A VANISHING THEOREM FOR DIFFERENTIAL OPERATORS IN POSITIVE CHARACTERISTIC

ALEXANDER SAMOKHIN

ABSTRACT. Let X be a smooth variety over an algebraically closed field k of characteristic p, and $\mathsf{F}\colon X \to X$ the Frobenius morphism. We prove that if X is an incidence variety (a partial flag variety in type $\mathbf{A}_n$) or a smooth quadric (in this case p is supposed to be odd) then $\mathrm{H}^i(X, \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)) = 0$ for $i > 0$. Using this vanishing result and the derived localization theorem for crystalline differential operators [3] we show that the Frobenius direct image $\mathsf{F}_*\mathcal{O}_X$ is a tilting bundle on these varieties provided that $p > h$, the Coxeter number of the corresponding group.

INTRODUCTION

Let X be a smooth variety over an algebraically closed field k of arbitrary characteristic, and $\mathcal{D}_X$ the sheaf of differential operators [8]. Recall that the sheaf $\mathcal{D}_X$ is equipped with a filtration by degree of an operator such that the associated graded sheaf is isomorphic to $\bigoplus(\mathsf{S}^i\Omega^1_X)^*$. Our goal is to study the cohomology vanishing of sheaves of differential operators in positive characteristic. We are primarily interested in the case when X is a homogeneous space of a semisimple algebraic group $\mathbf{G}$ over k, i.e. $X = \mathbf{G}/\mathbf{P}$ for some parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Below we survey some vanishing and non-vanishing results for sheaves of differential operators. If X is a smooth variety over a field k of characteristic zero then there is an isomorphism $(\mathsf{S}^i\Omega^1_X)^* = \mathsf{S}^i\mathcal{T}_X$ for $i \geq 0$, and it is known that symmetric powers of the tangent bundle of $\mathbf{G}/\mathbf{P}$ have vanishing higher cohomology groups [5]. In particular, this implies the vanishing of higher cohomology of the sheaf $\mathcal{D}_{\mathbf{G}/\mathbf{P}}$. This vanishing is also a consequence of the Beilinson–Bernstein localization theorem [2]. If the field k is of characteristic $p > 0$ then the situation is less clear. A crucial tool to prove the vanishing theorem for symmetric powers of the tangent bundle of $\mathbf{G}/\mathbf{P}$ in characteristic zero is the Grauert–Riemenschneider theorem, which does not hold in characteristic p in general. The characteristic p counterpart of the cohomology vanishing of symmetric powers of the tangent bundle of $\mathbf{G}/\mathbf{P}$ is inaccessible at the moment, though it is believed to be true (e.g., [6]). There are cases when such a theorem is known – for example, for flag varieties $\mathbf{G}/\mathbf{B}$ in good characteristic [20]. These results, however, do not give any information about cohomology groups of the sheaf $\mathcal{D}_{\mathbf{G}/\mathbf{P}}$, since for a variety X in positive characteristic the sheaves of graded rings $\bigoplus(\mathsf{S}^i\Omega^1_X)^*$ and $\bigoplus \mathsf{S}^i\mathcal{T}_X$ are no longer isomorphic. The latter sheaf is isomorphic to the associated graded ring of the sheaf of crystalline differential operators D_X [3], and the vanishing theorem in the case of flag varieties gives that $\mathrm{H}^i(\mathbf{G}/\mathbf{B}, \mathrm{D}_{\mathbf{G}/\mathbf{B}}) = 0$ for $i > 0$ and good p. On the other hand, some non-vanishing results are known for both the sheaves $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$ and $\bigoplus(\mathsf{S}^i\Omega^1_{\mathbf{G}/\mathbf{B}})^*$. It follows from [18] that for the flag variety $\mathbf{SL}_5/\mathbf{B}$ the former sheaf has a non-vanishing higher cohomology group. Moreover, the latter sheaf has always a non-vanishing cohomology group unless $\mathbf{G}/\mathbf{B}$ is isomorphic to $\mathbb{P}^1$ ([9], [22]).

From now on let k be an algebraically closed field of characteristic $p > 0$. Recall that for a variety X over k the sheaf of differential operators $\mathcal{D}_X$ is a union of sheaves of matrix algebras [9]. Precisely, $\mathcal{D}_X = \bigcup_{m \geq 1} \mathcal{E}nd_{\mathcal{O}_X^{p^m}}(\mathcal{O}_X)$. The main results of this paper are:

Theorem 0.1. *Let X be a partial flag variety of type $(1, n, n+1)$ of the group $\mathbf{SL}_n$. In other words, $X = \{(l \subset H \subset V)\}$, where l and H are a line and a hyperplane in the vector space V of dimension $n + 1$, respectively. Then $\mathrm{H}^i(X, \mathcal{E}nd_{\mathcal{O}_X^p}(\mathcal{O}_X)) = 0$ for $i > 0$.*

Theorem 0.2. *Let $\mathsf{Q}_n \subset \mathbb{P}^{n+1}$ be a smooth quadric of dimension n. Assume that p is an odd prime. Then $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{E}nd_{\mathcal{O}_{\mathsf{Q}_n}^p}(\mathcal{O}_{\mathsf{Q}_n})) = 0$ for $i > 0$.*

For a variety X the sheaf $\mathcal{E}nd_{\mathcal{O}_X^p}(\mathcal{O}_X)$ is also called the "sheaf of small differential operators" (as it is isomorphic to the central reduction of the sheaf D_X), which explains the title of the paper. The proof of Theorems 0.1 and 0.2 uses properties of sheaves of crystalline differential operators [3] and vanishing theorems for line bundles on the cotangent bundles of $\mathbf{G}/\mathbf{P}$. Independently, in a recent paper [21] A. Langer, using different methods, described the decomposition of Frobenius push-forwards $\mathsf{F}_{m*}\mathcal{O}_{\mathsf{Q}_n}$ into a direct sum of indecomposable bundles. This gives another proof of Theorem 0.2.

Our interest in the vanishing of higher cohomology groups of sheaves $\mathcal{E}nd_{\mathcal{O}_{\mathbf{G}/\mathbf{P}}^p}(\mathcal{O}_{\mathbf{G}/\mathbf{P}})$ is twofold. First, as we saw above, it is related to the vanishing of higher cohomology of the sheaf $\mathcal{D}_{\mathbf{G}/\mathbf{P}}$. In the case when $\mathbf{P} = \mathbf{B}$, the Borel subgroup, the vanishing of higher cohomology groups of $\mathcal{D}_{\mathbf{G}/\mathbf{B}}$ implies D-affinity of the flag variety $\mathbf{G}/\mathbf{B}$ ([1], [9], [21], [27]). On the other hand, it has implications for the derived category of coherent sheaves: if the vanishing holds for a given $\mathbf{G}/\mathbf{P}$ then the derived localization theorem [3] implies that the Frobenius pushforward $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ is a tilting bundle on $\mathbf{G}/\mathbf{P}$, provided that $p > h$, the Coxeter number of $\mathbf{G}$. We discuss these questions in a greater detail in Section 6 (see also [12], [16], [21], [26]).

We would like to conclude the introductory part with a conjecture. There are a number of previously known results (see the list below), to which the present paper gives an additional evidence, that suggest that one could hope for the vanishing of higher cohomology of the sheaves of small differential operators on homogeneous spaces $\mathbf{G}/\mathbf{P}$.

Conjecture: Let $\mathbf{G}/\mathbf{P}$ be the homogeneous space of a semisimple simply connected algebraic group $\mathbf{G}$ over k. Then for sufficiently large p one has $\mathrm{H}^i(\mathbf{G}/\mathbf{P}, \mathcal{E}nd_{\mathcal{O}_{\mathbf{G}/\mathbf{P}}^p}(\mathcal{O}_{\mathbf{G}/\mathbf{P}})) = 0$ for $i > 0$.

Surprisingly enough, this cohomology vanishing is known, apart from the above theorems, in very few cases, namely for projective spaces, the flag variety $\mathbf{SL}_3/\mathbf{B}$ by [9], the flag variety in type $\mathbf{B}_2$ by [1]. To the best of our knowledge, no counterexamples exist either. Finally, let us remark that the counterexample to the D-affinity from [18], which is based on the study of a particular $\mathcal{D}$-module on the grassmannian $\mathrm{Gr}_{2,5}$, does not rule out the possibility of the conjectural vanishing in this case. The first step to further test the conjecture is to check it for $\mathrm{Gr}_{2,5}$; this will be a subject of a forthcoming paper [28].

1. Preliminaries

1. **The Frobenius morphism.** Throughout we fix an algebraically closed field k of characteristic $p > 0$. Let X be a scheme over k. The absolute Frobenius morphism F_X is an endomorphism of X that acts identically on the topological space of X and raises functions on X to the p-th power:

$$(1.1) \qquad \mathsf{F}_X : X \to X, \qquad f \in \mathcal{O}_X \mapsto f^p \in \mathcal{O}_X.$$

Denote X' the scheme obtained by base change from X along the Frobenius morphism $\mathsf{F} : \operatorname{Spec}(k) \to \operatorname{Spec}(k)$. Then the relative Frobenius morphism $\mathsf{F} : X \to X'$ is a morphism of k-schemes. The schemes X and X' are isomorphic as abstract schemes.

For a quasicoherent sheaf $\mathcal{E}$ on X one has, the Frobenius morphism being affine:

$$\mathrm{H}^i(X, \mathcal{E}) = \mathrm{H}^i(X', \mathsf{F}_* \mathcal{E}).$$

2. **Koszul resolutions.** Let V be a finite dimensional vector space over k with a basis $\{e_1, \dots, e_n\}$. Recall that the *r-th exterior power* $\bigwedge^r \mathsf{V}$ of V is defined to be the r-th tensor power $\mathsf{V}^{\otimes r}$ of V divided by the vector subspace spanned by the elements:

$$u_1 \otimes \cdots \otimes u_r - (-1)^{\operatorname{sgn}\sigma} u_{\sigma_1} \otimes \cdots \otimes u_{\sigma(r)}$$

for all the permutations $\sigma \in \Sigma_r$ and $u_1, \dots, u_r \in V$. Similarly, the *r-th symmetric power* $\mathsf{S}^r \mathsf{V}$ of V is defined to be the r-th tensor power $\mathsf{V}^{\otimes r}$ of V divided by the vector subspace spanned by the elements

$$u_1 \otimes \cdots \otimes u_r - u_{\sigma_1} \otimes \cdots \otimes u_{\sigma(r)}$$

for all the permutations $\sigma \in \Sigma_r$ and $u_1, \dots, u_r \in \mathsf{V}$.

Let

$$(1.2) \qquad 0 \to \mathsf{V}' \to \mathsf{V} \to \mathsf{V}'' \to 0$$

be a short exact sequence of vector spaces. For any $n > 0$ there is a functorial exact sequence (the Koszul resolution, ([13], II.12.12))

$$(1.3) \qquad \cdots \to \mathsf{S}^{n-i} \mathsf{V} \otimes \overset{i}{\bigwedge} \mathsf{V}' \to \cdots \to \mathsf{S}^{n-1} \mathsf{V} \otimes \mathsf{V}' \to \mathsf{S}^n \mathsf{V} \to \mathsf{S}^n \mathsf{V}'' \to 0.$$

Another fact about symmetric and exterior powers is the following ([11], Exercise 5.16). For a short exact sequence such as (1.2) one has for each n the filtrations

$$(1.4) \qquad \mathsf{S}^n \mathsf{V} = \mathrm{F}_n \supset \mathrm{F}_{n-1} \supset \dots \quad \text{and} \quad \overset{n}{\bigwedge} \mathsf{V} = \mathrm{F}'_n \supset \mathrm{F}'_{n-1} \supset \dots$$

such that

(1.5)
$$\mathrm{F}_i/\mathrm{F}_{i-1} \simeq \mathsf{S}^{n-i}\mathsf{V}' \otimes \mathsf{S}^i\mathsf{V}''$$

and

(1.6)
$$\mathrm{F}'_i/\mathrm{F}'_{i-1} \simeq \bigwedge^{n-i}\mathsf{V}' \otimes \bigwedge^i\mathsf{V}''$$

When either V' or V'' is a one-dimensional vector space, these filtrations on exterior powers of V degenerate into short exact sequences. If V'' is one-dimensional then one obtains:

(1.7)
$$0 \to \bigwedge^r\mathsf{V}' \to \bigwedge^r\mathsf{V} \to \bigwedge^{r-1}\mathsf{V}' \otimes \mathsf{V}'' \to 0.$$

Similarly, if V' is one-dimensional, the above filtration degenerates to give a short exact sequence:

(1.8)
$$0 \to \bigwedge^{r-1}\mathsf{V}'' \otimes \mathsf{V}' \to \bigwedge^r\mathsf{V} \to \bigwedge^r\mathsf{V}'' \to 0.$$

3. Differential operators.

3.1. *"True" differential operators.* The material here is taken from [8] and [9]. Let X be a smooth scheme over k. Consider the product $X \times X$ and the diagonal $\Delta \subset X \times X$. Let $\mathcal{J}_\Delta$ be the sheaf of ideals of Δ.

Definition 1.1. *An element $\phi \in \mathcal{E}nd_k(\mathcal{O}_X)$ is called a differential operator if there exists some integer $n \geq 0$ such that*

(1.9)
$$\mathcal{J}_\Delta^n \cdot \phi = 0.$$

One obtains a sheaf $\mathcal{D}_X$, the sheaf of differential operators on X. Denote $\mathcal{J}_\Delta^{(n)}$ the sheaf of ideals generated by elements a^n, where $a \in \mathcal{J}_\Delta$. There is a filtration on the sheaf $\mathcal{D}_X$ given by

(1.10)
$$\mathcal{D}_X^{(n)} = \{\phi \in \mathcal{E}nd_k(\mathcal{O}_X) \colon \mathcal{J}_\Delta^{(n)} \cdot \phi = 0\}.$$

Since k has characteristic p, one checks:

(1.11)
$$\mathcal{D}_X^{(p^n)} = \mathcal{E}nd_{\mathcal{O}_X^{p^n}}(\mathcal{O}_X).$$

Indeed, the sheaf $\mathcal{J}_\Delta$ is generated by elements $a \otimes 1 - 1 \otimes a$, where $a \in \mathcal{O}_X$, hence the sheaf $\mathcal{J}_\Delta^{(p^n)}$ is generated by elements $a^{p^n} \otimes 1 - 1 \otimes a^{p^n}$. This implies (1.12). One also checks that this filtration exhausts the whole $\mathcal{D}_X$, so one has (Theorem 1.2.4, [9]):

(1.12)
$$\mathcal{D}_X = \bigcup_{n \geq 1} \mathcal{E}nd_{\mathcal{O}_X^{p^n}}(\mathcal{O}_X).$$

The filtration from (1.12) was called the p–filtration in *loc.cit.* By definition of the Frobenius morphism one has $\mathrm{H}^i(X, \mathcal{E}nd_{\mathcal{O}_X^p}(\mathcal{O}_X)) = \mathrm{H}^i(X', \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X))$. The sheaf $\mathcal{D}_X$ contains divided powers of vector fields (hence the name "true") as opposed to the sheaf of PD-differential operators D_X that is discussed in the next paragraph.

3.2. *Crystalline differential operators.* The material of this subsection is taken from [3]. We recall, following *loc.cit.* the basic properties of crystalline differential operators (differential operators without divided powers, or PD-differential operators in the terminology of Berthelot and Ogus).

Let X be a smooth variety, $\mathcal{T}_X^*$ the cotangent bundle, and $\mathrm{T}^*(X)$ the total space of $\mathcal{T}_X^*$. Denote $\pi \colon \mathrm{T}^*(X) \to X$ the projection.

Definition 1.2. *The sheaf D_X of crystalline differential operators on X is defined as the enveloping algebra of the tangent Lie algebroid, i.e., for an affine open $U \subset X$ the algebra $\mathrm{D}(U)$ contains the subalgebra $\mathcal{O}$ of functions, has an $\mathcal{O}$-submodule identified with the Lie algebra of vector fields $Vect(U)$ on U, and these subspaces generate $\mathrm{D}(U)$ subject to relations $\xi_1\xi_2 - \xi_2\xi_1 = [\xi_1,\xi_2] \in Vect(U)$ for $\xi_1,\xi_2 \in Vect(U)$, and $\xi \cdot f - f \cdot \xi = \xi(f)$ for $\xi \in Vect(U)$ and $f \in \mathcal{O}(U)$.*

Let us list the basic properties of the sheaf D_X [3]:

- The sheaf of non-commutative algebras $\mathsf{F}_* \mathrm{D}_X$ has a center, which is isomorphic to $\mathcal{O}_{\mathrm{T}^*(X')}$, the sheaf of functions on the cotangent bundle to the Frobenius twist of X. The sheaf $\mathsf{F}_* \mathrm{D}_X$ is finite over its center.
- This makes $\mathsf{F}_* \mathrm{D}_X$ a coherent sheaf on $\mathrm{T}^*(X')$. Thus, there exists a sheaf of algebras $\mathbb{D}_X$ on $\mathrm{T}^*(X')$ such that $\pi_* \mathbb{D}_X = \mathsf{F}_*\mathrm{D}_X$ (by abuse of notation we denote the projection $\mathrm{T}^*(X') \to X'$ by the same letter π). The sheaf $\mathbb{D}_X$ is an Azumaya algebra over $\mathrm{T}^*(X')$ of rank $p^{2\dim(X)}$.
- There is a filtration on the sheaf $\mathsf{F}_*\mathrm{D}_X$ such that the associated graded ring $\mathrm{gr}(\mathsf{F}_*\mathrm{D}_X)$ is isomorphic to $\mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)} = \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$.
- Let $i\colon X' \hookrightarrow \mathrm{T}^*(X')$ be the zero section embedding. Then $i^*\mathbb{D}_X$ splits as an Azumaya algebra, the splitting bundle being $\mathsf{F}_*\mathcal{O}_X$. In other words, $i^*\mathbb{D}_X = \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)$.

Finally, recall that the sheaf D_X acts on $\mathcal{O}_X$ and that this action is not faithful in general. It gives rise to a map $\mathrm{D}_X \to \mathcal{D}_X$; its image is the sheaf of "small differential operators" $\mathcal{E}nd_{\mathcal{O}_X^p}(\mathcal{O}_X)$.

3.3. *Cartier descent.* Let $\mathcal{E}$ be an $\mathcal{O}_X$-module equipped with an integrable connection ∇, and let $U \subset X$ be an open subset with local coordinates $t_1, \ldots, t_d$. Let $\partial_1, \ldots, \partial_d$ be the local basis of derivations that is dual to the basis (dt_i) of Ω_X^1. The connection ∇ is said to have zero p-curvature over U if and only if for any local section s of $\mathcal{E}$ and any i one has $\partial_i^p s = 0$. For any $\mathcal{O}_{X'}$-module $\mathcal{E}$ the Frobenius pullback $\mathsf{F}^*\mathcal{E}$ is equipped with a canonical integrable connection with zero p-curvature. The Cartier descent theorem (Theorem 5.1, [19]) states that the functor F^* induces an equivalence between the category of $\mathcal{O}_{X'}$-modules and that of $\mathcal{O}_X$-modules equipped with an integrable connection with zero p-curvature.

On the other hand, if an $\mathcal{O}_X$-module $\mathcal{E}$ is equipped with an integrable connection with zero p-curvature then it has a structure of left D_X-module. Given the Cartier descent theorem, we see that the Frobenius pullback $\mathsf{F}^*\mathcal{E}$ of a coherent sheaf $\mathcal{E}$ on X' is a left D_X-module. We will use the Cartier descent in Section 6.

2. Vanishing theorems for line bundles

Of crucial importance for us are the vanishing theorems for cotangent bundles of homogeneous spaces. Let $\mathbf{G}$ be a connected, simply connected, semisimple algebraic group over k, $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$, and $\mathbf{T}$ a maximal torus. Let $R(\mathbf{T}, \mathbf{G})$ be the root system of $\mathbf{G}$ with respect to $\mathbf{T}$,

R^+ the subset of positive roots, $S \subset R^+$ the simple roots, and h the Coxeter number of $\mathbf{G}$ that is equal to $\sum m_i$, where m_i are the coefficients of the highest root of $\mathbf{G}$ written in terms of the simple roots α_i. By $\langle \cdot, \cdot \rangle$ we denote the natural pairing $X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$, where $X(\mathbf{T})$ is the group of characters (also identified with the weight lattice) and $Y(\mathbf{T})$ the group of one parameter subgroups of $\mathbf{T}$ (also identified with the coroot lattice). For a subset $I \subset S$ let $\mathbf{P} = \mathbf{P}_I$ denote the associated parabolic subgroup. Recall that the group of characters $X(\mathbf{P})$ of $\mathbf{P}$ can be identified with $\{\lambda \in X(\mathbf{T}) | \langle \lambda, \alpha^\vee \rangle = 0,$ for all $\alpha \in I\}$. In particular, $X(\mathbf{B}) = X(\mathbf{T})$. A weight $\lambda \in X(\mathbf{B})$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in S$. A dominant weight $\lambda \in X(\mathbf{P})$ is called $\mathbf{P}$-regular if $\langle \lambda, \alpha^\vee \rangle > 0$ for all $\alpha \notin I$, where $\mathbf{P} = \mathbf{P}_I$ is a parabolic subgroup. A weight λ defines a line bundle $\mathcal{L}_\lambda$ on $\mathbf{G}/\mathbf{B}$. Line bundles on $\mathbf{G}/\mathbf{B}$ that correspond to dominant weights are ample. If a weight λ is $\mathbf{P}$-regular then the corresponding line bundle is ample on $\mathbf{G}/\mathbf{P}$.

Recall that the prime p is *a good prime* for $\mathbf{G}$ if p is coprime to all the coefficients of the highest root of $\mathbf{G}$ written in terms of the simple roots. In particular, if $\mathbf{G}$ is a simple group of type $\mathbf{A}$ then all primes are good for $\mathbf{G}$; if $\mathbf{G}$ is either of the type $\mathbf{B}$ or $\mathbf{D}$ then good primes are $p \geq 3$.

We will often use the Kempf vanishing theorem [10]:

Theorem 2.1. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, and $\mathcal{L}$ an effective line bundle on $\mathbf{G}/\mathbf{P}$. Then $\mathrm{H}^i(\mathbf{G}/\mathbf{P}, \mathcal{L}) = 0$ for $i > 0$.*

Next series of statements concerns line bundles on cotangent bundles. We start with the vanishing theorem by Kumar et al. ([20], Theorem 5):

Theorem 2.2. *Let $\mathrm{T}^*(\mathbf{G}/\mathbf{P})$ be the total space of the cotangent bundle of a homogeneous space $\mathbf{G}/\mathbf{P}$, and $\pi \colon \mathrm{T}^*(\mathbf{G}/\mathbf{P}) \to \mathbf{G}/\mathbf{P}$ the projection. Assume that* char k *is a good prime for $\mathbf{G}$. Let $\lambda \in X(\mathbf{P})$ be a $\mathbf{P}$-regular weight. Then*

$$(2.1) \qquad \mathrm{H}^i(\mathrm{T}^*(\mathbf{G}/\mathbf{P}), \pi^*\mathcal{L}(\lambda)) = \mathrm{H}^i(\mathbf{G}/\mathbf{P}, \mathcal{L}_\lambda \otimes \pi_*\mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}) = 0,$$

for $i > 0$.

Here $\mathcal{L}_\lambda$ denotes a line bundle that corresponds to the weight λ. In particular, one has:

$$(2.2) \qquad \mathrm{H}^i(\mathrm{T}^*(\mathbf{G}/\mathbf{B}), \mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{B})}) = 0,$$

for $i > 0$.

It was proved in [7] and [23] that nilpotent orbits in type $\mathbf{A}_n$ are normal and have rational singularities. This implies (Propositions 4.6 and 4.9 of [23]):

Theorem 2.3. *Let $\mathbf{G} = \mathbf{SL}_n(k)$, and $\mathbf{P} \subset \mathbf{G}$ a parabolic subgroup. Then*

$$(2.3) \qquad \mathrm{H}^i(\mathrm{T}^*(\mathbf{G}/\mathbf{P}), \mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P})}) = 0,$$

for $i > 0$.

Yet another vanishing theorem was also proved in [20] (Theorem 6):

Theorem 2.4. *Let $\mathbf{P}_\alpha \subset \mathbf{G}$ be a minimal parabolic subgroup corresponding to a simple short root α in the root system of $\mathbf{G}$, and p a good prime for $\mathbf{G}$. Then*

$$(2.4) \qquad \mathrm{H}^i(\mathrm{T}^*(\mathbf{G}/\mathbf{P}_\alpha), \mathcal{O}_{\mathrm{T}^*(\mathbf{G}/\mathbf{P}_\alpha)}) = 0,$$

for $i > 0$.

Remark 2.1. One believes that the higher cohomology of $\mathcal{O}_{T^*(G/P)}$ must vanish for any parabolic subgroup $\mathbf{P}$ and suitable p ([6], p. 182). More generally, the vanishing as in (2.1) should hold for any dominant line bundle $\mathcal{L}_\lambda$.

Let now $\mathbf{G}$ be either of the type $\mathbf{B}$ or $\mathbf{D}$, and $\mathbf{P}$ a maximal parabolic subgroup of $\mathbf{G}$ such that the grassmannian $\mathbf{G}/\mathbf{P}$ is isomorphic to a smooth quadric. We will need a similar vanishing result as in Theorem 2.4 for such grassmannians.

Lemma 2.1. *Let Q_n be a smooth quadric of dimension n. Assume that p is odd. Then $H^i(T^*(Q_n), \mathcal{O}_{T^*(Q_n)}) = 0$ for $i > 0$.*

Proof. Let $j : Q_n \hookrightarrow \mathbb{P}(V)$ be the embedding of Q_n into the projective space $\mathbb{P}(V)$, the dimension of V being equal to $n + 2$. Consider the adjunction sequence

$$(2.5) \qquad 0 \to \mathcal{T}_{Q_n} \to j^*\mathcal{T}_{\mathbb{P}(V)} \to \mathcal{O}_{Q_n}(2) \to 0,$$

and tensor it with $\mathcal{O}_{Q_n}(-1)$:

$$(2.6) \qquad 0 \to \mathcal{T}_{Q_n}(-1) \to j^*\mathcal{T}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{Q_n}(1) \to 0.$$

Note that $\mathrm{Hom}(j^*\mathcal{T}_{\mathbb{P}(V)}(-1), \mathcal{O}_{Q_n}(1)) = H^0(Q_n, j^*\Omega^1_{\mathbb{P}(V)}(2)) = \mathrm{Ker}(V^* \otimes V^* \to S^2V^*/\langle q \rangle)$ (the last isomorphism comes from the Euler exact sequence on $\mathbb{P}(V)$ restricted to Q_n). Here $q \in S^2V^*$ is the quadratic form that defines the quadric Q_n (since p is odd, we can identify quadratic and bilinear forms via polarization) and the map $j^*\mathcal{T}_{\mathbb{P}(V)}(-1) \to \mathcal{O}_{Q_n}(1)$ in (2.6) corresponds to q. The kernel of this map, which is isomorphic to $\mathcal{T}_{Q_n}(-1)$, is also equipped with a symmetric form that makes the bundle $\mathcal{T}_{Q_n}(-1)$ an orthogonal vector bundle with trivial determinant. We then obtain an isomorphism of vector bundles $\mathcal{T}_{Q_n}(-1)$ and $\Omega^1_{Q_n}(1)$ (in fact, the fiber of $\mathcal{T}_{Q_n}(-1)$ over a point on Q_n is isomorphic to $l^\perp/l$, where l is the line corresponding to this point and $l^\perp$ is the orthogonal complement to l with respect to q). Therefore, $\mathcal{T}_{Q_n} = \Omega^1_{Q_n}(2)$. Dualizing the sequence (2.5) and tensoring it then with $\mathcal{O}_{Q_n}(2)$ we get:

$$(2.7) \qquad 0 \to \mathcal{O}_{Q_n} \to j^*\Omega^1_{\mathbb{P}(V)}(2) \to \mathcal{T}_{Q_n} \to 0.$$

Let $\pi : T^*(Q_n) \to Q_n$ be the projection. For any $i \geq 0$ there is an isomorphism $H^i(T^*(Q_n), \mathcal{O}_{T^*(Q_n)}) = H^i(Q_n, S^\bullet\mathcal{T}_{Q_n})$, the projection π being an affine morphism. One has $H^i(Q_n, \mathcal{O}_{Q_n}) = 0$ for $i > 0$. Let $k \geq 1$. The Koszul resolution (Section 2) for the bundle $S^k\mathcal{T}_{Q_n}$ looks as follows:

$$(2.8) \qquad 0 \to j^*S^{k-1}(\Omega^1_{\mathbb{P}(V)}(2)) \to j^*S^k(\Omega^1_{\mathbb{P}(V)}(2)) \to S^k\mathcal{T}_{Q_n} \to 0.$$

Consider the projective bundle $\mathbb{P}(\mathcal{T}_{\mathbb{P}(V)}(-1))$ over $\mathbb{P}(V)$. It is isomorphic to the variety of partial flags $\mathsf{Fl}_{1,2,n+1}$. Let π and q be the projections of $\mathsf{Fl}_{1,2,n+1}$ onto $\mathbb{P}(V)$ and $\mathsf{Gr}_{2,n+1}$, respectively. Note that for $k \geq 0$ one has $R^\bullet\pi_*\mathcal{O}_\pi(k) = S^k\Omega^1_{\mathbb{P}(V)}(k)$, where $\mathcal{O}_\pi(1)$ is the relatively ample invertible sheaf on $\mathbb{P}(\mathcal{T}_{\mathbb{P}(V)}(-1))$. Thus, by the projection formula, $S^k(\Omega^1_{\mathbb{P}(V)}(2)) = R^\bullet\pi_*(\mathcal{O}_\pi(k) \otimes \pi^*\mathcal{O}_{\mathbb{P}(V)}(k)) = R^\bullet\pi_*q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k)$. On the other hand, $H^i(\mathbb{P}(V), R^\bullet\pi_*q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k)) =$

$\mathrm{H}^i(\mathbb{P}(\mathcal{T}_{\mathbb{P}(V)}(-1)), q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k))$. Using the Kempf vanishing we get $\mathrm{H}^i(\mathbb{P}(\mathcal{T}_{\mathbb{P}(V)}(-1)), q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k))$ $= 0$ for $i > 0$, the line bundle $q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k)$ being effective. Now tensor the short exact sequence

$$(2.9) \qquad 0 \to \mathcal{O}_{\mathbb{P}(V)}(-2) \to \mathcal{O}_{\mathbb{P}(V)} \to j_*\mathcal{O}_{\mathsf{Q}_n} \to 0,$$

with $\mathsf{S}^k(\Omega^1_{\mathbb{P}(V)}(2))$. The long exact cohomology sequence gives:

$$(2.10) \qquad \cdots \to \mathrm{H}^i(\mathbb{P}(V), \mathsf{S}^k(\Omega^1_{\mathbb{P}(V)}(2)) \otimes \mathcal{O}_{\mathbb{P}(V)}(-2)) \to \mathrm{H}^i(\mathbb{P}(V), \mathsf{S}^k(\Omega^1_{\mathbb{P}(V)}(2))) \to$$
$$\to \mathrm{H}^i(\mathsf{Q}_n, j^*\mathsf{S}^k(\Omega^1_{\mathbb{P}(V)}(2)) \to \dots.$$

By the above, the cohomology groups in the middle term vanish for $i > 0$. As for the first term, one has an isomorphism $\mathrm{H}^i(\mathbb{P}(V), \mathsf{S}^k(\Omega^1_{\mathbb{P}(V)}(2)) \otimes \mathcal{O}_{\mathbb{P}(V)}(-2)) = \mathrm{H}^i(\mathsf{Fl}_{1,2,n+1}, q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k) \otimes \pi^*\mathcal{O}_{\mathbb{P}(V)}(-2))$. The latter group is isomorphic to $\mathrm{H}^i(\mathsf{Gr}_{2,n+1}, \mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k) \otimes \mathrm{R}^\bullet q_* \pi^* \mathcal{O}_{\mathbb{P}(V)}(-2))$. It is easy to see (for example, using the Koszul resolution of the structure sheaf of $\mathsf{Fl}_{1,2,n+1}$ inside the product $\mathbb{P}(V) \times \mathsf{Gr}_{2,n+1}$) that $\mathrm{R}^\bullet q_* \pi^* \mathcal{O}_{\mathbb{P}(V)}(-2) = \mathcal{O}_{\mathsf{Gr}_{2,n+1}}(-1)[-1]$. Thus, one obtains $\mathrm{H}^i(\mathsf{Fl}_{1,2,n+1}, q^*\mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k) \otimes \pi^*\mathcal{O}_{\mathbb{P}(V)}(-2)) = \mathrm{H}^{i-1}(\mathsf{Gr}_{2,n+1}, \mathcal{O}_{\mathsf{Gr}_{2,n+1}}(k-1)) = 0$ for $i \neq 1$. From the sequence (2.10) one has $\mathrm{H}^i(\mathsf{Q}_n, j^*\mathsf{S}^k(\Omega^1_{\mathbb{P}(V)}(2)) = 0$ for $i > 0$. Finally, the sequence (2.7) implies $\mathrm{H}^i(\mathsf{Q}_n, \mathsf{S}^k\mathcal{T}_{\mathsf{Q}_n}) = 0$ for $i > 0$, hence the statement.

$\square$

3. Cohomology of the Frobenius neighborhoods

Let X be a smooth variety over k. To compute the cohomology groups $\mathrm{H}^i(X', \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X))$ we will use the properties of sheaves $\mathbb{D}_X$ from Section 3.2. Keeping the previous notation, we get:

$$\mathrm{H}^j(X', \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)) = \mathrm{H}^j(X', i^*\mathbb{D}_X) = \mathrm{H}^j(\mathrm{T}^*(X'), i_*i^*\mathbb{D}_X) =$$
$$(3.1) \qquad = \mathrm{H}^j(\mathrm{T}^*(X'), \mathbb{D}_X \otimes i_*\mathcal{O}_{X'}),$$

the last isomorphism in (3.1) follows from the projection formula. Recall the projection $\pi\colon \mathrm{T}^*(X') \to X'$. Consider the bundle $\pi^*\mathcal{T}^*_{X'}$. There is a tautological section s of this bundle such that the zero locus of s coincides with X'. Hence, one obtains the Koszul resolution:

$$(3.2) \qquad 0 \to \cdots \to \overset{k}{\bigwedge}(\pi^*\mathcal{T}_{X'}) \to \overset{k-1}{\bigwedge}(\pi^*\mathcal{T}_{X'}) \to \cdots \to \mathcal{O}_{\mathrm{T}^*(X')} \to i_*\mathcal{O}_{X'} \to 0.$$

Tensor the resolution (3.2) with the sheaf $\mathbb{D}_X$. The rightmost cohomology group in (3.1) can be computed via the above Koszul resolution.

Lemma 3.1. *Fix $k \geq 0$. Then for any $j \geq 0$, the vanishing $\mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* \bigwedge^k(\pi^*\mathcal{T}_{X'})) = 0$ implies* $\mathrm{H}^j(\mathrm{T}^*(X'), \mathbb{D}_X \otimes \bigwedge^k(\pi^*\mathcal{T}_{X'})) = 0$.

Proof. Denote C^k the sheaf $\mathbb{D}_X \otimes \bigwedge^k(\pi^*\mathcal{T}_{X'})$. Take the direct image of C^k with respect to π. Using the projection formula we get:

$$\mathrm{R}^\bullet \pi_* C^k = \mathrm{R}^\bullet \pi_*(\mathbb{D}_X \otimes \bigwedge^k(\pi^*\mathcal{T}_{X'})) = \pi_*(\mathbb{D}_X \otimes \bigwedge^k(\pi^*\mathcal{T}_{X'}))$$
$$(3.3) \qquad = \mathsf{F}_*\mathrm{D}_X \otimes \bigwedge^k(\mathcal{T}_{X'}),$$

the morphism π being affine. Hence,

$$(3.4) \qquad \mathrm{H}^j(\mathrm{T}^*(X'), \mathbb{D}_X \otimes \bigwedge^k(\pi^*\mathcal{T}_{X'})) = \mathrm{H}^j(X', \mathsf{F}_*\mathrm{D}_X \otimes \bigwedge^k(\mathcal{T}_{X'})).$$

The sheaf $\mathsf{F}_*\mathrm{D}_X \otimes \bigwedge^k(\mathcal{T}_{X'})$ is equipped with a filtration that is induced by the filtration on $\mathsf{F}_*\mathrm{D}_X$, the associated sheaf being isomorphic to $\mathrm{gr}(\mathsf{F}_*\mathrm{D}_X) \otimes \bigwedge^k(\mathcal{T}_{X'}) = \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)} \otimes \bigwedge^k(\mathcal{T}_{X'})$. Clearly, for $j \geq 0$

$$(3.5) \qquad \mathrm{H}^j(X', \mathrm{gr}(\mathsf{F}_*\mathrm{D}_X) \otimes \bigwedge^k(\mathcal{T}_{X'})) = 0 \quad \Rightarrow \quad \mathrm{H}^j(X', \mathsf{F}_*\mathrm{D}_X \otimes \bigwedge^k(\mathcal{T}_{X'})) = 0.$$

There are isomorphisms:

$$\mathrm{H}^j(X', \mathsf{F}_*\pi_*\mathcal{O}_{\mathrm{T}^*(X)} \otimes \bigwedge^k(\mathcal{T}_{X'})) = \mathrm{H}^j(X, \pi_*\mathcal{O}_{\mathrm{T}^*(X)} \otimes \mathsf{F}^* \bigwedge^k(\mathcal{T}_{X'})) =$$

$$(3.6) \qquad = \mathrm{H}^j(\mathrm{T}^*(X), \pi^*\mathsf{F}^* \bigwedge^k(\mathcal{T}_{X'})),$$

the last isomorphism following from the projection formula. Finally, $\mathrm{H}^j(\mathrm{T}^*(X), \pi^*\mathsf{F}^* \bigwedge^k(\mathcal{T}_{X'})) = \mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* \bigwedge^k(\pi^*\mathcal{T}_{X'}))$, hence the statement of the lemma. $\qquad\square$

Remark 3.1. *Assume that for a given X one has $\mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* \bigwedge^k(\pi^*\mathcal{T}_{X'})) = 0$ for $j > k \geq 0$. The spectral sequence $\mathrm{E}_1^{p,q} = \mathrm{H}^p(\mathrm{T}^*(X'), \mathbb{D}_X \otimes \bigwedge^q(\pi^*\mathcal{T}_{X'}))$ converges to $\mathrm{H}^{p-q}(\mathrm{T}^*(X'), \mathbb{D}_X \otimes i_*\mathcal{O}_{X'})$. Lemma 3.1 and the resolution (3.2) then imply:*

$$(3.7) \qquad \mathrm{H}^j(\mathrm{T}^*(X'), \mathbb{D}_X \otimes i_*\mathcal{O}_{X'}) = \mathrm{H}^j(X', \mathcal{E}nd(\mathsf{F}_*\mathcal{O}_X)) = 0$$

for $j > 0$, and

$$(3.8) \qquad \mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* i_*\mathcal{O}_{X'}) = 0$$

for $j > 0$.

Remark 3.2. The complex $\tilde{C}^k \colon = \mathsf{F}^*\pi^* \bigwedge^k(\mathcal{T}_{X'})$ is quasiisomorphic to $\mathsf{F}^* i_*\mathcal{O}_{X'}$, the structure sheaf of the Frobenius neighborhood of the zero section X' in the cotangent bundle $\mathrm{T}(X')$. Below we show that if X is either a smooth quadric (for odd p) or a partial flag variety then

$$(3.9) \qquad \mathrm{H}^j(\mathrm{T}^*(X), \mathsf{F}^* i_*\mathcal{O}_{X'}) = 0$$

for $j > 0$.

4. Quadrics

Theorem 4.1. *Let Q_n be a smooth quadric of dimension n. Assume that p is odd. Then $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{E}nd_{\mathcal{O}_{\mathsf{Q}_n}^p}(\mathcal{O}_{\mathsf{Q}_n})) = 0$ for $i > 0$.*

Proof. Let V be a vector space of dimension $n+2$. A smooth quadric $\mathsf{Q}_n \subset \mathbb{P}(V)$ is a homogeneous space, and a hypersurface of degree two in $\mathbb{P}(V)$. To keep the notation simple, we will ignore the Frobenius twist of Q_n and will be dealing with the absolute Frobenius morphism $\mathsf{F} : \mathsf{Q}_n \to \mathsf{Q}_n$; this will not change the cohomology groups in question.

Lemma 4.1 shows that $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^r \mathcal{T}_{\mathsf{Q}_n} \otimes \mathsf{F}_* \pi_* \mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)}) = 0$ for $i > r \geq 0$. Lemma 3.1 then gives $\mathrm{H}^i(\mathrm{T}^*(\mathsf{Q}_n), \mathbb{D}_{\mathsf{Q}_n} \otimes \bigwedge^r(\pi^* \mathcal{T}_{\mathsf{Q}_n})) = 0$ for $i > r \geq 0$. Using Remark 2, we get:

$$\tag{4.1} \mathrm{H}^i(\mathsf{Q}_n, \mathcal{E}nd_{\mathcal{O}^p_{\mathsf{Q}_n}}(\mathcal{O}_{\mathsf{Q}_n})) = 0$$

for $i > 0$. $\qquad\qquad\square$

Lemma 4.1. *Let $\mathcal{L}$ be an effective line bundle on Q_n. Then for $i > r \geq 0$ one has $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^r \mathcal{T}_{\mathsf{Q}_n} \otimes \mathsf{F}_* \pi_* \mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)} \otimes \mathcal{L}) = 0$.*

Proof. Recall that $\pi_* \mathcal{O}_{\mathrm{T}^*(\mathsf{Q}_n)} = \mathsf{S}^\bullet \mathcal{T}_{\mathsf{Q}_n}$. To simplify the notation, put also $\mathbb{L} = \mathsf{F}_* \mathsf{S}^\bullet \mathcal{T}_{\mathsf{Q}_n} \otimes \mathcal{L}$. If $r = 0$ and $\mathcal{L}$ is ample then the statement follows from Theorem 2.2. Indeed, by the projection formula one has:

$$\tag{4.2} \mathrm{H}^i(\mathsf{Q}_n, \mathbb{L}) = \mathrm{H}^i(\mathsf{Q}_n, \mathsf{S}^\bullet \mathcal{T}_{\mathsf{Q}_n} \otimes \mathcal{L}^{\otimes p}) = 0$$

for $i > 0$. If $\mathcal{L} = \mathcal{O}_{\mathsf{Q}_n}$ then Lemma 2.1 implies:

$$\tag{4.3} \mathrm{H}^i(\mathsf{Q}_n, \mathbb{L}) = \mathrm{H}^i(\mathsf{Q}_n, \mathsf{S}^\bullet \mathcal{T}_{\mathsf{Q}_n}) = 0,$$

for $i > 0$. Let $r \geq 1$. We argue by induction on r, the base of induction being $r = 1$. Denote $j : \mathsf{Q}_n \hookrightarrow \mathbb{P}(V)$ the embedding. Recall the adjunction sequence (see Lemma 2.1)

$$\tag{4.4} 0 \to \mathcal{T}_{\mathsf{Q}_n} \to j^* \mathcal{T}_{\mathbb{P}(V)} \to \mathcal{O}_{\mathsf{Q}_n}(2) \to 0,$$

and consider the Euler sequence on $\mathbb{P}(V)$ restricted to Q_n:

$$\tag{4.5} 0 \to \mathcal{O}_{\mathsf{Q}_n} \to V \otimes \mathcal{O}_{\mathsf{Q}_n}(1) \to j^* \mathcal{T}_{\mathbb{P}(V)} \to 0.$$

The sequence (4.4) gives rise to a short exact sequence (see Section 1):

$$\tag{4.6} 0 \to \bigwedge^r \mathcal{T}_{\mathsf{Q}_n} \to j^* \bigwedge^r \mathcal{T}_{\mathbb{P}(V)} \to \bigwedge^{r-1} \mathcal{T}_{\mathsf{Q}_n} \otimes \mathcal{O}_{\mathsf{Q}_n}(2) \to 0.$$

Consider first the case $r = 1$. Tensoring the sequence (4.4) with $\mathbb{L}$, one obtains:

$$\tag{4.7} 0 \to \mathcal{T}_{\mathsf{Q}_n} \otimes \mathbb{L} \to \mathcal{T}_{\mathbb{P}(V)} \otimes \mathbb{L} \to \mathcal{O}_{\mathsf{Q}_n}(2) \otimes \mathbb{L} \to 0.$$

As in (4.2), one has $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{O}_{\mathsf{Q}_n}(2) \otimes \mathbb{L}) = 0$ for $i > 0$. Tensoring the sequence (4.5) with $\mathbb{L}$, we get:

$$\tag{4.8} 0 \to \mathbb{L} \to V \otimes \mathcal{O}_{\mathsf{Q}_n}(1) \otimes \mathbb{L} \to \mathcal{T}_{\mathbb{P}(V)} \otimes \mathbb{L} \to 0.$$

Again by (4.2) and (4.3), one has $\mathrm{H}^i(\mathsf{Q}_n, \mathbb{L}) = \mathrm{H}^i(\mathsf{Q}_n, V \otimes \mathcal{O}_{\mathsf{Q}_n}(1) \otimes \mathbb{L}) = 0$ for $i > 0$. Hence, $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{T}_{\mathbb{P}(V)} \otimes \mathbb{L}) = 0$ for $i > 0$. From (4.7) we conclude that $\mathrm{H}^i(\mathsf{Q}_n, \mathcal{T}_{\mathsf{Q}_n} \otimes \mathbb{L}) = 0$ for $i > 1$.

Now fix $m > 1$. Assume that for $r \leq m$ one has $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^r \mathcal{T}_{\mathsf{Q}_n} \otimes \mathbb{L}) = 0$ for $i > r$. Let us prove that $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^{r+1} \mathcal{T}_{\mathsf{Q}_n} \otimes \mathbb{L}) = 0$ for $i > r + 1$. Tensoring the sequence (4.6) (for $r + 1$) with $\mathbb{L}$, we get:

$$\tag{4.9} 0 \to \bigwedge^{r+1} \mathcal{T}_{\mathsf{Q}_n} \otimes \mathbb{L} \to \bigwedge^{r+1} \mathcal{T}_{\mathbb{P}(V)} \otimes \mathbb{L} \to \bigwedge^r \mathcal{T}_{\mathsf{Q}_n} \otimes \mathcal{O}_{\mathsf{Q}_n}(2) \otimes \mathbb{L} \to 0.$$

By the inductive assumption one has $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^r \mathcal{T}_{\mathsf{Q}_n} \otimes \mathcal{O}_{\mathsf{Q}_n}(2) \otimes \mathbb{L}) = 0$ for $i > r$. Let us show that $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^{r+1} \mathcal{T}_{\mathbb{P}(V)} \otimes \mathbb{L}) = 0$ for $i > 0$. Clearly, this will imply $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^{r+1} \mathcal{T}_{\mathsf{Q}_n} \otimes \mathbb{L}) = 0$ for $i > r+1$. From the Euler sequence on $\mathbb{P}(V)$ one obtains:

$$(4.10) \qquad 0 \to \bigwedge^{l-1} \mathcal{T}_{\mathbb{P}(V)} \to \bigwedge^{l} V \otimes \mathcal{O}_{\mathbb{P}(V)}(l) \to \bigwedge^{l} \mathcal{T}_{\mathbb{P}(V)} \to 0.$$

Gluing together these short exact sequences for $l = 1, \ldots r+1$, one arrives at the long exact sequence:

$$(4.11) \qquad 0 \to \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \bigwedge^{2} V \otimes \mathcal{O}_{\mathbb{P}(V)}(2) \to \cdots \to \bigwedge^{r+1} \mathcal{T}_{\mathbb{P}(V)} \to 0.$$

Tensoring the above sequence with $\mathbb{L}$, and using Theorem 2.2 and Lemma 2.1, we conclude that $\mathrm{H}^i(\mathsf{Q}_n, \bigwedge^{r+1} \mathcal{T}_{\mathbb{P}(V)} \otimes \mathbb{L}) = 0$ for $i > 0$. Hence the statement. $\qquad \square$

5. **Partial flag varieties**

Let V be a vector space over k of dimension $n+1$, and X the flag variety $\mathrm{F}_{1,n,n+1}$, a smooth divisor in $\mathbb{P}(V) \times \mathbb{P}(V^*)$ of bidegree $(1,1)$. As in the previous section, we omit the superscript at X meaning the Frobenius twist and consider the absolute Frobenius morphism.

Theorem 5.1. $\mathrm{H}^i(X, \mathcal{E}nd_{\mathcal{O}_X^p}(\mathcal{O}_X)) = 0$ *for* $i > 0$.

Proof. The proof is similar to that of Theorem 4.1. Consider the line bundle $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$ over $\mathbb{P}(V) \times \mathbb{P}(V^*)$ (the symbol $\boxtimes$ denotes the external tensor product). Then X is isomorphic to the zero locus of a section of $\mathcal{O}_{\mathbb{P}(V)}(1) \boxtimes \mathcal{O}_{\mathbb{P}(V^*)}(1)$. Consider the adjunction sequence:

$$(5.1) \qquad 0 \to \mathcal{T}_X \to \mathcal{T}_{\mathbb{P}(V) \times \mathbb{P}(V^*)} \otimes \mathcal{O}_X \to \mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1) \to 0.$$

Note that $\mathcal{T}_{\mathbb{P}(V) \times \mathbb{P}(V^*)} = \mathcal{T}_{\mathbb{P}(V)} \boxplus \mathcal{T}_{\mathbb{P}(V^*)}$, the symbol $\boxplus$ denoting the external direct sum. One has the Euler sequences:

$$(5.2) \qquad 0 \to \mathcal{O}_{\mathbb{P}(V)} \to V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \to \mathcal{T}_{\mathbb{P}(V)} \to 0,$$

and

$$(5.3) \qquad 0 \to \mathcal{O}_{\mathbb{P}(V^*)} \to V^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \to \mathcal{T}_{\mathbb{P}(V^*)} \to 0.$$

Hence the short exact sequence:

$$0 \to \mathcal{O}_{\mathbb{P}(V) \times \mathbb{P}(V^*)} \oplus \mathcal{O}_{\mathbb{P}(V) \times \mathbb{P}(V^*)} \to V^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \boxplus \mathcal{O}_{\mathbb{P}(V)}(1) \otimes V \to$$
$$(5.4) \qquad \to \mathcal{T}_{\mathbb{P}(V) \times \mathbb{P}(V^*)} \to 0.$$

Show that $\mathrm{H}^i(X, \bigwedge^r \mathcal{T}_X \otimes \mathsf{F}_* \pi_* \mathcal{O}_{\mathrm{T}^*(X)}) = 0$ for $i > r \geq 0$. If $r = 0$ then by Theorem 2.3 one has:

$$(5.5) \qquad \mathrm{H}^i(X, \mathsf{F}_* \pi_* \mathcal{O}_{\mathrm{T}^*(X)}) = \mathrm{H}^i(X, \pi_* \mathcal{O}_{\mathrm{T}^*(X)}) = \mathrm{H}^i(X, \mathsf{S}^\bullet \mathcal{T}_X) = 0$$

for $i > 0$. Let now $r \geq 1$. The sequence (5.1) gives rise to a short exact sequence:

$$(5.6) \qquad 0 \to \bigwedge^{r} \mathcal{T}_X \to \bigwedge^{r} \mathcal{T}_{\mathbb{P}(V) \times \mathbb{P}(V)} \otimes \mathcal{O}_X \to \wedge^{r-1} \mathcal{T}_X \otimes (\mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1)) \to 0.$$

We argue again by induction on r. Let first $r = 1$. Tensoring the sequence (5.1) with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$, we get:

$$0 \to \mathcal{T}_X \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to$$

(5.7)
$$\to (\mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1)) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to 0.$$

By Theorem 2.2, one has for $i > 0$:

(5.8) $\qquad \mathrm{H}^i(X, (\mathcal{O}_X(1) \boxtimes \mathcal{O}_X(1)) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = \mathrm{H}^i(X, (\mathcal{O}_X(p) \boxtimes \mathcal{O}_X(p)) \otimes \mathsf{S}^\bullet\mathcal{T}_X) = 0.$

On the other hand, tensoring sequence (5.4) (considered first on $\mathbb{P}(V)\times\mathbb{P}(V^*)$, and then restricted to X) with $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$, one obtains:

$$0 \to \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X^{\oplus 2} \to (V^* \otimes \mathcal{O}_{\mathbb{P}(V^*)}(1) \boxplus \mathcal{O}_{\mathbb{P}(V)}(1) \otimes V) \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to$$

(5.9)
$$\to \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X \to 0.$$

The leftmost term in the above sequence has vanishing higher cohomology by (5.5). The middle term is the direct sum of several copies of the bundle $\mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X$ tensored with an effective line bundle on X. Let $\mathcal{L}$ be an arbitrary effective line bundle on X (i.e. the one isomorphic to either $\mathcal{O}_X(k) \boxtimes \mathcal{O}_X$ or $\mathcal{O}_X \boxtimes \mathcal{O}_X(k)$ for some $k \geq 0$). For any $l \geq 1$ there is a filtration on $\mathsf{S}^l(\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X)$ that comes from the sequence (5.1), the graded factors of this filtration being isomorphic to $\mathsf{S}^i\mathcal{T}_X \otimes (\mathcal{O}_X(l-i) \boxtimes \mathcal{O}_X(l-i))$ for $0 \leq i \leq l$. Tensoring $\mathsf{S}^l(\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X)$ with $\mathcal{L}$, the graded factors $\mathsf{S}^i\mathcal{T}_X \otimes (\mathcal{O}_X(l-i) \boxtimes \mathcal{O}_X(l-i))$ are tensored with $\mathcal{L}$. For $i < l$ the higher cohomology of the corresponding graded factor vanish by Theorem 2.2. On the other hand, the higher cohomology of $\mathsf{S}^l(\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathcal{O}_X) \otimes \mathcal{L}$ are easily seen to vanish (use the Koszul resolutions associated to the Euler sequences (5.2) and (5.3), and the Kempf vanishing). One obtains $\mathrm{H}^i(X, \mathsf{S}^\bullet\mathcal{T}_X \otimes \mathcal{L}) = 0$ for $i > 1$. Hence, the cohomology of the middle term in (5.9) vanish for $i > 1$, and, therefore, $\mathrm{H}^i(X, \mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = 0$ for $i > 1$. Coming back to the sequence (5.7), we get $\mathrm{H}^i(X, \mathcal{T}_X \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = 0$ for $i > 1$. The inductive step is completely analogous to that in the proof of Lemma 4.1 (one uses the Euler sequences (5.2) and (5.3) to ensure that $\mathrm{H}^i(X, \bigwedge^{r+1}\mathcal{T}_{\mathbb{P}(V)\times\mathbb{P}(V^*)} \otimes \mathsf{F}_*\mathsf{S}^\bullet\mathcal{T}_X) = 0$ for $i > r+1$). This allows to complete the induction argument. $\qquad\square$

6. Derived equivalences

Definition 6.1. *A coherent sheaf $\mathcal{E}$ on a smooth variety X over an algebraically closed field k is called a* tilting generator *of the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X if the following holds:*

(1) *The sheaf $\mathcal{E}$ is a tilting object in $\mathcal{D}^b(X)$, that is, for any $i \geq 1$ one has $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = 0$.*
(2) *The sheaf $\mathcal{E}$ generates the derived category $\mathcal{D}^-(X)$ of complexes bounded from above, that is, if for some object $\mathcal{F} \in \mathcal{D}^-(X)$ one has $\mathrm{R}Hom^\bullet(\mathcal{F}, \mathcal{E}) = 0$, then $\mathcal{F} = 0$.*

Tilting generators are a tool to construct derived equivalences. One has:

Lemma 6.1. ([14], *Lemma 1.2*) *Let X be a smooth scheme, $\mathcal{E}$ a tilting generator of the derived category $\mathcal{D}^b(X)$, and denote $R = \mathcal{E}nd(\mathcal{E})$. Then the algebra R is left-Noetherian, and the correspondence $\mathcal{F} \mapsto \mathrm{R}Hom^\bullet(\mathcal{E}, \mathcal{F})$ extends to an equivalence*

$$\tag{6.1} \mathcal{D}^b(X) \to \mathcal{D}^b(R - \mathrm{mod}^{\mathrm{fg}})$$

between the bounded derived category $\mathcal{D}^b(X)$ of coherent sheaves on X and the bounded derived category $\mathcal{D}^b(R - \mathrm{mod}^{\mathrm{fg}})$ of finitely generated left R-modules.

We now recall the derived localization theorem of [3]. Let $\mathbf{G}$ be a semisimple algebraic group over k, $\mathbf{G}/\mathbf{B}$ the flag variety, and $\mathcal{U}(\mathfrak{g})$ the universal enveloping algebra of the corresponding Lie algebra. The center of $\mathcal{U}(\mathfrak{g})$ contains the "Harish–Chandra part" $\mathfrak{Z}_{\mathrm{HC}} = \mathcal{U}(\mathfrak{g})^{\mathbf{G}}$. Denote $\mathcal{U}(\mathfrak{g})_0$ the central reduction $\mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{Z}_{\mathrm{HC}}} \mathbf{k}$, where $\mathbf{k}$ is the trivial $\mathfrak{g}$-module. Consider the category $\mathrm{D}_{\mathbf{G}/\mathbf{B}}$-mod of coherent $\mathrm{D}_{\mathbf{G}/\mathbf{B}}$-modules and the category $\mathcal{U}(\mathfrak{g})_0$-mod of finitely generated modules over $\mathcal{U}(\mathfrak{g})_0$. The derived localization theorem (Theorem 3.2, [3]) states:

Theorem 6.1. *Let char $k = p > h$, where h is the Coxeter number of the group $\mathbf{G}$. Then there is an equivalence of derived categories:*

$$\tag{6.2} \mathrm{D}^b(\mathrm{D}_{\mathbf{G}/\mathbf{B}} - \mathrm{mod}) \simeq \mathrm{D}^b(\mathcal{U}(\mathfrak{g})_0 - \mathrm{mod}),$$

There are also "unbounded" versions of Theorem 6.1, the bounded categories in (6.2) being replaced by categories unbounded from above or below (see Remark 2 on p. 18 of [3]). Thus extended, Theorem 6.1 implies:

Lemma 6.2. *Let $\mathbf{G}$ be a semisimple algebraic group over k, and $X = \mathbf{G}/\mathbf{B}$ the flag variety. Let char $k = p > h$, where h is the Coxeter number of the group $\mathbf{G}$. Then the bundle $\mathsf{F}_* \mathcal{O}_X$ satisfies the condition (2) of Definition 6.1.*

Proof. We need to show that for an object $\mathcal{E} \in \mathcal{D}^-(X')$ the equality $\mathrm{R}Hom^\bullet(\mathcal{E}, \mathsf{F}_*\mathcal{O}_X) = 0$ implies $\mathcal{E} = 0$. By adjunction we get:

$$\tag{6.3} \mathbb{H}^\bullet(X, (\mathsf{F}^*\mathcal{E})^\vee) = 0.$$

On the other hand,

$$(\mathsf{F}^*\mathcal{E})^\vee = \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{F}^*\mathcal{E}, \mathcal{O}_X) = \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}(\mathsf{F}^*\mathcal{E}, \mathsf{F}^*\mathcal{O}_{X'}) =$$
$$\tag{6.4} = \mathsf{F}^*\mathcal{R}\mathcal{H}om_{\mathcal{O}_{X'}}(\mathcal{E}, \mathcal{O}_{X'}) = \mathsf{F}^*\mathcal{E}^\vee.$$

By the Cartier descent (see Subsection 3.3), the object $\mathsf{F}^*\mathcal{E}^\vee$ is an object of the category $\mathrm{D}^-(\mathrm{D}_X - \mathrm{mod})$. Now $\mathsf{F}^*\mathcal{E}^\vee$ is annihilated by the functor $\mathrm{R}\Gamma$. Under our assumptions on p, this functor is an equivalence of categories by Theorem 6.1. Hence, $\mathsf{F}^*\mathcal{E}^\vee$ is quasiisomorphic to zero, and therefore so are $\mathcal{E}^\vee$ and $\mathcal{E}$. $\qquad\square$

Corollary 6.1. *Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$, and let $p > h$. Then the bundle $\mathsf{F}_*\mathcal{O}_{\mathbf{G}/\mathbf{P}}$ is a generator in $\mathrm{D}^b(\mathbf{G}/\mathbf{P})$.*

Proof. Denote $Y = \mathbf{G}/\mathbf{P}$, and let $\pi : X = \mathbf{G}/\mathbf{B} \to Y$ be the projection. As before, one has to show that for any object $\mathcal{E} \in \mathrm{D}^-(Y')$ the equality $\mathrm{R}Hom^\bullet(\mathcal{E}, \mathsf{F}_*\mathcal{O}_Y) = 0$ implies $\mathcal{E} = 0$. Notice that $\mathrm{R}^\bullet\pi_*\mathcal{O}_X = \mathcal{O}_Y$. The condition $\mathrm{R}Hom^\bullet(\mathcal{E}, \mathsf{F}_*\mathcal{O}_Y) = 0$ then translates into:

$$\tag{6.5} \mathrm{R}Hom^\bullet(\mathcal{E}, \mathsf{F}_*\mathcal{O}_Y) = \mathrm{R}Hom^\bullet(\mathcal{E}, \mathsf{F}_*\mathrm{R}^\bullet\pi_*\mathcal{O}_X) = \mathrm{R}Hom^\bullet(\mathcal{E}, \mathrm{R}^\bullet\pi_*\mathsf{F}_*\mathcal{O}_X) = 0.$$

By adjunction we get:

$$(6.6) \qquad \mathrm{R}Hom^{\bullet}(\pi^*\mathcal{E}, \mathsf{F}_*\mathcal{O}_X) = 0.$$

Lemma 6.2 then implies that the object $\pi^*\mathcal{E} = 0$ in $\mathrm{D}^-(X')$. Applying to it the functor π_* and using the projection formula, we get $\mathrm{R}^{\bullet}\pi_*\pi^*\mathcal{E} = \mathcal{E} \otimes \mathrm{R}^{\bullet}\pi_*\mathcal{O}_{X'} = \mathcal{E}$, hence $\mathcal{E} = 0$, q.e.d. $\qquad \square$

Corollary 6.2. *Let X be the variety of partial flags $\mathrm{F}_{1,n,n+1}$. Assume that $p > n+1$. Then $\mathsf{F}_*\mathcal{O}_X$ is a tilting bundle.*

Proof. Follows from Theorem 5.1 and Corollary 6.1. $\qquad \square$

Corollary 6.3. *Let Q_n be a smooth quadric of dimension n. Assume that $p > n - (-1)^k$, where $k \equiv n(2)$. Then $\mathsf{F}_*\mathcal{O}_{\mathsf{Q}_n}$ is a tilting bundle.*

Proof. Follows from Theorem 4.1 and Corollary 6.1. $\qquad \square$

Remark 6.1. The derived categories of coherent sheaves on quadrics and partial flag varieties $\mathrm{F}_{1,n,n+1}$ were described in [17] in terms of exceptional collections. By [21], decomposition of the Frobenius pushforward of the structure sheaf on a quadric gives rise to the full exceptional collection from [17]. The relation of the Frobenius pushforward of the structure sheaf on $\mathrm{F}_{1,n,n+1}$ and the full exceptional collection given in *loc.cit.* will be a subject of a future paper (for $n = 2$ the variety $\mathrm{F}_{1,2,3}$ is isomorphic to the flag variety $\mathbf{SL}_3/\mathbf{B}$ and the decomposition of $\mathsf{F}_*\mathcal{O}_{\mathbf{SL}_3/\mathbf{B}}$ into the direct sum of indecomposables was found in [12]).

References

[1] H. H. Andersen, M. Kaneda, *On the* D-*affinity of the flag variety in type* B_2, Mansucripta Math., **103** (2000), pp. 393–399

[2] A. Beilinson, J. Bernstein, *Localisation de $\mathfrak{g}$-modules*, C.R.Acad. Sci. Paris, Ser. I **292** (1981), pp. 15–18

[3] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Localization of modules for a semisimple Lie algebras in prime characteristic*, Ann. of Math. (2) **167** (2008), no. 3, pp. 945–991

[4] R. Bezrukavnikov, I. Mirković, D. Rumynin, *Singular localization and intertwining functors for reductive Lie algebras in prime characteristic*, Nagoya Math. J. 184 (2006), pp. 1–55

[5] W. Borho, H. Kraft, *Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen*, Comment. Math. Helv. **54** (1979), no. 1, pp. 61–104.

[6] M. Brion, S. Kumar, *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, v. 231, (2005) Birkhauser

[7] S. Donkin, *The normality of closures of conjugacy classes of matrices*, Invent. Math. **101** (1990), no. 3, pp. 717–736.

[8] A. Grothendieck, EGA IV, *Etude locale des schémas et des morphismes de schémas*, Inst. Hautes Etudes Sci. Publ. Math. **32** (1967)

[9] B. Haastert, *Über Differentialoperatoren und $\mathbb{D}$-Moduln in positiver Charakterik*, Manuscripta Math., **58** (1987), no. 4, pp. 385–415

[10] W. J. Haboush, *A short proof of the Kempf vanishing theorem*, Invent. Math. **56** (1980), no. 2, pp. 109–112

[11] R. Hartshorne, *Algebraic geometry*, Springer Verlag, Heidelberg, New York (1977), GTM 52

[12] Y. Hashimoto, M. Kaneda, D. Rumynin, *On localization of $\overline{D}$-modules*, in "Representations of algebraic groups, quantum groups, and Lie algebras", pp. 43–62, Contemp. Math., **413**, Amer. Math. Soc., Providence, RI, (2006)

[13] J. C. Jantzen, *Representations of algebraic groups*, 2nd ed. (2003), AMS

[14] D. Kaledin, *Derived equivalences by quantization*, Geom. Funct. Anal. **17** (2008), no. 6, pp. 1968–2004

[15] N. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turritin*, Publ. Math. Inst. Hautes Etud. Sci., **39** (1970), pp. 175–232

[16] M. Kaneda, J. Ye, *Equivariant localization of $\overline{D}$-modules on the flag variety of the symplectic group of degree 4*, J. Algebra **309** (2007), no. 1, pp. 236–281

[17] M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), pp. 479 – 508

[18] M. Kashiwara, N. Lauritzen, *Local cohomology and $\mathcal{D}$-affinity in positive characteristic*, C. R. Acad. Sci. Paris, Ser. I, **335** (2002), pp. 993–996

[19] N. Katz, *Nilpotent connections and the monodromy theorem: applications of a result of Turritin*, Publ. Math. Inst. Hautes Etud. Sci., **39** (1970), pp. 175–232

[20] S. Kumar, N. Lauritzen, J.F.Thomsen, *Frobenius splitting of cotangent bundles of flag varieties*, Invent. Math. **136** (1999), pp. 603–621

[21] A. Langer, D-*affinity and Frobenius morphism on quadrics*, Int. Math. Res. Not., 2008, no. 1, Art. ID rnm 145, 26 pp.

[22] N. Lauritzen, V. Mehta, *Differential operators and the Frobenius pull back of the tangent bundle on* G/P, Manuscripta Math. **98** (1999), no. 4, pp. 507–510

[23] V. Mehta, W. Van der Kallen, *A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices*, Compositio Math. **84** (1992) no. 2, pp. 211–221

[24] A. Samokhin, *On the* D-*affinity of quadrics in positive characteristic*, C.R. Acad. Sci. Paris, **344** (2007), Ser. I, pp. 377–382

[25] A. Samokhin, *A vanishing theorem in positive characteristic and tilting equivalences*, IHES preprint, March 2007, available at www.ihes.fr

[26] A. Samokhin, *Tilting bundles via the Frobenius morphism*, arXiv:0904.1235v2

[27] A. Samokhin, *On the* D-*affinity of flag varieties in positive characteristic*, arXiv:0906.1555v3

[28] A. Samokhin, *The Frobenius morphism on homogeneous spaces*, in preparation

INSTITUTE FOR INFORMATION TRANSMISSION PROBLEMS, MOSCOW, RUSSIA
E-mail address: `alexander.samokhin@gmail.com`