A Hausdorff Dimension Estimate for Kernel Sections of Non-autonomous Evolution Equations

V. CHEPYZHOV & M. VISHIK

Dedicated to Ciprian Foias

1. Introduction. This paper studies the kernel of a non-autonomous evolution equation:

$$\partial_t u = A(u,t), \quad u\Big|_{t=\tau} = u_\tau, \tag{1}$$

and the corresponding process $\{U(t,\tau),\ t > \tau,\ t \in \mathbf{R}\}$ acting in a Banach space E, $U(t,\tau)u_\tau = u(t)$, where $u(t)$ is a solution of (1) (see Section 2). By definition, the kernel K consists of all bounded complete trajectories of the equation:

$$K = \{u(\cdot) \mid u(t),\ t \in \mathbf{R}, \text{ is a solution of (1)},\ \|u(t)\|_E \le M_u\ \forall t \in \mathbf{R}\}.$$

Let $K(s) = \{u(s) \mid u(\cdot) \in K\}$, $K(s) \in E$, be the kernel section at time s. In the case of an autonomous evolution equation (when $A(u,t) = A(u)$), the set $K(s) = \mathcal{A}$ does not depend on s, and it forms the maximal invariant set of the corresponding semigroup $\{S(t),\ t \ge 0\}$, $S(t) = U(t,0)$, $S(t)\mathcal{A} = \mathcal{A}$, $\forall t \ge 0$. It is well known that the maximal invariant set of a continuous and asymptotically compact semigroup coincides with the global attractor of this semigroup (see [21], [18], [1]). The kernel and kernel sections are natural generalizations of the notion of maximal invariant set for a non-autonomous dynamical system. If a process $\{U(t,\tau),\ t \ge \tau,\ \tau \in \mathbf{R}\}$ acting in a Banach space E is continuous and possesses a compact uniformly attracting set $P \Subset E$, then the kernel K of this process is non-empty and the kernel section $K(s)$ is compact in E for any $s \in \mathbf{R}$ (see Section 2, Theorem 2.2). We show that kernel sections satisfy some properties that are analogous to the invariance and attracting properties of the attractor of a semigroup. We prove that $U(t,\tau)K(\tau) = K(t)$, $t \ge \tau$, $\tau \in \mathbf{R}$, and, for any bounded set B in E,

$$\mathbf{dist}_E(U(\tau,\tau-T)B\ ,\ K(\tau)) \to 0 \qquad (T \to +\infty),$$

where τ is an arbitrary fixed number (Theorem 2.3).

Indiana University Mathematics Journal ©, Vol. 42, No. 3 (1993)

Let $\mathcal{K} = \bigcup_{s\in\mathbf{R}} K(s)$ be the union of all the kernel sections of a process $\{U(t,\tau)\}$. In the case of the non-autonomous equations of mathematical physics, the closure $\bar{\mathcal{K}}$ in E may have infinite Hausdorff dimension, as shown by many examples (Section 6). Nevertheless, the kernel section $K(s)$ has finite Hausdorff dimension for any $s \in \mathbf{R}$ (Sections 3 and 4). The upper bound for the Hausdorff dimension of the kernel sections has a form analogous to that of the corresponding autonomous evolution equations given in [8].

In Section 5 we present Hausdorff dimension estimates for kernel sections encountered in the following problems of mathematical physics:

(i) The two-dimensional Navier-Stokes system with time-dependent external force $\varphi(x,t)$ ($\varphi \in C_b(\mathbf{R}, H)$);

(ii) the non-autonomous reaction-diffusion system with time-dependent nonlinear interaction function $f(u,t)$ and with external force $\varphi(x,t)$;

(iii) the damped hyperbolic equation with time-dependent terms.

Parallel results for autonomous problems can be found in [8], [2], [14] (see also [18] and [1]).

Note that, in the particular case when the symbol $\sigma_0(t)$ of a non-autonomous equation depends almost periodically on time t (see [5]), the uniform attractor $\mathcal{A}$ of the corresponding process coincides with the union of all kernel sections $\mathcal{K}_\sigma(0)$ of all the equations with $\sigma = \sigma(t)$, from the hull $\mathcal{H}(\sigma_0)$ of the symbol $\sigma_0 : \mathcal{A} = \bigcup_{\sigma\in\mathcal{H}(\sigma_0)} \mathcal{K}_\sigma(0)$ (see [3, 4, 5, 6, 7]). In the case of quasiperiodic dependence on time, estimates for the Hausdorff dimension of uniform attractors were also obtained in [3, 4, 5, 6, 7].

2. Kernel of a process generated by a non-autonomous evolution equation. Let $\{U(t,\tau),\ t \geq \tau,\ \tau \in \mathbf{R}\} = \{U(t,\tau)\}$ be a process acting in a Banach space E. Thus, $U(t,\tau) : E \to E$, $U(t,s)U(s,\tau) = U(t,\tau)$, $U(\tau,\tau) = I$ $\forall t \geq s \geq \tau$, $\tau \in \mathbf{R}$. A function $u(s)$, $s \in \mathbf{R}$, is said to be a *complete trajectory* of the process $\{U(t,\tau)\}$ if

$$U(t,\tau)u(\tau) = u(t) \qquad \forall t \geq \tau,\ \tau \in \mathbf{R}. \tag{2}$$

A complete trajectory $u(s)$ of a process $\{U(t,\tau)\}$ is said to be bounded if the set $\{u(s) \mid s \in \mathbf{R}\}$ is bounded in the norm of E, i.e., $\|u(s)\|_E \leq C_u$ $\forall s \in \mathbf{R}$ (see [17], [9, 10], and [11], where the concept of a process was introduced and some important properties of processes were established).

Definition 2.1. The *kernel* K of a process $\{U(t,\tau)\}$ consists of all bounded complete trajectories of the process $\{U(t,\tau)\}$:

$$K = \big\{u(\cdot) \mid u(\cdot) \text{ satisfies (2) and } \|u(s)\| \leq C_u \ \forall s \in \mathbf{R}\big\}.$$

Definition 2.2. The *section* $K(s) \subset E$ of a kernel K at time $s \in \mathbf{R}$ is defined by:

$$K(s) = \{u(s) \mid u(\cdot) \in K\}. \tag{3}$$

Remark 2.1. If the process $\{U(t,\tau)\}$ is generated by a semigroup $\{S(t), t \geq 0\}$, i.e., $U(t,\tau) = U(t-\tau,0) = S(t-\tau)\ \forall t \geq \tau,\ \tau \in \mathbf{R}$, then the kernel K consists of all the bounded complete trajectories $u(s)$, $s \in \mathbf{R}$, of this semigroup: $S(t)u(s) = u(s+t), \forall t \geq 0,\ s \in \mathbf{R}$. In this case, the sections $K(s)$ do not depend on s; thus, $K(s) = K(0)$. If the semigroup satisfies some extra conditions, then $K(0)$ coincides with the global attractor $\mathcal{A}$ of the semigroup $\{S(t)\}$. In the case of a general process $\{U(t,\tau)\}$, the kernel sections $K(s)$ depend on $s \in \mathbf{R}$.

We shall investigate processes generated by evolution equations of the form:

$$\partial_t u = A(u,t), \quad u\big|_{t=\tau} = u_\tau, \qquad u_\tau \in E,\ t \geq \tau,\ \tau \in \mathbf{R}. \tag{4}$$

Here $A(u,t)$ is a family of nonlinear operators depending on t, $t \in \mathbf{R}$, with domain E_1 not depending on t and with range E_0: $A(u,t) : E_1 \times \mathbf{R} \to E_0$, where E_1, E_0, and E are Banach spaces. Usually $E_1 \subseteq E \subseteq E_0$, and E_1 is dense in E. The meaning of the expression "$u(t)$ is a solution of the problem (4)" is to be discussed separately in each particular case. To begin with, assume that the problem (4) is uniquely solvable for any $\tau \in \mathbf{R}$, $u_\tau \in E$, and has the solution $u(t) \in E$, $t \geq \tau$. Consider a two-parameter family of mappings $\{U(t,\tau),\ t \geq \tau,\ \tau \in \mathbf{R}\}$ defined by the formula:

$$U(t,\tau)u_\tau = u(t), \qquad (u_\tau = u(\tau)), \tag{5}$$

where $u(t)$ is a solution of (4). It is clear that $\{U(t,\tau)\}$ is a process on E. Thus, according to (2), the kernel K of the process $\{U(t,\tau)\}$ (or the kernel of the equation (4)) consists of all bounded solutions $u(t) \in E$ of (4) defined for all $t \in \mathbf{R}$. Evidently the following holds.

Proposition 2.1. *Let K be the kernel of the process $\{U(t,\tau)\}$. Then*

$$U(t,\tau)K(\tau) = K(t) \qquad \forall t \geq \tau,\ \tau \in \mathbf{R}. \tag{6}$$

The next theorem contains conditions under which the kernel K of the process $\{U(t,\tau)\}$ is non-empty. As in [11], we introduce the notion of a uniformly asymptotically compact process. A set $P \subset E$ is said to be a *uniformly attracting set* of a process $\{U(t,\tau)\}$ if for any bounded set $B \subset E$,

$$\sup_{\tau \in \mathbf{R}} \operatorname{dist}_E(U(T+\tau,\tau)B\ ,\ P) \to 0 \qquad (T \to +\infty). \tag{7}$$

Here, for any $X \subseteq E$ and $Y \subseteq E$ we define

$$\operatorname{dist}_E(X,Y) = \sup_{x \in X} \operatorname{dist}_E(x,Y) = \sup_{x \in X}\ \inf_{y \in Y} \|x-y\|_E.$$

Definition 2.3. A process $\{U(t,\tau)\}$ possessing a compact uniformly attracting set is said to be *uniformly asymptotically compact.*

Theorem 2.2. *Let $\{U(t,\tau)\}$ be a uniformly asymptotically compact process acting in a space E, with a compact, uniformly attracting set $P \Subset E$. Each mapping $U(t,\tau) : E \to E$ is assumed continuous. Then the kernel K of the process $\{U(t,\tau)\}$ is non-empty, the kernel sections $K(s)$ are all compact, and*

$$K(s) \subseteq P \qquad \forall s \in \mathbf{R}. \tag{8}$$

Proof. Consider the set:

$$\mathcal{A}_T(\tau) = \overline{\bigcup_{s \geq T} U(\tau,\tau - s)P}, \qquad T \geq 0. \tag{9}$$

(The bar indicates closure in E.) Then we clearly have:

$$\mathcal{A}_{T_1}(\tau) \supseteq \mathcal{A}_{T_2}(\tau) \qquad \forall T_1 \leq T_2. \tag{10}$$

The uniform attracting property (7) implies that

$$\operatorname{dist}_E(\mathcal{A}_T(\tau), P) \to 0 \qquad (T \to +\infty). \tag{11}$$

Set

$$\mathcal{A}_\infty(\tau) = \bigcap_{T \geq 0} \mathcal{A}_T(\tau).$$

Since P is a compact set, then by (10) and (11) the set $\mathcal{A}_\infty(\tau)$ is also compact and non-empty, and $\mathcal{A}_\infty(\tau) \subseteq P$. Let us show that

$$\mathcal{A}_\infty(\tau) = K(\tau) \qquad \forall \tau \in \mathbf{R}, \tag{12}$$

where $K(\tau)$ is the section of the kernel K of the process $\{U(t,\tau)\}$ at time τ. Pick any bounded complete trajectory $u(s)$ of the process $\{U(t,\tau)\}$. Then, according to (7), $u(\tau) \in P$ $\forall \tau \in \mathbf{R}$. Indeed, $u(\tau) = U(\tau,\tau - s)u(\tau - s)$. The set $B = \{u(\tau - s),\ s \geq 0\}$ is bounded in E. Property (7) implies that $\operatorname{dist}_E(u(\tau),P) \leq \operatorname{dist}_E(U(\tau,\tau - s)B\ ,\ P) \to 0$ $(s \to +\infty)$, i.e., $\operatorname{dist}_E(u(\tau),P) = 0$ and $u(\tau) \in P$ $\forall \tau \in \mathbf{R}$. On the other hand, from the inclusion $u(\tau - T) \in P$ it follows that $u(\tau) = U(\tau,\tau - T)u(\tau - T) \in U(\tau,\tau - T)P \subseteq \mathcal{A}_T(\tau)$ $\forall T \geq 0$. Therefore, $u(\tau) \in \mathcal{A}_T(\tau)$ $\forall T \geq 0$. Hence $u(\tau) \in \mathcal{A}_\infty(\tau)$. Thus we have established that

$$K(\tau) \subseteq \mathcal{A}_\infty(\tau). \tag{13}$$

To prove the inverse inclusion, we need the following identity:

$$U(t,\tau)\mathcal{A}_\infty(\tau) = \mathcal{A}_\infty(t) \qquad \forall t \geq \tau,\ \tau \in \mathbf{R}. \tag{14}$$

By the definition of the set $\mathcal{A}_\infty(\tau)$,

$$u_\tau \in \mathcal{A}_\infty(\tau) \iff \exists\ s_n \to +\infty,\ \{x_n\} \subseteq P : U(\tau,\tau - s_n)x_n \to u_\tau \qquad (n \to +\infty). \tag{15}$$

Using the continuity of the mapping $U(t,\tau)$, we have

$$U(t,\tau - s_n)x_n = U(t,\tau)U(\tau,\tau - s_n)x_n \to U(t,\tau)u_\tau \qquad (n \to +\infty).$$

Therefore, by (15), $U(t,\tau)u_\tau \in \mathcal{A}_\infty(t)$, i.e., $U(t,\tau)\mathcal{A}_\infty(\tau) \subseteq \mathcal{A}_\infty(t)$. Let us verify the inverse inclusion. Let u_t be any element of $\mathcal{A}_\infty(t)$. Then, according to (15), there exist two sequences $s_n \to +\infty$ and $\{x_n\} \subseteq P$ such that $U(t,t-s_n)x_n \to u_t$, $n \to +\infty$. Note that $U(t,t-s_n)x_n = U(t,\tau)U(\tau,\tau-(\tau-t+s_n))x_n$. The set $\{x_n\}$ is bounded, thus $\{y_n = U(\tau,\tau - s'_n)x_n\}$, $s'_n = \tau - t + s_n$, is attracted to P when $n \to +\infty$. Therefore, for some sequence $\{\bar{y}_n\} \subset P$, we have $\|y_n - \bar{y}_n\|_E \to 0$, $n \to +\infty$. On the other hand, the set P is compact, and, by refining $\{\bar{y}_n\}$, we may assume that $\bar{y}_n \to u_\tau$, $n \to +\infty$ for some element $u_\tau \in P$. Hence, $y_n = U(\tau,\tau - s'_n)x_n \to u_\tau$, $n \to +\infty$. Owing to (15), we get $u_\tau \in \mathcal{A}_\infty(\tau)$. Finally, using the continuity of the mapping $U(t,\tau)$, we deduce:

$$u_t = \lim_{n\to\infty} U(t,\tau)U(\tau,\tau - s'_n)x_n = U(t,\tau)\lim_{n\to\infty} y_n = U(t,\tau)u_\tau,$$

$$u_\tau \in \mathcal{A}_\infty(\tau).$$

Hence, $u_t \in U(t,\tau)\mathcal{A}_\infty(\tau)$ and $U(t,\tau)\mathcal{A}_\infty(\tau) \supseteq \mathcal{A}_\infty(t)$, $t \geq \tau$. So (14) is proved.

Using (14), let us show that $\mathcal{A}_\infty(\tau) \subseteq K(\tau)$, $\forall \tau \in \mathbf{R}$. Indeed, let u_τ be any element of $\mathcal{A}_\infty(\tau)$. We shall construct a bounded complete trajectory $u(s)$, $s \in \mathbf{R}$, of the process $\{U(t,\tau)\}$ such that $u|_{s=\tau} = u_\tau$. We put $u(s) = U(s,\tau)u_\tau$, where $s \geq \tau$. Given (14), $u(s) \in \mathcal{A}_\infty(s) \subseteq P$ for any $s \geq \tau$. Let us extend $u(s)$ to $s \leq \tau$. The identity (14) implies that there exists $u_{\tau-1} \in \mathcal{A}_\infty(\tau-1)$ such that $U(\tau,\tau-1)u_{\tau-1} = u_\tau$. If we now put $u(s) = U(s,\tau-1)u_{\tau-1}$ for $s \in [\tau-1,\tau]$, we shall get $u(s) \in \mathcal{A}_\infty(s) \subseteq P$ for $s \geq \tau - 1$. Applying this procedure several times, one can construct $u(s) \in \mathcal{A}_\infty(s) \subseteq P$ for $s \geq \tau - n$, $n \in \mathbf{N}$. Letting $n \to +\infty$, we get a bounded complete trajectory $u(s)$ of the process $\{U(t,\tau)\}$, $s \in \mathbf{R}$, such that $u(s) \in \mathcal{A}_\infty(s) \subset P$ $\forall s \in \mathbf{R}$ and $u(\tau) = u_\tau$. Therefore, $u_\tau = u(\tau) \in K(\tau)$ and $\mathcal{A}_\infty(\tau) \subseteq K(\tau)$. Taking into account (13), we obtain the identity (12). The proof is complete. □

Corollary 2.1. *If a process $\{U(t,\tau)\}$ satisfies the conditions of Theorem 2.2, then the kernel section $K(\tau)$ is given by*

$$K(\tau) = \bigcap_{T\geq 0} \overline{\bigcup_{s\geq T} U(\tau,\tau - s)P}. \tag{16}$$

Theorem 2.3. *Let the conditions of Theorem 2.2 be satisfied. Then for any bounded set $B \subset E$ and for any fixed $\tau \in \mathbf{R}$*

$$\operatorname{dist}_E(U(\tau,\tau - T)B\ ,\ K(\tau)) \to 0 \qquad (T \to +\infty). \tag{17}$$

Proof. Assume to the contrary that for some bounded set B and for some $\tau \in \mathbf{R}$, (17) is not true. Therefore, there exist two sequences $\{T_n\}$, $T_n \to +\infty$ and $\{x_n\} \subseteq B$, such that

$$\operatorname{dist}_E(U(\tau,\tau - T_n)x_n\ ,\ K(\tau)) > \delta > 0 \qquad \forall n \in \mathbf{R}. \tag{18}$$

Identity (16) implies that, for some $T^1 = T^1(\delta)$,

$$\operatorname{dist}_E\Big(\overline{\bigcup_{s\geq T^1} U(\tau,\tau-s)P}\ ,\ K(\tau)\Big) < \delta/4. \tag{19}$$

If n is large enough, then

$$U(\tau,\tau-T_n)x_n = U(\tau,\tau-T^1)U(\tau-T^1,\tau-T_n)x_n.$$

The mapping $U(\tau,\tau-T^1)$ is continuous, so that it is uniformly continuous on a compact set. Therefore, for any $\varepsilon > 0$, there exist $\delta_1 > 0$ such that $\operatorname{dist}_E(y,P) \leq \delta_1$ implies that $\operatorname{dist}_E(U(\tau,\tau-T^1)y\ ,\ U(\tau,\tau-T^1)P) \leq \varepsilon$. Put $\varepsilon = \delta/4$ and find the corresponding $\delta_1 = \delta_1(\delta)$. The set $\{x_n\}$ is bounded. It follows from (7) that $\operatorname{dist}_E(U(\tau-T^1,\tau-T_n)x_n\ ,\ P) \leq \delta_1$ for large n. Hence,

$$\operatorname{dist}_E(U(\tau,\tau-T^1)U(\tau-T^1,\tau-T_n)x_n, U(\tau,\tau-T^1)P) < \delta/4. \tag{20}$$

From (19) and (20) it follows that

$$\begin{aligned}
&\operatorname{dist}_E\big(U(\tau,\tau-T_n)x_n\ ,\ K(\tau)\big)\\
&= \operatorname{dist}_E\big(U(\tau,\tau-T^1)U(\tau-T^1,\tau-T_n)x_n\ ,\ K(\tau)\big)\\
&\leq \operatorname{dist}_E\big(U(\tau,\tau-T^1)U(\tau-T^1,\tau-T_n)x_n\ ,\ U(\tau,\tau-T^1)P\big)\\
&\qquad\qquad + \operatorname{dist}_E\big(U(\tau,\tau-T^1)P\ ,\ K(\tau)\big)\\
&\leq \delta/4+\delta/4 = \delta/2.
\end{aligned}$$

This contradicts (18), which completes the proof. □

Remark 2.2. Property (6) of the kernel sections $K(\tau)$ is similar to the invariant property of the attractor of a semigroup. Property (17) is analogous to the attracting property of a semigroup's attractor. □

Remark 2.3. Let the process $\{U(t,\tau)\}$ satisfy the backward uniqueness property, i.e., $U(t,\tau)u_\tau = U(t,\tau)u'_\tau$ implies that $u_\tau = u'_\tau$. Then, under the conditions of Theorem 2.2, the sections $K(\tau)$ are homeomorphic (in E) to each other. The corresponding homeomorphism is given by $U(t,\tau)$, $U(t,\tau)K(\tau) = K(t)$, $K(t) = (U(t,\tau))^{-1}K(\tau)$, $t \geq \tau$.

3. On the finite dimensionality of sets connected by volume contraction mappings. In this section we study the Hausdorff dimension of sets X_i ($i \in \mathbf{Z}$) connected by a sequence $\{S_i\}_{i\in\mathbf{Z}}$ of volume contraction mappings. Theorem 3.1 generalizes the fundamental result of [13] and [8] concerning the estimate of the Hausdorff dimension of a compact set X that has an invariant volume contraction mapping S, $S : X \to X$, $S(X) = X$.

Let H be a Hilbert space and $Y \Subset H$ a compact subset. Given $d \in \mathbf{R}_+$ and $\varepsilon > 0$, we denote by $\mu(Y,d,\varepsilon)$ the infimum $\sum r_i^d$, where the infimum is taken over all the possible coverings of Y by balls $B_{r_i}(x_i)$ of radii $r_i \leq \varepsilon$ and with centers $x_i \in Y$. Let $\mu(Y,d)$ denote the *d-dimensional Hausdorff measure* $\mu(Y,d) = \lim_{\varepsilon\to 0+} \mu(Y,d,\varepsilon)$, and let dim (Y) denote the *Hausdorff dimension* of Y in the space E : dim $(Y) = \inf \{d : \mu(Y,d) = 0\}$ (see [18] and [1] for a detailed description).

Let $X \Subset H$ be a compact set. Consider a sequence of sets $\{X_i\}_{i\in\mathbf{Z}}$, $X_i \subseteq X$, and a sequence of mappings $\{S_i\}_{i\in\mathbf{Z}}$ such that $S_i : X_i \to X_{i+1}$, $i \in \mathbf{Z}$. We assume that each S_i is surjective, i.e.,

$$S_i(X_i) \twoheadrightarrow X_{i+1} \qquad \forall i \in \mathbf{Z}. \tag{21}$$

The sequence $\{S_i\}_{i\in\mathbf{Z}}$ is said to be *uniformly quasidifferentiable* on $\{X_i\}_{i\in\mathbf{Z}}$ if, for any S_i and any $u \in X_i$, there exists a bounded linear operator $L_i(u) \in \mathcal{L}(H,H)$ (the quasidifferential) such that

$$\left(\sup_{i\in\mathbf{Z}} \sup_{\substack{u,\, v\in X_i \\ 0<\|v-u\|_H<\varepsilon}} \frac{\|S_i(v) - S_i(u) - L_i(u)(v-u)\|_H}{\|v-u\|_H} \right) \to 0 \tag{22}$$

($\varepsilon \to 0+$). Let L be a bounded linear operator, $L \in \mathcal{L}(H,H)$. The contraction coefficient of d-dimensional volumes under the action of L is defined by

$$\omega_d(L) = \sup_{\xi_1,\dots,\xi_d;\ \|\xi_i\|\leq 1} \|L\xi_1 \wedge L\xi_2 \wedge \cdots \wedge L\xi_d\|, \tag{23}$$

where $\|\zeta_1 \wedge \zeta_2 \wedge \cdots \wedge \zeta_d\| = \left(\det\{(\zeta_i,\zeta_j)_{i=1,\dots d}^{j=1,\dots d}\}\right)^{1/2}$ denotes the volume of the parallelepiped spanned by the vectors $\zeta_1, \zeta_2, \dots, \zeta_d \in H$. One says that an operator L contracts d-dimensional volumes if $\omega_d(L) < 1$. Let us now formulate the Main Theorem.

Theorem 3.1. *Suppose X is compact in H, the sequence of mappings $\{S_i\}_{i\in\mathbf{Z}}$, $S_i : X_i \to X_{i+1}$, $X_i \subseteq X$, satisfies* (21) *and is uniformly quasidifferentiable on $\{X_i\}_{i\in\mathbf{Z}}$, with quasidifferentials $L_i(u) \in \mathcal{L}(H,H)$, $i \in Z$, $u \in X_i$. We also assume that*

$$\sup_{i\in\mathbf{Z}} \sup_{u\in X_i} \|L_i(u)\|_{\mathcal{L}(H,H)} = m < \infty, \tag{24}$$

$$\bar{\omega}_d = \sup_{i\in\mathbf{Z}} \sup_{u\in X_i} \omega_d(L_i(u)) = k < 1. \tag{25}$$

Then the Hausdorff dimension of X_i is finite and the following holds:

$$\dim(X_i) \leq d \qquad \forall i \in \mathbf{Z}. \tag{26}$$

The proof of Theorem 3.1 follows the main ideas of [13], [8], and [18]. In order to estimate the approximate d-dimensional measure $\mu(LB_1(0),d,\varepsilon)$ of the image by L of the unit ball $B_1(0)$, we rely on the following lemma:

Lemma 3.2. *Let k and m be positive numbers such that $k \leq m^d$. Assume $L \in \mathcal{L}(H,H)$ is a linear operator such that $\omega_d(L) \leq k$ and $\|L\|_{\mathcal{L}(H,H)} \leq m$. Then*

$$\mu(LB_1(0)\ ,\, d\ ,\, \sqrt{d}k^{1/d}) \leq \beta_d k$$

and

$$\mu(LB_1(0) + B_\eta(0)\ ,\, d\ ,\, (1+\eta R)\sqrt{d}k^{1/d}) \leq \beta_d(1+\eta R)^d k, \tag{27}$$

where $\beta_d = 2^{d-1}(d)^{d/2}$, $R = m^{d-1}/k$, and η is any positive number.

Lemma 3.2 follows directly from [18]. □

Proof of Theorem 3.1. It is enough to show (26) for $i = 0$. Note that we may assume that the number k in (25) is arbitrarily small. Indeed, for any integer p, the sequence of mappings $\{S_i^p\}_{i\in\mathbf{Z}}$ is well defined on the sequence of sets $\{X_{ip}\}$. Here

$$S_i^p = S_{ip+p-1} \cdot S_{ip+p-2}\ \cdot\ \cdots\ \cdot\ S_{ip+1} \cdot S_{ip} : X_{ip} \to X_{(i+1)p}.$$

Obviously the sequence $\{S_i^p\}_{i\in\mathbf{Z}}$ satisfies (21): $S_i^p(X_{ip}) = X_{(i+1)p}$, $\forall i \in \mathbf{Z}$. It is easy to check that (22) is valid with $\{L_i(u)\}$ replaced by $\{L_{i,p}(u)\}$, where

$$L_{i,p}(u) = L_{ip+p-1}(u_{p-1}) \cdot L_{ip+p-2}(u_{p-2})\ \cdot\ \cdots\ \cdot\ L_{ip+1}(u_1) \cdot L_{ip}(u),$$

and $u_1 = S_{ip}(u)$, $u_2 = S_{ip+1}(u_1)$,..., $u_{p-1} = S_{ip+p-1}(u_{p-2})$. Due to the multiplicative property of the contraction coefficient of d-dimensional volumes for products of linear operators, (24) and (25) imply

$$\omega_d(L_{i,p}(u)) \leq k^p\ ,\ \|L_{i,p}(u)\|_{\mathcal{L}(H,H)} = m^p$$

for any $u \in X_{ip}$, $i \in Z$. Thus, the value $\bar{\omega}_d$ corresponding to p can be made arbitrarily small by taking p sufficiently large.

Note that m and k from (24) and (25) satisfy the inequality $k \leq m^d$. Now, without any loss of generality we may assume that, in addition to (25), the following are valid:

$$2\sqrt{d}k^{1/d} \leq 1/2 \text{ and } \beta_d \cdot 2^d \cdot k \leq 1/2.$$

We put $\eta = R^{-1} = \left(m^{d-1}/k\right)^{-1}$. Lemma 3.2 and inequality (27) imply that

$$\begin{aligned} \mu(LB_1(0) + B_\eta(0), d, 1/2) &\leq \mu(LB_1(0) + B_\eta(0)\ ,\, d\ ,\, 2\sqrt{d}k^{1/d}) \\ &\leq \beta_d \cdot 2^d \cdot k \leq 1/2 \end{aligned} \tag{28}$$

if $\omega_d(L) \le k$ and $\|L\|_{\mathcal{L}(H,H)} \le m$. L is linear since, from (28), for any $r > 0$,

$$\mu(LB_r(0) + rB_\eta(0), d, r/2) \le r^d/2. \tag{29}$$

Given (22), one can find $\varepsilon_0 = \varepsilon_0(\eta) > 0$ such that for any $\varepsilon \le \varepsilon_0$ the supremum in (22) is less than or equal to η , i.e.,

$$\|S_i(v) - S_i(u) - L_i(u)(v-u)\|_H \le \eta \|v-u\|_H \tag{30}$$

for any $i \in \mathbf{Z}$, u, $v \in X_i$, $\|v-u\|_H \le \varepsilon$.

Let us consider an arbitrary covering of the set X_i by a finite number of balls $B_{r_j}(u_j)$ where $u_j \in X_i$, $r_j \le \varepsilon$, $j = 1, \dots, N$:

$$X_i \subseteq \bigcup_{j=1}^{N} (B_{r_j}(u_j) \cap X_i)).$$

By (21) we have

$$X_{i+1} \subseteq \bigcup_{j=1}^{N} S_i(B_{r_j}(u_j) \cap X_i). \tag{31}$$

Taking into account (30), we obtain

$$S_i\left(B_{r_j}(u_j) \cap X_i\right) \subseteq S_i(u_j) + L_i(u_j)B_{r_j}(0) + r_jB_\eta(0).$$

Hence, by (31),

$$\begin{aligned} \mu(X_{i+1}, d, \varepsilon/2) &\le \sum_{j=1}^{N} \mu(L_i(u_j)B_{r_j}(0) + r_jB_\eta(0), d, \varepsilon/2) \\ &\le \sum_{j=1}^{N} \mu(L_i(u_j)B_{r_j}(0) + r_jB_\eta(0), d, r_j/2) \\ &\le 1/2 \sum_{j=1}^{N} r_j^d. \end{aligned} \tag{32}$$

Here we have used inequality (29) with $L_i(u_j)$ instead of L. According to (24) and (25) the operators $L_i(u_j)$ satisfy the conditions of Lemma 3.2. By taking in (32) the infimum for all the coverings of X_i by balls $B_{r_j}(u_j)$ with radii $r_j \le \varepsilon$, we find

$$\mu(X_{i+1}, d, \varepsilon/2) \le 1/2 \cdot \mu(X_i, d, \varepsilon), \qquad \forall i \in \mathbf{Z}. \tag{33}$$

Iterating (33) ℓ times gives

$$\mu(X_{i+\ell}, d, \varepsilon/(2^\ell)) \le (1/2)^\ell \cdot \mu(X_i, d, \varepsilon), \qquad \forall \ell \in \mathbf{N}. \tag{34}$$

By the assumptions of Theorem 3.1, $X_i \subseteq X$, thus, $\mu(X_i, d, \varepsilon) \le \mu(X, d, \varepsilon)$, and we have

$$\mu(X_{i+\ell}, d, \varepsilon/(2^\ell)) \le (1/2)^\ell \cdot \mu(X, d, \varepsilon), \qquad \forall \ell \in \mathbf{N},\ i \in \mathbf{Z}.$$

Choose $i = -\ell$. Thus,

$$\mu(X_0, d, \varepsilon/(2^\ell)) \le (1/2)^\ell \cdot \mu(X, d, \varepsilon), \qquad \forall \ell \in \mathbf{N}. \tag{35}$$

Finally, note that the set X is compact; therefore, $\mu(X,d,\varepsilon) < \infty$ $\forall \varepsilon > 0$. Letting $\ell \to +\infty$ in (35), we obtain $\mu(X_0,d) = 0$, i.e., $\dim(X_0) \le d$. The proof is complete. □

4. Dimension estimates for kernel sections of evolution equations. In this section we apply Theorem 3.1 to estimate the Hausdorff dimension of the kernel sections $K(s)$ of a process $\{U(t,\tau)\}$ generated by problem (4), where $E = H$ is a Hilbert space. Suppose that the process $\{U(t,\tau)\}$ is uniformly quasidifferentiable on $\{K(\tau)\}_{\tau\in\mathbf{R}}$, i.e., there exists a family of bounded linear operators (quasidifferentials) $\{U'(t,\tau;u) : u \in K(\tau),\ t \ge \tau,\ \tau \in \mathbf{R}\}$, $U'(t,\tau;u) : H \to H$, such that

$$\|U(t,\tau)v - U(t,\tau)u - U'(t,\tau;u)(v-u)\|_H \le \gamma(t-\tau, \|v-u\|_H)\cdot\|v-u\|_H, \tag{36}$$

where u, $v \in K(\tau)$, $\gamma(s,\xi) \to 0$ $(\xi \to 0+)$ for all $s \ge 0$. The function $\gamma(s,\xi)$ does not depend on u, $v \in K(\tau)$ and $\tau \in \mathbf{R}$. We shall assume that the quasidifferential $\{U'(t,\tau;u)\}$ is generated by the variation equation corresponding to (4):

$$\partial_t v = A_u(u(t),t)v,\ v|_{t=\tau} = v_\tau,\ v_\tau \in E,\ t \ge \tau,\ \tau \in \mathbf{R}, \tag{37}$$

i.e., $U'(t,\tau;u_\tau)v_\tau = v(t)$, where $v(t)$ is the solution of problem (37), and $u(t) = U(t,\tau)u_\tau$ is the solution of problem (4) with initial condition $u_\tau \in K(\tau)$.

Let $d \in \mathbf{N}$ and $L : H_2 \to H$. The d-dimensional trace of the operator L (see [8]) is defined by the formula $\mathrm{Tr}_d L = \sup_Q \mathrm{Tr}\, LQ$, where the supremum is taken over all the orthogonal projectors Q in H on the space QH of dimension d belonging to the domain H_2 of the operator L. We recall that $\mathrm{Tr}\, LQ = \sum_{j=1}^d (L\varphi_j, \varphi_j)$, where $\varphi_1, \ldots, \varphi_d$ is an orthonormal system in QH. Let us introduce

$$\tilde{q}_d = \liminf_{T\to+\infty} \sup_{\tau\in\mathbf{R}} \sup_{u_\tau\in K(\tau)} \left(\frac{1}{T} \int_\tau^{\tau+T} \mathrm{Tr}_d(A_u(u(s),s))\, ds \right),$$

where $u(t) = U(t,\tau)u_\tau$.

Theorem 4.1. *Let the process $\{U(t,\tau),\ t \ge \tau,\ \tau \in \mathbf{R}\}$ generated by the problem (4) be uniformly quasidifferentiable on $\{K(\tau)\}_{\tau\in\mathbf{R}}$. Assume the set $\bigcup_{\tau\in\mathbf{R}} K(\tau)$ is precompact in H. Assume also*

$$\sup_{u_\tau \in K(\tau)} \|U'(t,\tau;u_\tau)\|_{\mathcal{L}(H,H)} \le C(t-\tau), \qquad t > \tau, \tag{38}$$

and

$$\tilde{q}_d < 0, \tag{39}$$

where $U'(t,\tau;u_\tau)v_\tau$, $u_\tau \in K(\tau)$, $v_\tau \in E$, satisfy the variation equation (37). *Then*

$$\dim(K(\tau)) \leq d \qquad \forall \tau \in \mathbf{R}.$$

Proof. Fix $\tau \in \mathbf{R}$, $u_\tau \in K(\tau)$ and find the contraction coefficient $\omega_d(U'(t,\tau;u_\tau))$. According to (23) we chose an arbitrary set of vectors $\xi_1,\ldots,\xi_d \in H$ and find $\|v_1(t)\wedge v_2(t)\wedge\cdots\wedge v_d(t)\|$, where $v_j(t) = U'(t,\tau;u_\tau)\xi_j$ is the solution of problem (37) with the initial condition $v_j|_{t=\tau} = \xi_j$. Thanks to the identity proved in [8] and [18] we get

$$(40)\quad \|v_1(t)\wedge\cdots\wedge v_d(t)\| = \|\xi_1\wedge\cdots\wedge\xi_d\|\cdot\exp\left(\int_\tau^t \operatorname{Tr}(A_u(u(s),s)\cdot Q_d(s))\,ds\right),$$

where $u(s) = U(s,\tau)u_\tau$ and $Q_d(s) = Q_d(s,\tau,u_\tau;\xi_1,\xi_2,\ldots,\xi_d)$ is the projector onto the space spanned by $v_1(s),v_2(s),\ldots,v_d(s)$.

By the definition of the d-trace we have

$$(41)\qquad \begin{aligned}\omega_d(U'(t,\tau;u_\tau)) &= \sup_{\xi_1,\ldots,\xi_d;\ \|\xi_i\|\leq 1}\|v_1(t)\wedge\cdots\wedge v_d(t)\|\\ &\leq \exp\left(\int_\tau^t \operatorname{Tr}_d(A_u(u(s),s))\,ds\right).\end{aligned}$$

Let us introduce the contraction coefficient of d-dimensional volumes under the action of the mapping $U(t,\tau)$:

$$\bar{\omega}_d(t,\tau) = \sup_{u_\tau\in K(\tau)}\omega_d(U'(t,\tau;u_\tau)).$$

By (41)

$$(42)\qquad \bar{\omega}_d(t,\tau) \leq \exp(q_d(t,\tau)(t-\tau)),$$

where

$$q_d(t,\tau) = \sup_{u_\tau\in K(\tau)}\frac{1}{t-\tau}\left(\int_\tau^t \operatorname{Tr}_d(A_u(u(s),s))\,ds\right).$$

Now consider the sequence of sets $\{X_i\}_{i\in\mathbf{Z}}$, $X_i = K(\tau+iT)$, and the sequence of mappings $\{S_i\}_{i\in\mathbf{Z}}$, $S_i = U((i+1)T+\tau, iT+\tau)$, where $T>0$. Proposition 2.1 implies that $S_i(X_i) = X_{i+1}$ $\forall i\in\mathbf{Z}$. It is clear by (36) that the sequence of mappings $\{S_i\}_{i\in\mathbf{Z}}$ is uniformly quasidifferentiable on $\{X_i\}_{i\in\mathbf{Z}}$. By (38) we also have that

$$\begin{aligned}\sup_{i\in\mathbf{Z}}\sup_{u\in X_i}\|L_i(u)\|_{\mathcal{L}(H,H)} &\leq \sup_{\tau\in\mathbf{R}}\sup_{u\in K(\tau)}\|U'(T+\tau,\tau;u)\|_{\mathcal{L}(H,H)}\\ &\leq C(T),\end{aligned}$$

where $L_i(u) = U'((i+1)T+\tau, iT+\tau; u)$ is the quasidifferential of S_i at $u \in X_i$. Finally, according to condition (39), there exists a positive number $\delta > 0$ such that, for some $T > 0$,

$$q_d(T+\tau,\tau) = \sup_{u_\tau \in K(\tau)} \frac{1}{T}\left(\int_\tau^{\tau+T} \mathrm{Tr}_d\left(A_u(u(s),s)\right) ds\right) < -\delta < 0$$

for all $\tau \in \mathbf{R}$. Thus, by (42),

$$\begin{aligned}
\bar{\omega}_d &= \sup_{i\in\mathbf{Z}} \sup_{u\in X_i} \omega_d(L_i(u)) \\
&\le \sup_{\tau\in\mathbf{R}} \sup_{u_\tau\in K(\tau)} \omega_d(U'(\tau+T,\tau;u_\tau)) \\
&= \sup_{\tau\in\mathbf{R}} \bar{\omega}_d(\tau+T,\tau) \le \sup_{\tau\in\mathbf{R}} \exp\left(q_d(\tau+T,\tau)T\right) \\
&\le \exp(-\delta T) < 1.
\end{aligned}$$

By the assumptions above, the set $X = \overline{\bigcup_{\tau\in\mathbf{R}} K(\tau)}$ is compact in H and $X_i \subseteq X$ $\forall i \in \mathbf{Z}$. All the conditions of Theorem 3.1 are verified, which implies that the Hausdorff dimension $\dim X_0 = \dim K(\tau) \le d$ for any $\tau \in \mathbf{R}$. Theorem 4.1 is thus completely proved. □

5. The estimates of dimension of kernel sections for equations of mathematical physics.

1. NAVIER-STOKES SYSTEM WITH TIME DEPENDENT EXTERNAL FORCE. Excluding the pressure p, the Navier-Stokes system can be rewritten in the form:

$$\partial_t u + Lu + B(u,u) = \varphi(x,t),\ x = (x_1,x_2) \in \Omega \Subset \mathbf{R}^2, \tag{43}$$

$$L = -\nu\Pi\Delta,\ B(u,u) = \Pi\sum_{i=1}^{2} u^i\partial_i u,\ \varphi = \Pi\varphi_0,\ u|_{\partial\Omega} = 0, \tag{44}$$

where $u = (u^1,u^2)$, $\varphi = (\varphi^1,\varphi^2)$ (see [15], [16], [19], [20]). By H (H_1) we denote, as usual, the closure of the set $V_0 = \{v : v \in (C_0^\infty(\Omega))^2,\ (\nabla,v) = 0\}$ in the norm $\|\ \|$ ($\|\ \|_1$) of the space $(L_2(\Omega))^2$ $((H_1(\Omega))^2)$. Π stands for the orthogonal projector on H in $(L_2(\Omega))^2$. We assume that $\varphi(\cdot,t) \in C_b(\mathbf{R},H)$. The initial conditions are posed at $t=\tau$:

$$u\big|_{t=\tau} = u_\tau, \qquad u_\tau \in H. \tag{45}$$

Just like in the autonomous case (when $\varphi \equiv \varphi(x)$), one can prove that the problem (43)–(45) has a unique solution $u(t)$:

$$u(t) \in C([\tau,+\infty),H) \cap L_2((\tau,\tau+T),H_1), \quad \partial_t u \in L_2((\tau,\tau+T),H_{-1}) \tag{46}$$

$\forall T > 0$. Here $H_{-1} = (H_1)^*$ is the dual space. The solution satisfies a priori estimates similar to those in the autonomous case. Thus the process $\{U(t,\tau), t \geq \tau\} : U(t,\tau)u_\tau = u(t)$, acting on H and corresponding to (43), (45) is defined by operators $U(t,\tau) : H \to H$. The operators $U(t,\tau) : H \to H$ are continuous in H for any $t \geq \tau$, $\tau \in \mathbf{R}$. The process $\{U(t,\tau)\}$ is uniformly compact (see [3]), i.e., there exists a set P that is compact and uniformly absorbing (with respect to τ). Therefore, we can apply Theorems 2.2 and 2.3 to the process $\{U(t,\tau)\}$. Let K be the kernel of the process $\{U(t,\tau)\}$ and $K(t_0)$ be the section of K at time $t = t_0$.

Theorem 5.1. *For any* $t_0 \in \mathbf{R}$,

$$\dim K(t_0) \leq \left\lceil \frac{C}{\nu^2}(M_{-1}(|\varphi|^2)^{1/2} \right\rceil, \tag{47}$$

where

$$M_{-1}(|\varphi|^2) = \liminf_{T\to\infty} \sup_{\tau\in\mathbf{R}} \frac{1}{T}\int_\tau^{\tau+T} |\varphi(s)|_{-1}^2\,ds,$$

a constant C *does not depend on* ν *and* t_0. *Here* $\lceil a \rceil$ *means the smallest integer greater than* a; $|\varphi(s)|_{-1} = \|\varphi(\cdot,s)\|_{-1}$.

Proof. Using standard methods (see [1, 2] and [18]), one can show that the process $\{U(t,\tau)\}$ is uniformly quasidifferentiable on $\{K(\tau)\}_{\tau\in\mathbf{R}}$, and the corresponding variation equation has the form

$$\partial_t v = -Lv - B(u(t),v) - B(v,u(t)) \equiv A_u(u(t),t)v, \quad v\Big|_{t=\tau} = v_\tau.$$

The estimates leading to (47) are analogous to those given in [18] for the autonomous case. First of all one has to establish the following inequality:

$$\mathrm{Tr}\,(A_u(u(t),t)Q_d) \leq -\frac{\nu c_0 d^2}{2|\Omega|} + \frac{1}{2\nu c_0}\|u(t)\|_1^2, \tag{48}$$

where the dimension of the image of the projector Q_d equals to d. Then, using the a priori estimate

$$\int_\tau^t \|u(s)\|_1^2 ds \leq \frac{\|u_\tau\|^2}{\nu} + \frac{1}{\nu^2}\int_\tau^t |\varphi(s)|_{-1}^2\,ds, \tag{49}$$

one gets

$$\begin{aligned} q_d(\tau+T,\tau) &= \sup_{u_\tau\in K(\tau)} \frac{1}{T}\int_\tau^{\tau+T} \mathrm{Tr}_d\,(A_u(u(s),s))\,ds \\ &\leq -\frac{\nu c_0 d^2}{2|\Omega|} + \frac{1}{2\nu^2 T c_0}\sup_{u_\tau\in K(\tau)} \|u_\tau\|^2 + \frac{1}{2\nu^3}\frac{1}{Tc_0}\int_\tau^{\tau+T} |\varphi(s)|_{-1}^2\,ds. \end{aligned}$$

Note that $\sup_{u_\tau \in K(\tau)} \|u_\tau\|^2 \le C_1$. Therefore,

$$\begin{aligned}\tilde{q_d} &= \liminf_{T\to+\infty} \sup_{\tau\in\mathbf{R}} q_d(\tau+T,\tau)\\ &\le -\frac{\nu c_0 d^2}{2|\Omega|} + \frac{1}{2\nu^3 c_0}\left(\liminf_{T\to+\infty} \sup_{\tau\in\mathbf{R}} \frac{1}{T}\int_\tau^{\tau+T} |\varphi(s)|_{-1}^2\, ds\right)\\ &= -\frac{\nu c_0 d^2}{2|\Omega|} + \frac{1}{2\nu^3 c_0} M_{-1}(|\varphi|^2).\end{aligned}$$

Thus (47) follows immediately from Theorem 4.1. □

2. REACTION-DIFFUSION SYSTEM DEPENDING ON TIME. The following system is considered:

$$\partial_t u = \nu a \Delta u - f(u,t) + \varphi(x,t), \quad u|_{\partial\Omega} = 0 \text{ (or } \partial u/\partial\nu\big|_{\partial\Omega} = 0), \tag{50}$$

where $x \in \Omega \Subset \mathbf{R}^n$, $a = \{a_i^j\}_{i=1,\dots,N}^{j=1,\dots,N}$ is an $N \times N$-matrix with a positive symmetric part $a + a* \ge \beta^2 I$, $\beta^2 > 0$, $f = (f^1,\dots,f^N)$, $\varphi = (\varphi^1,\dots,\varphi^N)$, $u = (u^1,\dots,u^N)$. We assume that $\varphi(\cdot,t) \in C_b(\mathbf{R},H)$, $H = (L_2(\Omega))^N$. Also let f, $f'_{u^i} \in C(\mathbf{R}^N \times \mathbf{R}, \mathbf{R}^N)$, and let the following conditions hold for all $t \in \mathbf{R}$, u, $v \in \mathbf{R}^N$:

$$\gamma_2|u|^p - C_2 \le f\cdot u \le \gamma_1 |u|^p + C_1, \quad \gamma_i > 0,\ p \ge 2, \tag{51}$$

$$f_u v\cdot v \ge -C_3 v\cdot v, \quad |f_u| \le C_4(|u|^{p-2}+1). \tag{52}$$

We also assume that

$$|f(u+z,t) - f(u,t) - f_u(u,t)z| \le C_5(1+|u|^{p_1}+|z|^{p_1})|z|^{1+\gamma}, \tag{53}$$

where $p_1 < 4/(n-2)$, and γ is positive and sufficiently small. We supply the system (50) with the initial conditions

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in H = (L_2(\Omega))^N. \tag{54}$$

Problem (50)–(54) has (for all $u_\tau \in H$) a unique solution $u(t) \in C_b([\tau,+\infty),H) \cap L_2((\tau,\tau+T), (H_0^1(\Omega))^N)$ $\forall T \in \mathbf{R}$. (See [1, 2]). Thus, a process $\{U(t,\tau)\}$ acting on $(L_2(\Omega))^N$ corresponds to problem (50)–(54). It was shown in [3] that the process $\{U(t,\tau)\}$ possesses a uniformly absorbing set B_0, $B_0 \Subset H$, and the mappings $U(t,\tau) : H \to H$ are continuous for any t, τ, $t \ge \tau$, $\tau \in \mathbf{R}$. Thus, Theorems 2.2 and 2.3 are applicable to the process $\{U(t,\tau)\}$. The kernel K of $\{U(t,\tau)\}$ is non-empty, its sections $K(t)$ are compact in H, and $K(t) \subseteq B_0$ $\forall t \in \mathbf{R}$.

Theorem 5.2. *Under the assumptions on f and φ above, the Hausdorff dimension of a section $K(t_0)$ satisfies:*

$$\dim K(t_0) \leq \left\lceil \frac{C}{\nu^{n/2}} \right\rceil \qquad \forall t_0 \in \mathbf{R}. \tag{55}$$

Proof. The uniform quasidifferentiability result for the process $\{U(t,\tau)\}$ follows from the assumptions on the functions f and φ. The verification of this property is quite analogous to the corresponding proof for the autonomous case given in [2, 1]. Note that condition (53) is essential. In order to obtain (55), it is sufficient to estimate the trace of the operator $A_u(u(t),t) = \nu a\Delta - f'(u,t)$:

$$\begin{aligned} \mathrm{Tr}\,(A_u(u(t),t)Q_d) &\leq \sum_{j=1}^{d}(\nu a\Delta\varphi_j,\varphi_j) - \sum_{j=1}^{d}(f'_u(u(t),t)\varphi_j,\varphi_j) \\ &\leq -\nu\beta^2\sum_{j=1}^{d}\|\varphi_j\|_1^2 + C_3\sum_{j=1}^{d}\|\varphi_j\|^2. \end{aligned} \tag{56}$$

(We have used condition (52).) By means of the Courant variational principle (see [18]), we have

$$\sum_{j=1}^{d}\|\varphi_j\|_1^2 \geq \sum_{j=1}^{d}\lambda_j,$$

where $\lambda_1, \lambda_2, \ldots, \lambda_j, \ldots$ are the eigenvalues of the operator $-\Delta$ with boundary condition (50), in non-decreasing order. It is well known that $\lambda_j \geq C_6 j^{2/n}$, $C_6 > 0$; therefore,

$$\sum_{j=1}^{d}\|\varphi_j\|_1^2 \geq C_7 d^{1+2/n}, \qquad C_7 > 0. \tag{57}$$

Substituting (57) into (56), we obtain

$$\mathrm{Tr}\,(A_u(u(t),t)Q_d) \leq -\nu C_8 d^{1+2/n} + C_3 d. \tag{58}$$

Finally, we infer from (58) that

$$\tilde{q}_d \leq -\nu C_8 d^{1+2/n} + C_3 d.$$

Thus, $\tilde{q}_d < 0$ if $d \geq \lceil C/\nu^{n/2} \rceil$, where $C = (C_3/C_8)^{n/2}$. Theorem 4.1 implies that $\dim K(t_0) \leq \lceil C/\nu^{n/2} \rceil$, $\forall t_0 \in \mathbf{R}$. □

3. NON-AUTONOMOUS HYPERBOLIC EQUATION WITH DISSIPATION. Let us consider the equation

$$(59)\qquad \partial_t^2 u + \gamma \partial_t u = \Delta u - f(u,t) + \varphi(x,t), \quad u|_{\partial\Omega} = 0, \quad x \in \Omega \Subset \mathbf{R}^3,$$

where $\gamma > 0$. For the sake of brevity we restrict ourselves to the case $n = 3$. It is assumed that $f(u,t) \in C^2(\mathbf{R}\times\mathbf{R})$, $\varphi(\cdot,t) \in C_b(\mathbf{R}, L_2(\Omega))$, and the following conditions hold for all $(t,u) \in \mathbf{R}\times\mathbf{R}$:

$$(60)\qquad F \geq -mu^2 - C_m, \quad F = F(u,t) = \int_0^u f(v,t)\,dv,$$

$$(61)\qquad fu - cF + mu^2 \geq -C_m, \quad \text{where } m > 0 \text{ is sufficiently small, } c > 0,$$

$$(62)\qquad |f'_u| \leq C(1+|u|^\rho), \quad |f'_t| \leq C(1+|u|^{\rho+1}), \quad 0 < \rho < 2,$$

$$(63)\qquad F'_t \leq \beta^2 F + C, \quad \beta > 0 \text{ is sufficiently small,}$$

$$(64)\qquad |f'_u(u,t) - f'_u(u_1,t)| \leq C(|u|^{2-\delta} + |u_1|^{2-\delta} + 1)|u-u_1|^\delta, \quad 0 \leq \delta \leq 1.$$

(We recall that, in the general case $\Omega \Subset \mathbf{R}^n$, the restriction on ρ is $0 < \rho < 2/(n-2)$ when $n \geq 3$ and $0 < \rho$ when $n = 2$.) The initial conditions posed at $t = \tau$ are

$$(65)\qquad u\big|_{t=\tau} = u_\tau, \quad \partial_t u\big|_{t=\tau} = p_\tau.$$

We denote $y(t) = (u(t), \partial_t u(t)) = (u(t), p(t))$, $y_\tau = (u_\tau, p_\tau) = y(\tau)$. $E = H_0^1(\Omega)\times L_2(\Omega)$ is the space of functions y with the norm $\|y\|_E^2 = \|u\|_1^2 + \|p\|^2$. Analogously, we introduce the space E_1 with the norm $\|y\|_{E_1}^2 = \|u\|_2^2 + \|p\|_1^2$. It is convenient now to introduce the new variables: $w = (u,v)^\top = R_\alpha y = (u, u_t + \alpha u)^\top$, $u_t = \partial_t u$, $\alpha = \min(\gamma/4, \lambda_1/(2\gamma))$, where λ_1 is the first eigenvalue of the operator $-\Delta u$, $u\big|_{\partial\Omega} = 0$. Using these variables the equation (59) is equivalent to the following system:

$$(66)\qquad \partial_t w = A_\alpha w = L_\alpha w - G(w,t), \quad w\big|_{t=\tau} = w_\tau,$$

where $w_\tau \in E$,

$$L_\alpha = \begin{pmatrix} -\alpha I & I \\ \Delta + \alpha(\gamma-\alpha) & -(\gamma-\alpha)I \end{pmatrix},$$

$G(w,t) = (0, f(u,t) - \varphi(x,t))^\top$. It follows from the conditions on $f(u,t)$ and $\varphi(x,t)$ that problem (60) generates a process $\{U(t,\tau)\}$ by $U(t,\tau)w_\tau = w(t)$, $U(t,\tau): E \to E$. The process $\{U(t,\tau)\}$ is continuous and uniformly asymptotically compact, i.e., there exists a set $P \Subset E$ that is compact and uniformly attracting (w.r.t. τ). (See [12], [18], [3]). By Theorem 2.2 and 2.3 the process $\{U(t,\tau)\}$ possesses a non-empty kernel K such that $K(t) \subseteq P$ $\forall t \in \mathbf{R}$. Moreover, any bounded complete trajectory $w(s)$, $s \in \mathbf{R}$, of the process is uniformly bounded in E_1: $\|w(t)\|_{E_1} \leq M \ \forall t \in \mathbf{R}$, where the constant M does not depend on $u(t)$. Therefore, by the Sobolev embedding theorem,

$$\|u(t)\|_{C_b} \leq M_1 \ \forall t \in \mathbf{R}, \quad (u(\cdot), \partial_t u(\cdot)) = w(\cdot) \in K, \tag{67}$$

where $u(t)$ is any complete solution of the equation (59) and M_1 does not depend on $u(t)$.

Estimates for $\dim K(t_0)$ can be established similar to the autonomous case given in [14] (see also [18]).

Theorem 5.3. *For the Hausdorff dimension of a kernel section $K(t_0)$ of the process $\{U(t,\tau)\}$ generated by problems* (59), (65), *the following estimation holds:*

$$\dim K(t_0) \leq \left\lceil \frac{C}{\alpha^3} \right\rceil, \qquad \forall t_0 \in \mathbf{R}, \tag{68}$$

where $C = C(M_1)$, C does not depend on t_0.

Proof. Condition (64) implies that the operators $\{U(t,\tau)\}$ are uniformly quasidifferentiable on $\{K(\tau)\}_{\tau \in \mathbf{R}}$ and that the quasidifferentials $U'(t,\tau; w_\tau) z_\tau = z(t)$ satisfy the variation equation of problem (66):

$$\partial_t z = L_\alpha z - G'_w(w,t) z = A'_{\alpha w}(w(t),t) z, \quad w\big|_{t=\tau} = w_\tau, \quad z = (r,q), \tag{69}$$

$G'_w(w,t)z = (0, f'_u(u,t) r)$ (see [18]). Let us estimate the trace

$$\operatorname{Tr} A'_{\alpha w}(w(t),t) Q_d(t) = \sum_{j=1}^{d} (A'_{\alpha w}(w(t),t)\zeta_j, \zeta_j)_E. \tag{70}$$

Here $\zeta_j = (r_j, q_j)$ is an orthonormal system in $Q_d(t)E$ and $Q_d(t)$ is the projector on the space spanned by $z_1(t), \ldots, z_d(t)$, where $z_j(t)$ is the solution of (69) with initial condition $z_j(\tau) \in E$ $(j = 1, \ldots, d)$. Let us estimate the right-hand side of (70)

$$\begin{aligned} (A'_{\alpha w}(w(t),t)\zeta_j, \zeta_j)_E &= (L_\alpha \zeta_j, \zeta_j) - (f'_u(u,t) r_j, q_j) \\ &\leq -(\alpha/2)\|\zeta_j\|_E^2 + C(M_1)\|r_j\|\|q_j\| \\ &\leq -(\alpha/4)\left(\|r_j\|_1^2 + \|q_j\|^2\right) + (C_1(M_1)/\alpha)\|r_j\|^2. \end{aligned} \tag{71}$$

We have chosen the parameter α in such a way that the operator L_α is negative: $(L_\alpha \zeta_j, \zeta_j) \leq -(\alpha/2)\|\zeta_j\|_E^2$. Observe that it is essential that

$$\sup\{\|f'_u(u(t),t)\|_{C_b} : (u(\cdot), \partial_t u(\cdot)) = w(\cdot) \in K, \ t \in \mathbf{R}\} \leq C_1(M_1)$$

(see (67)). The system ζ_j is orthonormal in E; therefore, from (71), it follows that

$$\operatorname{Tr} A'_{\alpha w}(w(t),t)Q_d(t) \le -(\alpha/4)d + (C_1(M_1)/\alpha)\sum_{j=1}^{d}\|r_j\|^2 \tag{72}$$
$$\le -(\alpha/4)d + (C_1(M_1)/\alpha)\sum_{j=1}^{d}\lambda_j^{-1}$$
$$\le -(\alpha/4)d + (C_2(M_1)/\alpha)d^{1/3},$$

where λ_j, $(j = 1,...,d)$ are the first d eigenvalues of the operator $-\Delta u$, $u|_{\partial\Omega} = 0$, in nondecreasing order, $\lambda_j \ge c_0 j^{2/3}$. Note that we have used the inequality

$$\sum_{j=1}^{d}\|r_j\|^2 \le \sum_{j=1}^{d}\lambda_j^{-1},$$

which is proved in [18]. The right-hand side of (72) is negative if $d \ge \lceil C/\alpha^3 \rceil$, where $C = (4C_2)^{3/2}$. Finally, from Theorem 4.1 we infer (68). □

6. Some conclusive remarks.

1. Consider the set $\mathcal{K}$ consisting of all values of all complete trajectories $u(t) \in K$, where K is the kernel of some processes, i.e.,

$$\mathcal{K} = \bigcup_{\tau\in\mathbf{R}} K(\tau).$$

The closure $\bar{\mathcal{K}}$ in E can have infinite Hausdorff dimension

$$\dim \bar{\mathcal{K}} = +\infty \tag{73}$$

for all the problems described in Section 4. For example, let us show (73) for the Navier-Stokes system. We put

$$u(x,t) = \sum_{j=1}^{\infty}[a_j(x)\cos(\lambda_j t) + b_j(x)\sin(\lambda_j t)], \tag{74}$$

where $u = (u^1, u^2)$ and $a_j(x) = (a_j^1(x), a_j^2(x))$, $b_j(x) = (b_j^1(x), b_j^2(x))$ are smooth linear independent vector functions such that $a_j\,\big|_{\partial\Omega} = 0$, $(\nabla, a_j) = 0$, $b_j\,\big|_{\partial\Omega} = 0$, $(\nabla, b_j) = 0$. We assume that the series (74) and its derivatives with respect to x and t converge rapidly. We also assume that the frequencies λ_j $(j = 1,2,\ldots)$ are rationally independent. Set

$$\varphi(x,t) = \partial_t u(x,t) + Lu(x,t) + B(u(x,t),u(x,t)). \tag{75}$$

Evidently, $\varphi(\cdot,t) \in C_b(\mathbf{R},H)$. Problem (43)–(45) with such an external force $\varphi(x,t)$ possesses a non-empty kernel with compact sections $K(\tau)$. Obviously, $u(\cdot,\cdot) \in K$. It is easy to show that the projection $u_N(x,t)$ of the function $u(x,t)$ onto the $2N$-dimensional space, spanned by the vectors $\{(a_j(x), b_j(x)), j = 1,\ldots,N\}$, provides a dense subset of the N-dimensional torus $\mathbf{T}^N \subset H$. Therefore, the set $\overline{\mathbf{Im}u} = \overline{\{u(\cdot,t) : t \in \mathbf{R}\}}$ has a Hausdorff dimension larger than N for any $N \in \mathbf{N}$, i.e., $\dim \overline{\mathbf{Im}}u = \infty$. Evidently $\overline{\mathbf{Im}}u \subseteq \bar{\mathcal{K}}$, and therefore $\dim \bar{\mathcal{K}} = +\infty$.

2. It was shown in [11] and in [3, 4, 5, 6, 7] that a uniformly asymptotically compact process $\{U(t,\tau)\}$ acting in a space E possesses the compact uniform attractor $\mathcal{A}$. The uniform attractor is the smallest closed set in E, which is uniformly attracting (with respect to $\tau \in \mathbf{R}$) when $T = t - \tau \to +\infty$. Note that the kernel K or, to be precise, the corresponding set $\mathcal{K} = \bigcup_{\tau \in \mathbf{R}} K(\tau)$ belongs to $\mathcal{A}$, but, in general, does not coincide with $\mathcal{A}$. Uniform attractors for non-autonomous evolution equations with almost periodic symbols were studied in [3, 4, 5, 6, 7], where some questions concerning the Hausdorff dimension of attractors were considered.

3. In this paper we presented some upper bounds for the Hausdorff dimension of the kernel sections of non-autonomous equations. Analogous results are also true for the fractal dimension. We refer the reader to [18], where the corresponding techniques for autonomous equations are presented.

REFERENCES

[1] A. V. BABIN AND M. I. VISHIK, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[2] ———, *Attractors of evolutionary partial differential equations and estimates of their attractors*, Uspekhi Mat. Nauk **4** (1983), 133-187.

[3] V. V. CHEPYZHOV AND M. I. VISHIK, *Non-autonomous dynamical systems and their attractors*, (Appendix), Asymptotic behavior of solutions of evolutionary equations, by M. I. Vishik, Cambridge University Press, 1992.

[4] ———, *Nonautonomous evolution equations and their attractors*, Russ. J. Math. Physics **2** (1993).

[5] ———, *Attractors for nonautonomous evolution equations with almost periodic symbols*, C. A. Acad. Sci. Paris **316** (1993), 357-361.

[6] ———, *Nonautonomous evolutionary equations with almost periodic symbols*, Proceedings of the conference dedicated to L. Amerio (to appear).

[7] ———, *Attractors of non-autonomous systems and processes*, J. Math. Pures Appl. (to appear).

[8] P. CONSTANTIN, C. FOIAS, AND R. TEMAM, *Attractors Representing Turbulent Flows*, Memories of Amer. Math. Soc., vol. 53, 1985.

[9] C. M. DAFERMOS, *Semi-flows associated with compact and almost uniform processes*, Math. Systems Theory **8** (1974), 142-149.

[10] ———, *Almost periodic processes and almost periodic solutions of evolutional equations*, Proceedings of a University of Florida International Symposium, Academic Press, New York, 1977, pp. 43-57.

[11] A. HARAUX, *Systèmes dinamiques dissipatifs et applications*, Masson, Paris, 1991.

[12] ———, *Two Remarks on Dissipative Hyperbolic Problems*, Coll. de France Seminar, Pitman Res. Notes Math., Vol. 6, 1985.

[13] A. DOUADY AND J. OESTERLE, *Dimension de Hausdorff des attracteurs*, C. A. Acad. Sci. Paris Ser. A **290** (1980), 1135-1138.

[14] J. M. GHIDAGHLIA AND R. TEMAM, *Attractors for damped nonlinear hyperbolic equations*, J. Math. Pures Appl. **66** (1987), 273-319.

[15] O. A. LADYZHENSKAYA, *Mathematical problems in the dynamics of a viscous incompressible liquid*, Nauka, Moscow, 1970.

[16] J.-L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gautier-Villars, Paris, 1969.

[17] G. R. SELL, *Nonautonomous differential equations and topological dynamics I, II*, Amer. Math. Soc. **127** (1967), 241-262, 263-283.

[18] R. TEMAM, *Infinite-dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1988.

[19] ———, *On the theory and numerical analysis of the Navier-Stokes equations*, Springer-Verlag, New York, 1974.

[20] M. I. VISHIK AND A. V. FURSIKOV, *Mathematical problems of statistical hydromechanics*, Kluwer Academic Publishers, Dordrecht,, 1987.

[21] J. K. HALE, *Asymptotic Behaviour of Dissipative Systems*, Amer. Math. Soc., Providence, Rhode Island, 1988.

This research was supported in part by the Swedish Natural Scientific Research Council.

Institute for Problems of Information Transmission
Russian Academy of Science
Ermolovoy 19
Moscow 101447, GSP-4
RUSSIA
E-MAIL: CHEP@IPPI.MSK.SU

Received: May 10th, 1993.