Hausdorff Dimension Estimation for Attractors of Nonautonomous Dynamical Systems in Unbounded Domains: An Example

V. V. CHEPYZHOV
Institute for Problems of Information Transmission (*IPPI*)
AND
M. A. EFENDIEV
Freie Universität Berlin

Abstract

The subject of this paper is the asymptotic behavior of a class of nonautonomous, infinite-dimensional dynamical systems with an underlying unbounded domain. We present an approach that is able to overcome both the law of compactness of the trajectories and the continuity of the spectrum of the linear part of the equations under consideration, providing nevertheless existence of uniform attractors. Moreover, our approach allows us to estimate the Hausdorff dimension of attractors of nonautonomous equations in terms of physical parameters.© 2000 John Wiley & Sons, Inc.

1 Processes and Their Attractors Related to Nonautonomous Equations

We begin with some preliminaries. Let E be a Banach space. In E we consider the nonautonomous Cauchy problem

$$(1.1) \qquad \begin{cases} \partial_t u = A(u,t) \\ u|_{t=\tau} = u_\tau,\ u_\tau \in E,\ \tau \in \mathbb{R},\ t \geq \tau\,, \end{cases}$$

where $A(u,t) : E_1 \times \mathbb{R} \to E_0$ is a family of nonlinear operators and E_1 and E_0 are Banach spaces. We assume that the embeddings $E_1 \subset E \subset E_0$ are everywhere dense. In this paper, as a model of (1.1), we consider the reaction-diffusion equation (RDE) with a quasi-periodic external force of the form

$$(1.2) \qquad \begin{cases} \frac{\partial u}{\partial t} = \nu \Delta u - f(u) - \lambda_0 u - g_0(x,t)\,, \quad x \in \mathbb{R}^n\,, \\ u|_{t=\tau} = u_\tau\,, \end{cases}$$

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and the two-dimensional Navier-Stokes system in the strip $\Omega \subset \mathbb{R}^2$, that is,

$$\begin{cases} \partial_t \bar{u} + \nu \Delta \bar{u} + \sum_{i=1}^2 u_i \partial_i \bar{u} + \operatorname{grad} p = \bar{g}_0(x,t), \quad x \in \Omega, \\ \bar{u}|_{t=\tau} = u_\tau, \ \operatorname{div} \bar{u} = 0, \ \bar{u}|_{\partial\Omega} = 0, \end{cases} \tag{1.3}$$

where

$$\begin{aligned} \Omega &= \{(x_1,x_2) : -\infty < x_1 < +\infty, \ 0 \le x_2 \le d\}, \\ \bar{u}(t,x_1,x_2) &= (u_1(t,x_1,x_2), u_2(t,x_1,x_2)), \\ \bar{g}_0(t,x_1,x_2) &= (g_{01}(t,x_1,x_2), g_{02}(t,x_1,x_2)), \\ \bar{u}_\tau(x) &= (u_{1\tau}(x), u_{2\tau}(x)). \end{aligned}$$

We always assume that the initial value problem (1.1) has a unique solution $u(t) \in E$ for all $t > \tau$ and $\forall \tau \in \mathbb{R}$ and $u_\tau \in E$.

Consider the two-parameter family of maps $\{U(t,\tau)\}$, $U(t,\tau) : E \to E$, such that

$$U(t,\tau) : u_\tau \to u(t).$$

DEFINITION 1.1 A family of maps $\{U(t,\tau)\}$ is called a *process* if

(i) $U(\tau,\tau) = \mathrm{Id}$ (identity) and
(ii) $U(t,s) \circ U(s,\tau) = U(t,\tau)$ for all $t \ge s \ge \tau, \ \tau \in \mathbb{R}$.

In this paper we are mainly interested in processes generated by nonautonomous evolution equations such as (1.2) and (1.3). It is clear that a process is a natural generalization of a semigroup $S_t : E \to E$, which corresponds to autonomous evolution equations; that is, $A(u,t) \equiv A_0(u)$. Note that in this case, it is easy to see that

$$U(t,\tau) = U_0(t-\tau).$$

Our main goal is to study the large-time asymptotics of a process $\{U(t,\tau)\}$ generated by equation (1.1), that is, the behavior of trajectories $u(t) = U(t,\tau)u_\tau$ of (1.1) when $t-\tau$ tends to infinity. As we will see below, the large-time dynamics of a process can be described in terms of attractors (we give a precise definition of *attractor* later). We follow reference [4]. As was shown in [4], an adequate theory of attractors for nonautonomous equations is obtained by considering a family of processes $\{U_g(t,\tau)\}$ instead of a single process where $g \in \Sigma$ is a functional parameter. For the convenience of the reader, we recall basic definitions from [4]. Indeed, following [4], consider the family of Cauchy equations

$$\partial_t u = A_{g(t)}(u), \tag{1.4}$$

$$u|_{t=\tau} = u_\tau, \quad u_\tau \in E, \tag{1.5}$$

where for any fixed $t \in \mathbb{R}$, $A_{g(t)}(u)$ is, in general, a nonlinear operator acting from a Banach space E_1 to a Banach space E_0 and $E_1 \subset E \subset E_0$ (everywhere dense). We assume that a functional parameter $g(t)$ belongs to a certain closed set Σ in

$C_b(\mathbb{R},W)$, where $(C_b(\mathbb{R},W)$ denotes the space of bounded continuous functions on $\mathbb{R}$ with values in a certain metric space W, with $T(h)\Sigma=\Sigma$, where

$$T(h)g(t) := g(t+h)\,, \quad h\in\mathbb{R}\,. \tag{1.6}$$

We suppose that the problem (1.4)–(1.5) is well-posed for any symbol $g(t)\in\Sigma$ so that any solution $u(t)\in E$ (we specify in each case what we mean by "solution") can be represented as

$$\begin{gathered} u(t)=U_g(t,\tau)u_\tau\,, \\ g=g(t)\in\Sigma,\; u_\tau\in E\,,\; \tau\in\mathbb{R}\,,\; t\geq\tau\,. \end{gathered} \tag{1.7}$$

Due to the uniqueness theorem for (1.4)–(1.5), operators defined by (1.7) define a process and satisfy the following translation identity:

$$U_{T(h)g}(t,\tau)=U_g(t+h,\tau+h)\,, \quad \forall h\geq 0\,,\; t\geq\tau\,,\; \tau\in\mathbb{R}\,. \tag{1.8}$$

Let $S(t): E\times\Sigma\to E\times\Sigma$ be the family of operators defined by

$$S(t)(u,g)=(U_g(t,0)u,T(t)g)\,, \quad t\geq 0\,,\; (u,g)\in E\times\Sigma\,. \tag{1.9}$$

It is not difficult to see that the family of operators $\{S(t)\}$ defined by (1.9) forms a semigroup on the extended phase space $E\times\Sigma$. We use this fact throughout this paper.

Before formulating the main result for a family of processes $\{U_g(t,\tau)\}$ from a dynamical viewpoint, we recall some definitions (see [4, 5]). Let E be the Banach space as before, and denote by $\beta(E)$ the set of all bounded subsets of E.

DEFINITION 1.2 A set $B_0\subset E$ is called a *uniformly* (with respect to Σ) *absorbing* (or *attracting*) set for the family of processes $\{U_g(t,\tau)\}$ if for any $\tau\in\mathbb{R}$ and any $B\subset\beta(E)$ there exists $T=T(\tau,B)\geq\tau$ such that

$$\bigcup_{g\in\Sigma} U_g(t,\tau)B\subset B_0\,, \quad \forall t\geq T \text{ (absorbing property)},$$

and

$$\lim_{t\to+\infty}\sup_{g\in\Sigma}\operatorname{dist}_E(U_g(t,\tau)B,P)=0\,, \quad \forall\tau\in\mathbb{R},\ B\subset\beta(E) \text{ (attracting property)}.$$

DEFINITION 1.3 A closed set $M_\Sigma\neq\varnothing$ is called a *uniform* (with respect to Σ) *attractor* of the $\{U_g(t,\tau)\}$ if it is uniformly attracting and is contained in any closed uniformly attracting set $\tilde{M}$ (minimality property).

Following Haraux [6], a family of processes possessing a compact, uniformly absorbing (uniformly attracting) set are called uniformly compact (or uniformly asymptotically compact) processes. Now we are in position to formulate the main result on the existence of attractors for a family of processes. Let $\Pi_1: E\times\Sigma\to E$ be the projector defined by $\Pi_1(u,g))=u$.

THEOREM 1.4 [4]. *Let a family of processes* $\{U_g(t,\tau)\}$, $g \in \Sigma$, *acting in the Banach space* E *be uniformly asymptotically compact and* $(E \times \Sigma, E)$ *continuous. Then the semigroup* $\{S(t)\} : E \times \Sigma \to E \times \Sigma$ *defined by* (1.9) *possesses the compact attractor* $\mathcal{M}^*$. *Moreover,* $\mathcal{M} = \Pi_1 \mathcal{M}^*$ *is the uniform attractor of the family of processes* $\{U_g\}$.

For a proof see [4].

We will apply Theorem 1.4 for the family of Cauchy problems (1.2) and (1.3). We emphasize that in [4], the Cauchy problems (1.2) and (1.3) for bounded domain were considered. However, nontrivial difficulties arise in the case of an unbounded domain. Indeed, in contrast to a bounded domain,

- the operators of a process $\{U_g\}$ or semigroup $\{S(t)\}$ corresponding to equations (1.2) and (1.3) are not compact,
- the Laplace operator has a continuous spectrum, so that one can't apply Galerkin's approximations for proving global existence in time solutions, and
- $H^1(\mathbb{R}^n)$ is not compactly embedded in $L_2(\mathbb{R}^n)$; therefore the operators U_g and $\{S(t)\}$ do not have absorbing sets that are compact in the original topology.

To simplify our presentation, we restrict ourselves to a family of Cauchy problems

$$\begin{cases} \partial_t u = \nu\Delta u - f(u) - \lambda_0 u - g(x,t), \quad x \in \mathbb{R}^n,\ g \in H(g_0), \\ u|_{t=\tau} = u_\tau \end{cases} \tag{1.10}$$

and

$$\begin{cases} \partial_t \bar{u} + \nu\Delta\bar{u} + \sum_{i=1}^2 u_i \partial_i \bar{u} + \operatorname{grad} p = \bar{g}(x,t), \quad x \in \Omega \subset \mathbb{R}^2,\ \bar{g} \in H(\bar{g}_0), \\ \bar{u}|_{t=\tau} = \bar{u}_\tau \end{cases} \tag{1.11}$$

with $\bar{g}(x,t) \in H(\bar{g}_0(x,t))$, where by $H(\bar{g}_0(x,t))$ we denote the hull of a given quasi-periodic function $\bar{g}_0(x,t)$ of t. Note that by definition

$$H(\bar{g}_0) := \overline{\{T(h)\bar{g}_0 : h \in \mathbb{R}\}}^{C_b(\mathbb{R},W)},$$

that is, the closure in $C_b(\mathbb{R},W)$ of the set of all translations of the given quasi-periodic function g_0. On the other hand, a quasi-periodic function $g_0(x,t)$ can be represented as

$$\bar{g}_0(x,t) = \tilde{g}_0(x, \alpha_1 t, \ldots, \alpha_k t)$$

where $\tilde{g}_0(x,\omega_1,\ldots,\omega_j+2\pi,\ldots,\omega_k) = \tilde{g}_0(x,\omega_1,\ldots,\omega_k)$ and the numbers $\alpha_1,\ldots,\alpha_k$ are rationally independent. When $\tilde{g}_0(\omega_1,\ldots,\omega_k)$ is a continuous function on $\mathbb{T}^k$ (k-dimensional torus), one can easily see that the hull $H(\bar{g}_0)$ is a set

$$H(\bar{g}_0(x,t)) = \{\tilde{g}_0(x,\alpha_1 t+\omega_{10},\ldots,\alpha_k t+\omega_{k0}); \omega_0 = (\omega_{10},\ldots,\omega_{k0}) \in \mathbb{T}^k\}. \tag{1.12}$$

Thus due to (1.12) it is reasonable to consider the torus $\mathbb{T}^k$ as the symbol space through the map

$$\mathbb{T}^k \ni \omega_0 \to \tilde{g}_0(x, \alpha t + \omega_0) := \tilde{g}_0(x, \alpha_1 t + \omega_{10}, \ldots, \alpha_k t + \omega_{k0}) \in C_b(\mathbb{R}, W)\,. \tag{1.13}$$

We set

$$T(h)\omega_0 = [\omega_0 + \alpha h] := \omega_0 + \alpha h (\operatorname{mod} \mathbb{T}^k)\,. \tag{1.14}$$

Obviously $T(h)\mathbb{T}^k = \mathbb{T}^k$.

2 Examples

2.1 Examples of Nonautonomous Equations of Mathematical Physics in Unbounded Domains Having Uniform Attractors

We consider reaction-diffusion equations (RDEs) in $\mathbb{R}^n$ of the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = \nu \Delta u - f(u) - \lambda_0 u - g(x,t)\,, & x \in \mathbb{R}^n\,, \\ u|_{t=\tau} = u_\tau\,, \end{cases} \tag{2.1}$$

with $g \in H(g_0(x,t))$ for some quasi-periodic functions (or, equivalently, $g(x,t) = \tilde{g}_0(x, \omega(t), \omega(t) = [\alpha t + \omega_0]$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, $\omega_0 \in \mathbb{T}^k$), $\lambda_0 > 0$. In Section 1 we noted the nontrivial difficulties that may arise in the case of unbounded domain. As we see below, these difficulties are overcome by the systematic use of weighted Sobolev spaces. Indeed, let $H_{0,\gamma}(\mathbb{R}^n)$ be the weighted Sobolev space, $\gamma > 0$; the norm in this space is defined by

$$\|u\|^2_{0,\gamma} = \int_{\mathbb{R}^n} (1 + |\varepsilon x|^2)^\gamma |u(x)|^2 \, dx\,, \tag{2.2}$$

where $\varepsilon > 0$ is a small enough but fixed number. Analogously, we define $H_{s,\gamma}(\mathbb{R}^n)$ by

$$\|u\|^2_{s,\gamma} = \sum_{|\ell| \le s} \|\partial^\ell u\|^2_{0,\gamma}\,. \tag{2.3}$$

CONDITION 2.1 *Let $f \in C^1(\mathbb{R}^n)$ satisfy the following*:

1. $f'(u) \ge -C$,
2. $f(u)u \ge 0$,
3. $|f(u)| \le C|u|^{1+\alpha}(1 + |u|^q)$, *where* $q, \alpha > 0$ *and for* $n > 2$,

   $$q + \alpha \le p_0 = \min\left\{\frac{4}{n}, \frac{2}{n-2}\right\},$$

   and $p_0 = 4/n$ *if* $n \le 2$.

   In addition, we assume that

4. $g_0 \in C^1(\mathbb{T}^k, H_0\gamma)$ *and* $\tilde{g}'_{\omega_j} := \frac{\partial \tilde{g}_0}{\partial \omega_j} \in C(\mathbb{T}^k, L_{q_1}(\mathbb{R}^n) \cap L_{q_2}(\mathbb{R}^n))$ *where* $q_1 = (1 - \delta)(\frac{n}{2} + 1)$ *and* $q_2 = (1 + \delta)$ *for some* $\delta > 0$.

THEOREM 2.2 *Let Condition* 2.1 *be satisfied and* $u_\tau \in H_{1,\gamma}(\mathbb{R}^n)$, $\gamma > 0$. *Then problem* (2.1) *has a unique solution* $u(t,x) \in L_2([\tau,T],H_{2,\gamma}) \cap L_\infty([\tau,+\infty),H_{1,\gamma})$, *with* $(1+|\varepsilon x|^2)^{\gamma/2}\partial_t u \in L_2([\tau,T],(H_{0,\gamma})^*)$ *for any* $T > 0$. *Moreover, the following estimates hold*:

$$\|u(t,x)\|^2_{0,\gamma} \le \|u(\tau,x)\|^2_{0,\gamma} e^{-\lambda_0\nu(t-\tau)} + C_1\|g\|^2_{C(\mathbb{T}^k,H_{0,\gamma})}, \tag{2.4}$$

(2.5)

$$\|u(t,x)\|^2_{0,\gamma} + \alpha_1 \int_\tau^t \|u\|^2_{0,\gamma}\, d\theta \le \|u(\tau,x)\|^2_{0,\gamma} + C_2\int_\tau^t \|g\|^2_{0,\gamma}\, d\theta + C_3(t-\tau),$$

$$\begin{aligned}(t-\tau)\|u(t,x)\|^2_{0,\gamma} + \alpha_2 \int_\tau^t (\theta-\tau)\|u(\theta,x)\|^2_{2,\gamma}\, d\theta \le \\ C_4\left(\int_\tau^t \|g\|^2_{0,\gamma} d\theta, \|u(\tau)\|^2_{0,\gamma})\right),\end{aligned} \tag{2.6}$$

$$\|u_1(t,x) - u_2(t,x)\|_{0,\gamma} \le C_5\|u_1(\tau,x) - u_2(\tau,x)\|_{0,\gamma}, \tag{2.7}$$

where $\alpha_1 > 0$, $\alpha_2 > 0$, C_2, C_3, C_4 *are functions of* $(t-\tau)$, *and* C_4 *also depends on* $(\|g\|_{0,\gamma,}\|u(\tau)\|_{0,\gamma})$, $u_\tau := u(\tau,x)$.

PROOF: Note that estimates (2.4)–(2.6) can be derived analogously to the autonomous case [2]. For example, to obtain (2.4), we multiply (2.1) by φu and integrate with respect to x, which leads to a differential inequality, verifying the estimate (2.4) and (2.5). To obtain (2.6), we multiply (2.1) by $(t-\tau)\varphi \cdot \Delta u$ and integrate with respect to x, which also leads to (2.6) using a differential inequality (see [2]). As for estimate (2.7), it is obtained in the standard manner, that is, by subtracting two solutions of (2.1) and using $f'(u) \ge -C$. Hence the process $U_{\omega_0}(t,\tau)u_\tau = u(t,x)$, where $u(t,x)$ is a solution of (2.1), $U_{\omega_0}(t,\tau) : H_{0,\gamma} \to H_{0,\gamma}$, $\gamma > 0$, $t \ge \tau$, $\tau \in \mathbb{R}$, and $\omega_0 \in \mathbb{T}^k$, is well-defined. ☐

PROPOSITION 2.3 *The family of processes* $\{U_{\omega_0}(t,\tau)\}$ *is uniformly bounded and uniformly asymptotically compact.*

PROOF: Indeed, from estimate (2.4) it follows that

$$\|U_{\omega_0}(t,\tau)u_\tau\|_{0,\gamma} \le C_0(\|u_\tau\|_{0,\gamma}), \quad \forall t \ge \tau,\ \tau \in \mathbb{R}, \tag{2.8}$$

where the constant C_0 depends on initial data $\|u_\tau\|$. This proves uniform boundedness with respect to $\omega_0 \in \mathbb{T}^k$. Moreover, the same estimate also implies that the set

$$B_0 = \left\{u \in H_{0,\gamma} : \|u\|^2_{0,\gamma} \le 2C_1 \cdot \|g\|^2_{C(\mathbb{T}^k,H_{0,\gamma})}\right\} \tag{2.9}$$

is a uniformly $H_{0,\gamma}$-absorbing set for the family $\{U_{\omega_0}(t,\tau)\}$.

On the other hand, from estimate (2.6), it follows that if $t > \tau$, $u_\tau \in H_{0,\gamma}$, then $U_{\omega_0}(t,\tau)u_\tau \in H_{1,\gamma}$ for $\forall \omega \in \mathbb{T}^k$. Moreover, the same estimates also implies that the set

$$B_1 = \bigcup_{\omega \in \mathbb{T}^k} \bigcup_{t \in \mathbb{R}} U_{\omega_0}(\tau+1,\tau)B_0 \tag{2.10}$$

is also uniformly $H_{0,\gamma}$-absorbing. Note that the set B_1 is bounded in $H_{1,\gamma}$. However, in contrast to the bounded case, we can't state that B_1 is compact in $H_{0,\gamma}$. Using the same trick as in [2], one can avoid this difficulty by the assumption

$$|f(u)| \leq C|u|^{1+\alpha}(1+|u|^{p_2}) \tag{2.11}$$

where α and q are due to Condition 2.1. We omit the details. This proves Proposition 2.3. Hence one can apply Theorem 1.4 for a family of processes $\{U_{\omega_0}(t,\tau)\}$ to yield existence of a uniform attractor $A_{\mathrm{RDE}} \subset H_{0,\gamma}$ for U_{ω_0}. □

2.2 Navier-Stokes Systems in the Two-Dimensional Strip with Quasi-Periodic External Force

Consider a family two-dimensional Navier-Stokes systems (after projecting out the pressure) with quasi-periodic external force, that is,

$$\begin{cases} \partial_t \bar{u} + \nu L\bar{u} + \Pi(\sum_{i=1}^2 u_i \partial_x^i \bar{u}) = \bar{g}(x,t), & x \in \Omega \subset \mathbb{R}^2,\ \bar{g} \in H(g_0), \\ \bar{u}|_{t=\tau} = \bar{u}_\tau,\ \bar{u}|_{\partial\Omega} = 0,\ \operatorname{div}\bar{u} = 0, \end{cases} \tag{2.12}$$

where by Π we denote the orthogonal projector $\Pi : (L_2(\Omega))^2 \to H$. Note that by $H(H_1)$ we denote the closure in $(L_2(\Omega))^2((H_1(\Omega))^2)$ with norm $\|\cdot\|$, $\|\cdot,\cdot\|_1$ of the set $V_0 = \{v \in (C_0^\infty(\Omega))^2 : \operatorname{div} v = 0\}$, $\bar{g}(t,x_1,x_2) = (g_1(t,x_1,x_2), g_2(t,x_1,x_2)) \in H(\bar{g}_0)$. Let $\bar{g}(t,x) = \tilde{g}_0(\omega(t),x)$, $\omega(t) = [\alpha t + \omega_0]$, $\bar{u}_\tau(x) = (u_{1\tau}(x), u_{2\tau}(x))$.

PROPOSITION 2.4 *Let $\bar{u}_\tau \in H$ and $\bar{g} \in C(\mathbb{T}^k, (L_2(\Omega))^2)$. Then there exists a unique solution $\bar{u}(x,t)$ of problem* (2.12) *such that*

$$\bar{u} \in \bigcap L_\infty(\mathbb{T}^k, H) \cap L_2(\mathbb{T}^k, H^1), \quad \partial_t u \in L_2(\mathbb{T}^k, H^{-1}),$$

and the following estimates hold:

$$\|\bar{u}\|^2 \leq e^{-\nu\lambda_1(t-\tau)}\|\bar{u}_\tau\|^2 + \nu^{-2}\lambda_1^{-1} \sup_{\omega \in \mathbb{T}^k} \|\bar{g}\|^2, \tag{2.13}$$

$$\nu \int_\tau^t \|\bar{u}\|_1 d\theta \leq \|\bar{u}_\tau\|^2 + (t-\tau)\nu^{-1}\lambda_1^{-1} \cdot \|\bar{u}_\tau\| \cdot \left(\int_{\mathbb{T}^k} \|\bar{g}\|^2 d\theta \right). \tag{2.14}$$

Hence problem (2.12) generates a family of processes $\{U_{\omega_0}(t,\tau) : t > \tau\}$, $\omega_0 \in \mathbb{T}^k$. Let $H_{1,\gamma} := H_{1,\gamma}(\Omega)$ be the Hilbert space defined by (2.3) and $H(\gamma) := H \cap H_{1,\gamma}$, $\gamma > 0$. Analogously to theorem 1.3 and theorem 4.1 in [1], one can show that $H(\gamma)$ is invariant under $\{U_{\omega_0}(t,\tau)\}$; furthermore, a family of processes $\{U_{\omega_0}(t,\tau)\}$ is uniformly asymptotically compact in $H(\gamma)$ if $\bar{g} \in H_{0,\gamma}$ and $\bar{u}_\tau \in H_{1,\gamma}$. Hence due to Theorem 1.4 a family of processes $\{U_{\omega_0}(t,\tau)\}$ possesses a uniform attractor in $H(\gamma)$. It remains to estimate the Hausdorff dimension of the attractors of equations (1.10) and (2.12).

3 Hausdorff Dimension Estimates for Uniform Attractors with Quasi-Periodic Symbols

3.1 Reaction-Diffusion Equations in $\mathbb{R}^n$

We start with RDEs in $\mathbb{R}^n$ with quasi-periodic external force. Due to the results in Section 2, a family of processes $\{U_{\omega_0}(t,\tau)\}$, $\omega_0 \in \mathbb{T}^k$, corresponding to (1.10) generates the semigroup $\{S(t)\}$,

$$S(t) : H_{0,\gamma} \times \mathbb{T}^k \to H_{0,\gamma} \times \mathbb{T}^k$$

$$S(t)(u_0,\omega_0) = (U_{\omega_0}(t,\tau)u_0, [\alpha t + \omega_0]), \quad u_0 \in H_{0,\gamma}, \, \omega_0 \in \mathbb{T}^k, \, t \geq 0, \tag{3.1}$$

which in turn corresponds to the following autonomous dynamical system:

$$\begin{cases} \partial_t u = \nu \Delta u - f(u) - \lambda_0 u - g(x,\omega), \quad \partial_t \omega = \alpha, \\ u|_{t=0} = u_0, \, \omega|_{t=0} = \omega_0, \, u_0 \in H_{0,\gamma}, \, \omega_0 \in \mathbb{T}^k, \end{cases} \tag{3.2}$$

or

$$\partial_t y = My, \quad y|_{t=0} = y_0, \tag{3.3}$$

where $y = (u,\omega) \in H_{0,\gamma} \times \mathbb{T}^k$ and

$$M(y) = (\nu \Delta u - f(u) - \lambda_0 u - g(x,\omega), \alpha). \tag{3.4}$$

As was shown in Section 2, the family of processes $\{U_{\omega_0}(t,\tau)\}$, $\omega_0 \in \mathbb{T}^k$, is uniformly asymptotically compact in $H_{0,\gamma}$ (Proposition 2.3). Therefore, due to Theorem 1.4, the semigroup $\{S(t)\}$ defined by (3.1) possesses a compact attractor $\mathcal{A}$ in $H_{0,\gamma} \times \mathbb{T}^k$. Moreover, the projection $\mathcal{A}_{\mathrm{RDE}}$, $\mathcal{A}_{\mathrm{RDE}} := \Pi_1 \mathcal{A}$, is a uniform attractor of the $\{U_{\omega_0}(t,\tau)\}$ and obviously

$$\dim \mathcal{A}_{\mathrm{RDE}} \leq \dim \mathcal{A} \tag{3.5}$$

where by $\dim \mathcal{A}_{\mathrm{RDE}}$ and $\dim \mathcal{A}$ we denote the Hausdorff dimension in $H_{0,\gamma}(H_{0,\gamma} \times \mathbb{T}^k)$ of the attractors of $\mathcal{A}_{\mathrm{RDE}}$ and $\mathcal{A}$, respectively. Hence to obtain an estimate for $\dim \mathcal{A}_{\mathrm{RDE}}$, it is sufficient to obtain an upper bound for $\dim \mathcal{A}$. We emphasize that the upper bound for $\dim \mathcal{A}$ is based on a well-known formula by Constantin, Foias, and Temam [8]. To this end we give a theorem of the differentiability of operators $S(t) : y_0 \to y(t)$ with respect to initial data $y_0 \in \mathcal{A}$.

THEOREM 3.1 *Suppose that Condition* 2.1 *holds. Then the operator* $S(t)$ *defined by* (3.1) *is uniformly differentiable on* $\mathcal{A}$ (*we denote its derivative by* $S'(t;y_0)$, $y_0 \in \mathcal{A}$) *with respect to the metric* $H_{0,0} = L_2(\mathbb{R}^n)$.

PROOF: Analogously to the bounded domain case, one can easily see that the differential $S'(t,y_0)$ at $y_0 = (u_0,\mu_0)$, $S'(t,y_0)z_0 = z(t)$, is the solution of the variational equation

(3.6)

$$\begin{cases} \partial_t v = \nu \Delta v - f'_u(u(t,x))v - \lambda_0 v + g'_\omega(x,\omega(t))\eta \\ \partial_t \eta = 0, \; v|_{t=0} = v_0, \; \eta|_{t=0} = \eta_0, \; u(t,x) = U_{\omega_0}(t,\tau)u_0, \; \omega(t) = [\alpha t + \omega_0] \end{cases}$$

with $v(t) = v(t,x) \in H_{0,0}$, $\eta \in \mathbb{R}^k$, $z(t) = (v(t,x), \eta)$. □

We denote by $M'(y(t))$,

$$M'(y(t))(v(t),\eta) := \nu\Delta v - f'_u(u(t))v - \lambda_0 v + g'_\omega(x,\omega(t))\eta, \quad y(t) = (u(t),\omega)\,. \tag{3.7}$$

Due to the formula by Constantin, Foias, and Teman [8], $\dim A \le d$ provided that, for any $y(t) = (u(t,x),\mu)$, $u(t,x) = U_{\omega_0}(t,\tau)u_0$, $u_0 \in A$, the following inequality holds:

$$\overline{\lim}_{t\to\infty}\frac{1}{t}\int_0^t \sup_{E_d}\sum_{j=1}^{d}(M'(y(t)z_j,z_j)d\tau < 0 \tag{3.8}$$

where E_d is any d-dimensional subspace in the Hilbert space $H_{0,0}\times\mathbb{R}^k$ and $z_j \in E_d$, $j = 1,\dots,d$, is any orthonormal family in $H_{0,0}\times\mathbb{R}^k$ belonging to $H_{2,0}\times\mathbb{R}^k$. Therefore, in order to obtain an upper bound for $\dim A$, we have to estimate $\sum_{j=1}^{d}(M'(y(t)z_j,z_j)_{L^2(\mathbb{R}^n)\times\mathbb{R}^k}$, where

$$z_j = (v_j,\eta_j) \in H_{0,0}\times\mathbb{R}^k, \quad (z_i,z_j)_{L^2(\mathbb{R}^n)\times\mathbb{R}^k} = \delta_{ij}\,.$$

To this end, we start with

$$(M'(y(t))z,z)_{L^2(\mathbb{R}^n)\times\mathbb{R}^k} = -\nu\|\bigtriangledown v\|^2 - \lambda_0\|v\|^2 - \int f'_u(u)v^2dx + \int (g'_\omega,\eta)_{\mathbb{R}^k} v\,dx, \tag{3.9}$$

where $z = (v,\eta) \in L^2(\mathbb{R}^n)\times\mathbb{R}^k$, $g'_\omega = (\frac{\partial g}{\partial\omega_1},\dots,\frac{\partial g}{\partial\omega_k})$.

It follows from (3.9) that for any positive $b > 0$ we have

$$\begin{aligned}(M'(y(t))z,z) &\le -\nu\|\bigtriangledown v\|^2 - \lambda_0\|v\|^2 - \int f'_u(u)v^2\,dx \\ &\quad + \int |g'_\omega|^{\frac{1-\delta}{2}}\cdot|v|\cdot|g'_\omega|^{\frac{1+\delta}{2}}\cdot|\eta|dx \\ &\le -\nu\|\bigtriangledown v\|^2 - \lambda_0\|v\|^2 - \int f'_u(u)v^2\,dx + \frac{b}{2}\int |g'_\omega|^{1-\delta}|v|^2\,dx \\ &\quad + \frac{|\eta|^2}{2b}\int |g'_\omega|^{1+\delta}\,dx\end{aligned}$$

where $0 < \delta < 1$. Let $G_1 = \max_{\omega\in\mathbb{T}^k}\int |g'_\omega|^{1+\delta}dx$. Thus

$$\begin{aligned}&(M'(y(t))z,z) \\ &\quad \ge -\nu\|\bigtriangledown v\|^2 - \lambda_0\|v\|^2 - \int f'_u(u)v^2\,dx + \frac{b}{2}\int |g'_\omega|^{1+\delta}|v|^2\,dx + \frac{G_1}{2b}|\eta|^2 \\ &\quad = (A_1 v,v) + (A_2\eta,\eta)\end{aligned}$$

where $A_j,\ j=1,2$, are quadratic forms defined by

$$(A_1 v, v) := -\nu\|\nabla v\|^2 - \lambda_0\|v\|^2 - \int f'_u(u)v^2 dx + \frac{b}{2}\int |g'_\omega|^{1-\delta}|v|^2\,dx, \tag{3.10}$$

$$(A_2\eta,\eta) := \frac{G_1}{2b}|\eta|^2\,. \tag{3.11}$$

We denote by $(\bar{A}z,z)$ an expression of the form

$$(\bar{A}_1 z,z) := (A_1 v,v) + (A_2\eta,\eta)\,, \quad z=(v,\eta)\,. \tag{3.12}$$

Let us recall that our basic task is to estimate the expression

$$\sum_{j=1}^{d}(M'(y(t))z_j,z_j)_{L_2(\mathbb{R}^n)\times\mathbb{R}^k} \quad \text{where } z_j=(v_j,\eta_j)\in L_2(\mathbb{R}^n)\times\mathbb{R}^k,\ (z_i,z_j)=\delta_{ij}\,.$$

PROPOSITION 3.2 *Let $\{z_j\}=(v_j,\eta_j)$, $j=1,\dots,d$, be any orthonormal system in $L_2(\mathbb{R}^n)\times\mathbb{R}^k$. Then there exists an integer number k_1, $0\le k_1\le k$, and $\{w_i\}\in L_2(\mathbb{R}^n)$, $\{\xi_m\}\in\mathbb{R}^k$, $i=1,\dots,d-k_1$, $m=1,\dots,k_1$, which are orthonormal in $L_2(\mathbb{R}^n)$ and $\mathbb{R}^k$, respectively, such that*

$$\sum_{j=1}^{d}(\bar{A}z_j,z_j) \le \sum_{i=1}^{d-k_1}(A_1 v_i,v_i) + \sum_{m=1}^{k_1}(A_2\eta_m,\eta_m)\,. \tag{3.13}$$

PROOF: Consider the subspace $L\subset\mathbb{R}^{d+k}$ of the form $L=\{\xi_1 v_1+\cdots+\xi_d v_d\}\times\mathbb{R}^k$, where $\xi_j\in\mathbb{R}^1$, $j=1,\dots,d$. In L there is a scalar product induced from $L_2(\mathbb{R}^n)\times\mathbb{R}^k$. Consider the restriction of $(\bar{A}z,z)$ to L. Note that $z_j=(v_j,\eta_j)\subset L$ and is orthonormal in L.

Then

$$\begin{aligned}
&\langle\bar{A}(\xi_1 v_1+\cdots+\xi_d v_d,\theta),(\xi_1 v_1+\cdots+\xi_d v_d,\theta)\rangle\\
&= (A_1(\xi_1 v_1+\cdots+\xi_d v_d,\xi_1 v_1+\cdots+\xi_d v_d)+(A_2\theta,\theta)\\
&= \sum_{i,j=1}^{d}(A_1 v_i,v_j)\xi_i\xi_j + (A_2\theta,\theta) = (B\xi,\xi)+(A_2\theta,\theta)\,.
\end{aligned} \tag{3.14}$$

From (3.14) it follows that the operator $\bar{A}$ is block diagonal,

$$\bar{A} = \begin{pmatrix} B & 0\\ 0 & A_2\end{pmatrix},$$

and it can be transformed to a diagonal form $\tilde{A}$,

$$\tilde{A} = \left\{\begin{matrix}\lambda_1 & & & \\ & \lambda_d & & \\ & & \nu_1 & \\ & & & \nu_k\end{matrix}\right\} \tag{3.15}$$

by orthogonal transformation in $L_2(\mathbb{R}^k)$ and $\mathbb{R}^k$, respectively. Let $w_1,\dots,w_d$ and $\nu_1 \dots,\nu_k$ be orthonormal in H and $\mathbb{R}^k$ eigenvectors of B and A_2, respectively. Obviously orthonormal eigenvectors of $\overline{A}$ have the form $\zeta_i = (w_i,0)$ and $\zeta_m = (0,\nu_m)$, $i = 1,\dots,d$, $m = 1,\dots,k$.

Then due to Courant's principle (see [8]), we have

$$\sum_{j=1}^{d}(\overline{A}z_j,z_j) \le \sum_{j=1}^{d}(\overline{A}\zeta_j,\zeta_j) \tag{3.16}$$

where $\{\zeta_j\}_{j=1,\dots,d}$ are eigenvectors of the matrix $\overline{A}$ in L corresponding to the greater eigenvalues of the block operator $\overline{A}$. Without loss of generality, we assume that (due to the block structure of $\overline{A}$) eigenvectors of $\overline{A}$ are

$$(w_1(t,y_0),0),\dots,(w_{d-k_1}(t,y_0),0),(0,\nu_1(t,y_0)),\dots,(0,\nu_{k_1}(t,y_0)),$$

where $0 \le k_1 \le k$, $k_1 \in \mathbb{N}$. Note that $A_2 = \frac{G_1}{2b}\mathrm{Id}$, where Id is the $(k\times k)$ identity matrix. This proves Proposition 3.2. □

COROLLARY 3.3 *There exists orthonormal in* $H = L_2(\mathbb{R}^n)$ *vectors* $w_1(t,y_0),\dots,w_{d-k_1}(t,y_0)$, $0 \le k_1 \le k$, *such that*

$$\begin{aligned}\sum_{j=1}^{d}(\overline{A}z_j,z_j) \le -\nu\sum_{j=1}^{d-k_1}\|\bigtriangledown w_j\|^2 - \sum_{j=1}^{d-k_1}\lambda_0\|w_j\|^2 - \sum_{j=1}^{d-k_1}\int f'_u(u)w_j^2\,dx \\ +\frac{b}{2}\sum_{j=1}^{d-k_1}\int |g'_w|^{1-\delta}|w_j|^2\,dx + \frac{G_1}{2b}k_1\,.\end{aligned} \tag{3.17}$$

Denote

$$f'_-(u) = \max\left(0, -f'_u(u) - \frac{\lambda_0}{2}\right). \tag{3.18}$$

Then obviously

$$\begin{aligned}\sum_{j=1}^{d}(\overline{A}z_j,z_j) \le -\frac{\lambda_0}{2}(d-k_1) + \frac{G_1}{2b}k_1 - \nu\int\sum_{j=1}^{d-k_1}|\bigtriangledown w_j|^2\,dx \\ +\sum_{j=1}^{d-k_1}\int f'_-(u)w_j^2\,dx + \frac{b}{2}\int |g'_w|^{1-\delta}\sum_{j=1}^{d-k_1}|w_j|^2\,dx\,.\end{aligned} \tag{3.19}$$

Using the Lieb-Thirring inequality (see [7]), that is,

$$\int\sum_{j=1}^{d-k_1}|\bigtriangledown w_j|^2 \ge C_0\int\left(\sum_{j=1}^{d-k_1}|w_j|^2\right)^{1+\frac{2}{n}}dx \tag{3.20}$$

where the constant C_0 does not depend on d and k_1, we obtain

$$\sum_{j=1}^{d}(\bar{A}z_j,z_j)\le -\frac{\lambda_0}{2}(d-k)+\frac{G_1}{2b}k-\nu C_0\int\left(\sum_{j=1}^{d-k_1}|w_j|^2\right)^{1+\frac{2}{n}}dx \tag{3.21}$$
$$+\int f'_-(u)\left(\sum_{j=1}^{d-k_1}|w_j|^2\right)dx+\frac{b}{2}\int|g'_\omega|^{1-\delta}\left(\sum_{j=1}^{d-k_1}|w_j|^2\right)dx.$$

Let $\rho(x):=\sum_{j=1}^{d-k_1}|w_j|^2$. Then (3.21) can be rewritten as

$$\sum_{j=1}^{d}(\bar{A}z_j,z_j)\le -\frac{\lambda_0}{2}(d-k)+\frac{G_1}{2b}k-\nu C_0\int|\rho(x)|^{1+\frac{2}{n}}dx \tag{3.22}$$
$$+\int f'_-(u)\rho(x)dx+\frac{b}{2}\int|g'_\omega|^{1-\delta}\rho(x)dx.$$

Due to the Hölder inequality, we have

$$\int f'_-(u)\rho(x)dx\le\left(\int|f'_-(u)|^{1+\frac{n}{2}}dx\right)^{\frac{2}{n+2}}\cdot\left(\int|\rho(x)|^{1+\frac{2}{n}}\right)^{\frac{n}{n+2}}. \tag{3.23}$$

In the same manner, we have

$$\int|g'_\omega|^{1-\delta}\rho(x)dx\le\left(\int|g'_\omega|^{(1-\delta)(1+\frac{n}{2})}\right)^{\frac{2}{2+n}}\cdot\left(\int|\rho(x)|^{1+\frac{2}{n}}\right)^{\frac{n}{n+2}}.$$

Let

$$V=V(t)=\left(\int|f'_-(u)|^{\frac{n}{2}+1}dx\right)^{\frac{2}{n+2}}+\frac{b}{2}\left(\int|g'_\omega|^{(1-\delta)\cdot(\frac{n}{2}+1)}dx\right)^{\frac{2}{n+2}},$$
$$q=q(t)=\left(\int|\rho(x)|^{1+\frac{2}{n}}dx\right)^{\frac{n}{n+2}},$$

It is clear that the right-hand side of (3.22) is not greater than $r(t)$ where

$$r(t):=-\frac{\lambda_0}{2}(d-k)+\frac{G_1}{2b}k+V(t)q(t)-\nu C_0(q(t))^{1+\frac{2}{n}}. \tag{3.24}$$

The function $r(t)$ for fixed t has a maximum with respect to q when $q=q_0$, where

$$q_0=C_3(n)\nu^{-\frac{n}{2}}\cdot V^{\frac{n}{2}}.$$

Therefore (3.22), (3.23), and (3.24) yield

$$\sum_{j=1}^{d}(\bar{A}z_j,z_j)\le -\frac{\lambda_0}{2}(d-k)+\frac{G_1}{2b}k+C_4\cdot\nu^{-\frac{n}{2}}V^{1+\frac{2}{n}}. \tag{3.25}$$

Thus to provide (3.8) it is sufficient that

$$\overline{\lim_{t\to+\infty}}\frac{1}{t}\int_0^t C_4\nu^{-\frac{n}{2}}V^{1+\frac{2}{n}}(\tau)d\tau-\frac{\lambda_0}{2}(d-k)+\frac{G_1}{2b}k<0. \tag{3.26}$$

To guarantee (3.26) we follow [2] but in a slightly different way. Note that (3.26) implies that $\dim\mathcal{A} \le d$ or, equivalently,

$$\dim\mathcal{A} \le k + \frac{G_1}{b}\lambda_0^{-1}k + 2C_4\lambda_0^{-1}\nu^{-\frac{n}{2}} \overline{\lim_{t\to\infty}} \frac{1}{t}\int_0^t V^{1+\frac{n}{2}}(\tau)d\tau + \varepsilon \tag{3.27}$$

for any ε. It remains to estimate the integral in (3.22). Let us recall that

$$V(t) = V_1(t) + V_2(t)$$

where

$$V_1(t) := \left(\int |f'_-(u)|^{\frac{n}{2}+1}\,dx\right)^{\frac{2}{n+2}} \quad \text{and} \quad V_2(t) := \frac{b}{2}\int |g'_\omega|^{(1-\delta)(\frac{n}{2}+1)}\,dx\,.$$

Note that $V_2(t) \le C_5 b$, where C_5 is some constant. $V_1(t)$ can be estimated in the following way. We consider two cases: (1) $f'(0) \ne 0$ and (2) $f'(0) = 0$. We show that in each case $|f'_-(u)| = 0$ for $|u| \le \delta$, where δ is sufficiently small.

Case 1. $f'(0) \ne 0$.

Obviously, for sufficiently small $\delta > 0$, we have $f(u) = f'(0)u + o(u)$ for $|u| \le \delta$ or $f(u)\cdot u = f'(0)u^2 + o(u^2)$ for $|u| \le \delta$. Since $f(u)\cdot u \ge 0$, we obtain that $f'(0) > 0$ for $|u| \le \delta$ and, due to continuity, $f'(u) \ge 0$ if $|u| \le \delta$. Hence $f'_-(u) = 0$.

Case 2. $f'(0) = 0$.

It follows from Condition 2.1 that

$$|f'(u)| \le |u|^\alpha C_1(u)\,. \tag{3.28}$$

Hence

$$|f'(u)| \le \delta^\alpha C_1 \quad \text{for } |u| \le \delta\,.$$

Choosing δ so small that

$$C_1\delta^{\alpha_0} \le \frac{\lambda_0}{2}\,,$$

we obtain that $|f'_-(u)| = 0$ for $\le |u| \le \delta$.

Let us recall that our main goal is to estimate

$$V_1(t) = \left(\int |f'_-(u)|^{\frac{n}{2}+1}\,dx\right)^{\frac{2}{n+2}}$$

where $u(t,x) = U_{w_0}(t,0)u_0$. Since $y(t) = (u(t,y_0),\omega(t))$ belongs to the attractor $\mathcal{A} \subset H_{0,\gamma}\times\mathbb{R}^k \subset H_{0,0}\times\mathbb{R}^k$, we have

$$\|u(t),\cdot)\|^2 = \int u^2(t,\cdot)dx \le C_{\mathcal{A}} \quad \text{for all } t \ge 0 \tag{3.29}$$

and $y(t) \in \mathcal{A}$. Therefore due to (3.29) the measure of all $\mathbb{R}^n$ for which $|u(t,x)| \ge \delta$ satisfies

$$\text{meas}\{x \in \mathbb{R}^n : |u(t,x)| \ge \delta\} \le C_{\mathcal{A}}\cdot\delta^{-2}\,. \tag{3.30}$$

Thus taking into account (3.29), (3.30), and $f'(u) \geq -C$, we conclude that

$$V_1(t) \leq C_A \delta^{-2} C^{\frac{n}{2}+1}.$$

On the other hand, $V_2(t) \leq C_7 \cdot b$. Thus we obtain

$$\dim A \leq k + \frac{G_1}{b} \lambda_0^{-1} k + 2C_4 \lambda_0^{-1} \nu^{-\frac{n}{2}} (C_A \delta^{-2} C^{\frac{n}{2}+1} + C_7 b)^{\frac{n}{2}+1}. \tag{3.31}$$

We recall that the parameter b in (3.31) is an arbitrary positive number. Analogously to [4], one can find an optimal value for b (depending on k) that yields

$$\dim A \leq k + C_8 k^{\frac{n}{n+2}} + C_9 \nu^{-\frac{n}{2}} \tag{3.32}$$

where C_8 and C_9 depend on δ, λ_0, C_A, $\|g\|_{C(\mathbb{T}^k, H_{0,0})}$, and n but not on k. As a result of the estimate (3.32), we obtain that the uniform attractor A_{RDE} admits

$$\dim A_{\mathrm{RDE}} \leq k + C_8 k^{\frac{n}{n+2}} + C_9 \nu^{-\frac{n}{2}}. \tag{3.33}$$

Remark 3.4. One can construct an example of a reaction-diffusion equation having a uniform attractor A^*_{RDE} such that

$$\dim A^*_{\mathrm{RDE}} \geq k. \tag{3.34}$$

In other words, the main term k in the estimates (3.33) is exact.

Remark 3.5. In the autonomous case, that is, $k = 0$, estimate (3.33) becomes the well-known upper bound $C \cdot \nu^{-n/2}$ for the Hausdorff dimension of the attractor of the autonomous reaction-diffusion equation (see [2]).

3.2 Two-Dimensional Navier-Stokes Equations with Quasi-Periodic External Force

We consider the family of Navier-Stokes systems (2.12) in the domain

$$\Omega = \{(x_1, x_2) : -\infty < x_1 < +\infty,\ 0 \leq x_2 \leq d\}$$

where $g(x,t) = \tilde{g}(x, \omega(t))$, $\omega(t) = [\alpha t + \omega_0]$, $\omega_0 \in \mathbb{T}^k$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$, and $\omega = (\omega_1, \ldots, \omega_k)$. As was shown in Section 2, this family is equivalent to the autonomous system

$$\begin{cases} \partial_t \bar{u} = -\nu L\bar{u} - \Pi \sum_{i=1}^2 u_i \frac{\partial \bar{u}}{\partial x_i} + \tilde{g}_0(x, \omega),\ \partial_t \omega = \alpha, \\ \bar{u}|_{t=0} = \bar{u}_0,\ \omega|_{t=0} = \omega_0,\ \bar{u}_0 \in H(\gamma),\ \omega_0 \in \mathbb{T}^k, \end{cases} \tag{3.35}$$

or, equivalently,

$$\partial_t \bar{y} = M(\bar{y}), \quad \bar{y} = (\bar{u}, \omega),$$

where

$$M(\bar{y}) = \left(-\nu L\bar{u} - \Pi \sum_{i=1}^{2} u_i \frac{\partial \bar{u}}{\partial x_i} + \tilde{g}_0(x, \omega), \alpha \right). \tag{3.36}$$

Due to Section 2, equation (3.35) generates a semigroup $\{S(t)\}$,

$$S(t) y_0 = y(t), \quad y_0 = (u_0,\ \omega_0),\ y(t) = (u(t), \omega(t)),$$

which has an attractor $A_{\rm NS} \subset H(\gamma) \times \mathbb{R}^k$.

Moreover, the operators are quasi-differentiable on the attractor $A \subset H(\gamma) \times \mathbb{T}^k$ (the case of unbounded domain makes no difference). Its quasi differential $S'(t,y_0)(\overline{v}_0,\eta) = (\overline{v}(t),\eta)$ is a solution of the variation equation

$$\begin{cases} \partial_t \overline{v} = -\nu L\overline{v} - B(\overline{u}(t),\overline{v}) - B(\overline{v},\overline{u}(t) + g'_\omega(x,\omega(t))\eta \\ \partial_t \eta \equiv 0,\ \eta = (\eta_1,\dots,\eta_k),\ \overline{v} = \overline{v}(x,t) = \overline{v}(t)\,, \end{cases} \tag{3.37}$$

where $B(\overline{u},\overline{v}) = \Pi \sum_{i=1}^2 u_i \partial_i \overline{u}$, $\overline{u}(t,x) = \overline{u}(t,y_0)$, $y_0 \in A$. Note that equation (3.37) can be rewritten in the form

$$\partial_t \overline{z} = M'(y(t))\overline{z}, \quad \overline{z}|_{t=0} = \overline{z}_0\,,\ \overline{z}_0 = (\overline{v},\eta)\,. \tag{3.38}$$

Here $\overline{y}(t) = (\overline{u}(t),\omega(t))$, $y_0 \in A$, $\omega(t) = [\alpha t + \omega_0]$, $\overline{u}(t) = U_{\omega_0}(t,0)\overline{u}_0$, is a solution of (3.35).

$$M'(\overline{y}(t))\overline{z} = (-\nu L\overline{v} - B(\overline{u}(t),\overline{v}) - B(\overline{v},\overline{u}(t)) + g'_\omega(x,\omega(t))\eta, 0)\,. \tag{3.39}$$

Let us recall that our basic task is to estimate the Hausdorff dimension of attractor $A_{\rm NS} \subset H(\gamma) \times \mathbb{R}^k$. We estimate the Hausdorff dimension of attractor $A_{\rm NS}$ with respect to the norm $H \times \mathbb{R}^k (H(0) = H)$. We follow the scheme developed for reaction-diffusion equations in $\mathbb{R}^n$. To apply the Constantin-Foias-Temam formula, we have to estimate

$$\sum_{j=1}^d (M'(y(t))\overline{z}_j, \overline{z}_j)_{(L_2(\Omega))}{}^2 \tag{3.40}$$

where E_d is d-dimensional space in $(H_2(\Omega))^2$ (the standard Sobolev space of order 2) and $\{\overline{z}_j\} \subset E_d,\ j = 1,\dots,d$, are the orthonormal family in $(L_2(\Omega))^2 \times \mathbb{R}^k$ belonging to $(H_2(\Omega))^2 \times \mathbb{R}^k$. It is easy to check that

$$\begin{aligned} (M'(y(t))\overline{z}_j,\overline{z}_j) &= -\nu(L\overline{v},\overline{v}) - (B(\overline{u}(t,y_0),\overline{v}),\overline{v}) \\ &\quad - (B(\overline{v},\overline{u}(t,y_0)),\overline{v}) + \int_\Omega (\vec{g}'_\omega,\eta)_{\mathbb{R}^k}\overline{v}\,dx \\ &\le -\nu(L\overline{v},\overline{v}) - (B(\overline{u}(t,y_0),\overline{v}),\overline{v}) - (B(\overline{v},\overline{u}(t,y_0)),\overline{v}) \\ &\quad + \frac{b}{2}\int_\Omega |\vec{g}'_\omega|^{1-\delta}|\overline{v}|^2\,dx + |\eta|^2 \frac{1}{2b}\int_\Omega |\vec{g}'_\omega|^{1+\delta}\,dx \\ &= (A_1\overline{v},\overline{v}) + (A_2\eta,\eta) \end{aligned} \tag{3.41}$$

where

$$\begin{aligned} (A_1\overline{v},\overline{v}) &:= -\nu(L\overline{v},\overline{v}) - (B(\overline{u}(t,y_0),\overline{v}),\overline{v}) - (B(\overline{v},\overline{u}),\overline{v}) \\ &\quad - \frac{b}{2}\int |\vec{g}'_\omega|^{1-\delta}|v|^2\,dx \end{aligned} \tag{3.42}$$

$$(A_2\eta,\eta) = \frac{G_1}{2b}(\eta,\eta)\,, \quad G_1 = \sup_{\omega\in\mathbb{T}^k}\int_\Omega |g'_\omega|^{1+\delta}\,dx\,. \tag{3.43}$$

Before we estimate $\sum_{j=1}^{d}(M'(y(t))\bar{z}_j,\bar{z}_j)$, note that

$$\lambda_1 =: \inf\{-\Delta v, v),\ v\in C_0^\infty(\Omega),\ \|v\|=1\} > 0\,. \tag{3.44}$$

We use this fact throughout this paper. Let us estimate (3.41) using Proposition 3.2. Indeed,

$$\begin{aligned}
\sum_{j=1}^{d}(M'(y(t)\bar{z}_j,z_j) &\le \sum_{j=1}^{d-k_1}(A_1w_j,w_j)+\sum_{m=1}^{k_1}(A_2\xi_m,\xi_m)\\
&= -\nu\sum_{j=1}^{d-k_1}(L\bar{w}_j,\bar{w}_j)+\sum_{j=1}^{d-k_1}(\bar{B}(\bar{u}(t,y_0),\bar{w}_j),\bar{w}_j)\\
&\quad+\sum_{j=1}^{d-k_1}(\bar{B}(\bar{w}_j,\bar{u}(t,y_0)),\bar{w}_j)+\frac{b}{2}\sum_{j=1}^{d-k_1}\int|g'_\omega|^{1-\delta}|\bar{w}_j|^2dx+\frac{G_1}{2b}k_1\\
&\le -\frac{\nu\lambda_1}{2}(d-k_1)+\frac{G_1}{2b}k-\frac{\nu}{2}\sum_{j=1}^{d-k_1}\int|\nabla\bar{w}_j|^2\,dx\\
&\quad+\sum_{j=1}^{d-k_1}\int 2|\nabla\bar{u}|\,|\bar{w}_j|^2\,dx+\frac{b}{2}\sum_{j=1}^{d-k_1}\int|g'_\omega|^{1-\delta}|\bar{w}_j|^2dx\\
&\le -\frac{\nu\lambda_1}{2}(d-k)+\frac{G_1}{2b}k-\frac{\nu}{2}\int\sum_{j=1}^{d-k_1}|\nabla\bar{w}_j|^2\,dx\\
&\quad+\sum_{j=1}^{d-k_1}\int 2|\nabla\bar{u}|\,|\bar{w}_j|^2\,dx+\frac{b}{2}\int|g'_\omega|^{1-\delta}\left(\sum_{j=1}^{d-k_1}|\bar{w}_j|^2\right)dx,
\end{aligned}$$

where $\bar{w}_j$ and ξ_m are due to Proposition 3.2 and $y_0\in A_{\mathrm{NS}}$. Denote $\rho(x):=\sum_{j=1}^{d-k_1}|\bar{w}_j|^2$.

Hence, due to the Lieb-Thirring inequality, we have

$$\begin{aligned}
&\sum_{j=1}^{d}(M'(\bar{y}(t))\bar{z}_j,\bar{z}_j)\\
&\quad\le -\frac{\nu\lambda_1}{2}(d-k)+\frac{G_1}{2b}k-\frac{\nu}{2}\int\sum_{j=1}^{d-k_1}|\nabla\bar{w}_j|^2\,dx\\
&\qquad+\frac{b}{2}\int|g'_\omega|^{1-\delta}\rho(x)dx+\int 2|\nabla\bar{u}|\cdot\rho(x)dx\\
&\quad\le -\frac{\nu\lambda_1}{2}(d-k)+\frac{G_1}{2b}k-\nu C_0\int_\Omega\rho^2(x)dx+\int_\Omega 2|\nabla u|\rho(x)dx\\
&\qquad+\frac{b}{2}\left(\varepsilon\int\rho^2(x)dx+\frac{1}{4\varepsilon}\int|g'_\omega|^{2-2\delta}dx\right)
\end{aligned} \tag{3.45}$$

$$\leq -\frac{\nu\lambda_1}{2}(d-k)+\frac{G_1}{2b}k + \int\left(2|\bigtriangledown u|\rho(x)dx-\nu C_0\rho^2(x)+\frac{\nu C_0}{2}\rho^2\right)dx + \frac{b^2}{8C_0\nu}\int |g'_\omega|^{2(1-\delta)}\,dx\,.$$

In (3.45) we chose $\varepsilon>0$ such that $b\varepsilon = C_0\nu$. Note that the expression

$$2|\bigtriangledown u|\rho-\frac{\nu C_0}{2}\rho^2 \tag{3.46}$$

has its maximum at $2|\bigtriangledown u|^2/C_0\nu$. Therefore we obtain from (3.45)

$$\sum_{j=1}^{d}(M'(\bar{y}(t))\bar{z}_j,\bar{z}_j)\leq -\frac{\nu\lambda_1}{2}(d-k)+\frac{G_1}{2b}k+\frac{b^2\cdot G_2}{8C_0\nu}+\frac{2}{C_0\nu}\int|\bigtriangledown u|^2\,dx\,. \tag{3.47}$$

Due to estimate (2.14) we have

$$\nu\int_0^t \|\bigtriangledown u\|^2 dt \leq \|\bar{u_0}\|^2 + t\nu^{-1}\lambda_1^{-1}G_1^2+2\nu^{-1}\lambda_1^{-1}\|\bar{u_0}\|\cdot G_1\,.$$

Hence (3.47) yields

$$\begin{aligned}\sum_{j=1}^{d}(M'(\bar{y}(t))\bar{z}_j,\bar{z}_j)\leq -\frac{\nu\lambda_1}{2}(d-k)+\frac{G_1}{2b}k+\frac{b^2G_2}{8C_0\nu}+\frac{2\|\bar{u_0}\|^2}{C_0\nu^2}\\ +\frac{2t}{C_0\nu^3}\lambda_1^{-1}G_1^2+\frac{\lambda_1^{-1}\|\bar{u_0}\|}{\nu^3}G_1\,.\end{aligned} \tag{3.48}$$

From (3.48) it follows that

$$\dim A \leq k+\frac{G_1\nu^{-1}}{2b\lambda_1}k+\frac{b^2G_2}{4C_0\lambda_1}\nu^{-2}+\frac{2G_1^2}{C_0\lambda_1^2}\nu^{-4}\,. \tag{3.49}$$

Recall that b is an arbitrary positive parameter in (3.49). It is not difficult to see that the optimal value for b in (3.49) is equal to

$$b(k)=\left(\frac{G_1\nu}{G_2}\cdot C_0\right)^{\frac{1}{3}}\cdot k^{\frac{1}{3}}\,.$$

Thus we have

$$\dim A\leq k+\beta\cdot k^{\frac{2}{3}}+\frac{2G_1^2}{C_0\cdot\lambda_1^2}\nu^{-4}\,. \tag{3.50}$$

Remark 3.6. This result is to be contrasted with the well-known estimates for the Hausdorff dimension of the attractor for the $N-S$ system in a bounded domain $\Omega\subset\mathbb{R}^2$:

(a) $\dim A\leq C\nu^{-2}$ (autonomous case, [3]) and
(b) $\dim A\leq k+C_1k^{1/3}+C_2\nu^{-2}$ (nonautonomous case).

In the case of $\Omega=\{(x_1,x_2):-\infty<x_1<+\infty,\ 0\leq x_2\leq d\}$, we have

(a_1) $\dim \mathcal{A} \le C\nu^{-4}$ and
(b_1) $\dim \mathcal{A} \le k + C_3 k^{2/3} + C_4\nu^{-4}$.

The presence of ν^{-4} (instead of ν^{-2}) is based on spectral arguments. Indeed, in the case of a bounded domain $\Omega \subset \mathbb{R}^2$, the spectrum of $-\Delta$ is discrete and enables us to prove (b) (see [3]). However, in the case $\Omega = \{(x_1,x_2) : -\infty < x_1 < +\infty,\ 0 \le x_2 \le d\}$, one can easily see that the spectrum of $-\Delta$ is

$$\sigma(-\Delta) = \left[\left(\frac{\pi}{d}\right)^2, \infty\right]. \tag{3.51}$$

A proof of (3.51) is based on the following simple argument. Let us consider the equation

$$-\Delta u + \lambda^2 u = g, \quad g \in L_2(\Omega), \tag{3.52}$$

or, equivalently,

$$\begin{cases} -\Delta u + \lambda^2 u = h,\ h = -g \in L_2(\Omega), \\ u(x_1,0) = u(x_1,d) = 0. \end{cases} \tag{3.53}$$

We seek a solution of (3.52) in the form

$$u(x_1,x_2) = \sum_{n=1}^{\infty} f_n(x_1) e_n(x_2) \tag{3.54}$$

where $e_n(x_2)$ is a solution of the following eigenvalue problem:

$$\begin{cases} e_n''(x_2) + \mu_n^2 e_n(x_2) = 0,\ 0 \le x_2 \le d, \\ e_n(0) = e_n(d) = 0, \end{cases} \tag{3.55}$$

with $\mu_n = \pi n/d$. It follows from (3.53), (3.54), and (3.55) that $f_n(x_1)$ satisfies the following equation:

$$f_n''(x_1) + \left(\lambda^2 - \left(\frac{\pi n}{d}\right)^2\right) f_n(x_1) = h_n(x_1) \tag{3.56}$$

where $-\infty < x_1 < +\infty$ and $h_n(x_1)$ is due to $h(x_1,x_2) = \sum_{n=1}^{\infty} h_n(x_1) e_n(x_2)$. Obviously $h_n(x_1) \in L_2(-\infty,+\infty)$, and if $\omega := \lambda^2 - (\pi/d)^2$ does not belong to the spectrum of the operator $-\Delta_2$ where $\Delta_2 := \frac{\partial^2}{\partial x_2^2}$, then $f_n(x_1) \in L_2(-\infty,+\infty)$. Thus from (3.56) it follows that, for $n = 1$, the $[(\pi/d)^2, \infty]$ belongs to the continuous spectrum of $-\Delta$. It is not difficult to prove that there is no other spectrum point.

Remark 3.7. Note that λ_1 as defined by (3.44) is equal to $(\pi/d)^2$. Indeed, $(\pi/d)^2$ is the first eigenvalue of $-\Delta_2$ with the Dirichlet boundary condition on $\Omega_2 = [0,d]$.

Consider

(3.57)

$$
\begin{aligned}
(-\Delta_2 u, u) &= -\int_{-\infty}^{\infty} dx_1 \int_0^d \left(\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\right) u(x_1,x_2)\,dx_2 \\
&= \int_{-\infty}^{\infty} dx_1 \int_0^d -\frac{\partial^2 u}{\partial x_1^2} u(x_1,x_2)\,dx_2 \\
&\quad + \int_{-\infty}^{\infty} dx_1 \int_0^d -\frac{\partial^2 u}{\partial x_2^2} u(x_1,x_2)\,dx_2 \\
&\geq \left(\int_{-\infty}^{\infty}\int_0^d u^2(x_1,x_2)\,dx_1 dx_2\right) \cdot \left(\frac{\pi}{d}\right)^2 + \int_{-\infty}^{\infty} dx_1 \int_0^d -\frac{\partial^2 u}{\partial x_1^2} u(x_1,x_2)\,dx_2 \\
&\geq \left(\frac{\pi}{d}\right)^2 \int_\Omega u^2(x_1,x_2)\,dx_1\,dx_2 .
\end{aligned}
$$

Therefore $\lambda_1 = (\pi/d)^2$. This proves the last remark.

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V. V. Chepyzhov
Institute for Problems of Information Transmissions (IPPI)
Bolshoy Karetnyi 19
GSP-4, 101447 Moscow
RUSSIA

M. A. Efendiev
Freie Universität Berlin
FB Mathematik und Informatik
14195 Berlin
Arnimallee 2-6
GERMANY
E-mail: efendiev@math.fu-berlin.de

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