4^* and negative at $t = t_2^*, t_5^*$. Define the point $t_3^* \in (t_2^*, t_4^*)$ where $x_*(t_3^*) = x_1^*$ and denote

$$\bar{f}_{n1}(t) = \widetilde{f}_x(t, x_*(t), x_5^*), \quad \bar{f}_{n2}(t) = \widetilde{f}_{\beta_0}(t, x_*(t), x_5^*), \quad n = 1, 4, 6$$
$$\bar{f}_{n1}(t) = \hat{f}_x(t, x_*(t), x_{n-1}^*), \quad \bar{f}_{n2}(t) = \hat{f}_{\alpha_0}(t, x_*(t), x_{n-1}^*), \quad n = 2, 5$$
$$\bar{f}_{31}(t) = \widetilde{f}_x(t, x_*(t), x_2^*), \quad \bar{f}_{32}(t) = \widetilde{f}_{\beta_0}(t, x_*(t), x_2^*);$$

the point t_3^* is important because the right hand side of equation (3) is $\widetilde{f}(t, x_*(t), x_2^*)$ for $t_2^* < t < t_3^*$ and $\widetilde{f}(t, x_*(t), x_5^*)$ for $t_3^* < t < t_4^*$. Now consider the auxiliary linear problems

$$\begin{array}{lll} u_n'(t) = \bar f_{n1}(t)u_n(t), & u_n(t_{n-1}^*) = 1, & n = 1,4, \\ v_n'(t) = \bar f_{n1}(t)v_n(t) + \bar f_{n2}(t), & v_n(t_{n-1}^*) = 0, & n = 1,4, \\ z_n'(t) = \bar f_{n1}(t)z_n(t) + \bar f_{n2}(t), & z_n(t_{n-1}^*) = 1, & n = 2,3,5,6, \end{array}$$

where $t_0^* = t_0$. Using their solutions of this problems and the quantity $J_3 = J_\beta(x_2^*, x_5^*)/(a + J_\beta(x_1^*, x_5^*))$, we define the number

$$\tilde\gamma = z_5(t_5^*)\big(v_4(t_4^*)+(z_3(t_3^*)z_2(t_2^*)(1-J_3)+J_3)u_4(t_4^*)(u_1(t_1^*)z_6(t_0+T)+v(t_1^*))\big).$$

Theorem 6.1. *If $|\tilde\gamma| < 1$, then the periodic solution x_* of* (1) *is asymptotically stable with respect to admissible perturbations of initial data.*

The effect of the local extrema is that for a short period of time near the moment t_3^* the periodic solution x_* and any its small perturbation satisfy to different equations (3): the difference is close to a non-zero constant. This generates the term J_3 in the expression for $\tilde\gamma$, which is not present in the expression for the quantity γ of Section 4 for the simpler solutions.

For example, consider the equation

$$(x + Px)' = (1 - 0.5\sin t - \cos 2t)e^{-(x+1)^2} + \lambda - e^{-0.0625}(0.5x + 1.625),$$

where the density of the Preisach measure is $\mu = 0.5$ in the triangle $-2 < \beta < \alpha < 0$ and $\mu = 0$ outside this triangle. The graphs of the three periodic solutions for $\lambda = 0.071$ are shown in Figure 1(b). Here $\tilde\gamma = 0.986$ for the topmost stable periodic solution and $\tilde\gamma = 1.095$ for the unstable solution, $J_3 = 0.166$. We note that the effect of local extrema is important in this example, since we obtain $\tilde\gamma = 1.009$ for the stable solution by setting $J_3 = 0$.

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