PERGAMON

Nonlinear Analysis 44 (2001) 811–819

Nonlinear Analysis

www.elsevier.nl/locate/na

A note on the fractal dimension of attractors of dissipative dynamical systems

V.V. Chepyzhov[a], A.A. Ilyin[b,*]

[a] *Institute for Problems of Information Transmission, Bolshoĭ Karetnyĭ 19, Moscow 101447, Russia*
[b] *Keldysh Institute of Applied Mathematics, Miusskaya Sq. 4, Moscow 125047, Russia*

Received 23 July 1998; accepted 27 May 1999

Keywords: Fractal dimension; Attractors; Navier–Stokes equations

1. Introduction

Let X be a compact set in a Hilbert space H: $X \Subset H$. We recall the definitions of the Hausdorff and fractal dimensions of X (see, for instance, [8]).

Definition 1.1. The Hausdorff dimension of X in H is the number

$$\dim_H X = \inf\{d \,|\, \mu(X,d) = 0\},$$

where $\mu(X,d) = \lim_{\varepsilon\to 0+} \mu(X,d,\varepsilon)$ and $\mu(X,d,\varepsilon) = \inf \sum r_i^d$, the infimum being taken over all coverings of X by balls $B(x_i, r_i)$ centred at x_i with radii $r_i \le \varepsilon$.

Definition 1.2. The fractal dimension of X in H is the number

$$\dim_F X = \limsup_{\varepsilon\to 0+} \frac{\log_2(N_X(\varepsilon))}{\log_2(1/\varepsilon)},$$

where $N_X(\varepsilon)$ is the minimum number of balls of radius ε which is necessary to cover X.

It is well known that $\dim_H(X) \le \dim_F(X)$, where the inequality can be strict.

In our work we shall be dealing with estimates of the fractal dimension of the invariant sets (attractors) of the semigroups generated by infinite-dimensional dynamical

* Corresponding author. Fax: 095-972-07-37.
E-mail addresses: chep@ippi.ras.ru (V.V. Chepyzhov), ilyin@spp.keldysh.ru (A.A. Ilyin).

PII: S0362-546X(99)00309-0

systems. It was shown in [8] that if the sums of the first m global Lyapunov exponents of the invariant set X have a majorant $f(m)$, then $\dim_H X \le d_0$, where $f(d_0)=0$. For the fractal dimension it was shown there that $\dim_F X \le \text{const}\, d_0$, where const > 1. In our work we show (see Theorem 2.1) that if $f(d)$ is concave, then $\dim_F X \le d_0$. In other words, the Hausdorff and fractal dimensions have the same upper bound.

Turning to the applications we first note that in all the examples of partial differential equations considered in [8] the function $f(d)$ is concave. Secondly, in the two particular examples (a reaction–diffusion system and two-dimensional Navier–Stokes system) we obtain $f(d)$ explicitly and, consequently, we obtain explicit upper bounds of the fractal dimension of the attractor in terms of the physical parameters.

Finally, we observe that Theorem 2.1 can be obtained by a careful analysis of the corresponding results in [8]. However, we prove Theorem 2.1 generalizing to the infinite-dimensional case the method of [2], where the Kaplan–Yorke estimate for the fractal dimension of the Lorenz attractor has been obtained.

2. Main estimate

Suppose that a semigroup S_t of continuous operators acts in a Hilbert space H and let X be a compact strictly invariant set of S_t: $S_tX = X$, $X \Subset H$. We suppose that the semigroup S_t is uniformly quasidifferentiable on X for each t, that is, for $u, v \in X$ there exists a linear operator $DS_t(u)$ such that

$$\|S_t(u) - S_t(v) - DS_t(u)(u-v)\| \le h(r)\|u-v\|, \tag{2.1}$$

where $\|u-v\| \le r$, $h(r) = o(1)$, and $\sup_{t\in[0,1]}\sup_{u\in X}\|DS_t(u)\|_{\mathscr{L}(H,H)} < \infty$.

We further suppose that the operator $L = DS_t(u)$ is compact. This involves no loss of generality (see [8], Proposition V.1.1) and we are doing so for the sake of simplicity only. Then the image of the unit ball $B(0,1)$ of H centred at the origin is the ellipsoid $\mathscr{E} = LB(0,1)$ with semiaxes $\alpha_1(t,u) \ge \alpha_2(t,u) \ge \dots$, that are the eigenvalues of the self-adjoint operator $(L^*L)^{1/2}$.

Following [8] we set for integer k

$$\omega_0(t,u) = 1,$$

$$\omega_k(t,u) = \alpha_1(t,u)\alpha_2(t,u)\cdots\alpha_k(t,u),$$

$$\bar{\omega}_k(t) = \sup_{u\in X} \omega_k(t,u).$$

Then $\lim_{t\to\infty} t^{-1}\log\bar{\omega}_k(t) = q(k)$ and hence for any $\varepsilon > 0$

$$\bar{\omega}_k(t) \le c_k e^{(q(k)+\varepsilon)t} \tag{2.2}$$

for some positive constants $c_k = c_k(\varepsilon)$, where $q(k)$ is the sum of the first k global Lyapunov exponents (see [8], Section V.2.3).

The above definitions can be generalized for an arbitrary real $d = k + s$, $0 \le s < 1$ by setting

$$\omega_d(t,u) = \omega_k(t,u)^{1-s}\omega_{k+1}(t,u)^s.$$

Using (2.2) we obtain the similar bound for $\bar{\omega}_d(t)$

$$\bar{\omega}_d(t) = \sup_{u \in X} \omega_k(t,u)^{1-s}\omega_{k+1}(t,u)^s \le c_d \mathrm{e}^{(q(d)+\varepsilon)t}, \tag{2.3}$$

where $c_d = c_d(\varepsilon) = c_k(\varepsilon)^{1-s}c_{k+1}(\varepsilon)^s$ and $q(d) = q(k+s) = (1-s)q(k) + sq(k+1)$.

Lemma 2.1. *Let $\mathscr{E}$ be an ellipsoid with semiaxes $\alpha_1 \ge \alpha_2 \ge \dots$. If $r \ge \alpha_{n+1}$, then the minimum number of balls of radius $r\sqrt{n+1}$ which is necessary to cover $\mathscr{E}$ is*

$$N_{\mathscr{E}}(r\sqrt{n+1}) \le 2^n \frac{\omega_k(\mathscr{E})}{r^k}, \tag{2.4}$$

where $\alpha_{k+1} \le r \le \alpha_k$ and $\omega_k(\mathscr{E}) = \alpha_1 \cdots \alpha_k$. (If $r > \alpha_1(\mathscr{E})$, then $N_{\mathscr{E}}(r\sqrt{n+1}) = 1$.)

In addition, for any $\eta > 0$

$$N_{\mathscr{E}+B(0,\,\eta)}(\eta + r\sqrt{n+1}) \le N_{\mathscr{E}}(r\sqrt{n+1}) \le 2^n \frac{\omega_k(\mathscr{E})}{r^k}. \tag{2.5}$$

Proof. Estimate (2.4) is the well-known covering lemma [3,8]. Estimate (2.5) follows from the observation that if N balls of radius ε cover $\mathscr{E}$, then N balls centred at the same points with radius $\varepsilon + \eta$ cover $\mathscr{E} + B(0,\eta)$. The proof is complete. □

We first assume that $q(d)$ is majorized by a linear function; the role of concavity will become clear in Corollaries 2.1 and 2.2.

Theorem 2.1. *Suppose that for $d \ge 0$, $q(d) \le f(d) = -ad + b$, where $a, b > 0$. Then*

$$\dim_F X \le d_0 = b/a. \tag{2.6}$$

Proof. We fix a real p, $0 < p < a$, and an integer $n > d_0$ so that $np + f(n) < 0$. Then there exists an $\varepsilon > 0$ (from (2.2)) such that

$$\mathrm{e}^{-npt} > c_n \mathrm{e}^{(f(n)+\varepsilon)t} \tag{2.7}$$

for $t \ge t(n,\varepsilon)$. We fix p, n, $t(n,\varepsilon)$, and also an arbitrary real $D > 0$ satisfying

$$D > \max_{0 \le d \le n} (d + f(d)/p) = b/p.$$

Then $-p(D-d) + f(d) < 0$ for $d \in [0,n]$ and hence $\bar{\omega}_d(t)\mathrm{e}^{-tp(D-d)} \to 0$ as $t \to \infty$. Using this we can choose $t^* \ge t(n,\varepsilon)$ so large that for $\sigma = \mathrm{e}^{-pt^*}$

$$2^{n+D}(n+1)^{D/2} \max_{0 \le d \le n} \bar{\omega}_d(t^*)\sigma^{D-d} < 1. \tag{2.8}$$

We now set $t = t^*$ in (2.1) and let r be so small that $h = \sigma\sqrt{n+1} = \mathrm{e}^{-pt^*}\sqrt{n+1}$. Then $N_X(r)$ balls cover X and by the invariance property

$$X = \bigcup_{j=1}^{N_X(r)} B(u_j, r) \cap X = \bigcup_{j=1}^{N_X(r)} S_{t^*}(B(u_j,r) \cap X). \tag{2.9}$$

We denote by $\mathscr{E}_j$ the ellipsoids $DS_{t^*}(u_j)B(0,1)$ with semiaxes $\alpha_1(t^*,u_j) \geq \cdots$ and observe that $\sigma \geq \alpha_n(t^*,u_j)$ (and more so $\sigma\sqrt{n+1} \geq \alpha_n(t^*,u_j)$). In fact, if we assume the contrary: $\sigma < \alpha_n(t^*,u_j)$, then

$$e^{-npt^*} = \sigma^n < (\alpha_n(t^*,u_j))^n \leq \omega_n(t^*,u_j) \leq \bar{\omega}_n(t^*) \leq c_n e^{(f(n)+\varepsilon)t^*},$$

which contradicts (2.7).

By the differentiability of S_t we have the following inclusion for the right-hand side of (2.9):

$$S_{t^*}(B(u_j,r) \cap X) \subset S_{t^*}(u_j) + r(\mathscr{E}_j + B(0,h)). \tag{2.10}$$

We consider the covering of the set $r(\mathscr{E}_j + B(0,h)) = r(\mathscr{E}_j + B(0,\sigma\sqrt{n+1}))$ by balls of radius $2r\sigma\sqrt{n+1}$. Then, in view of (2.5) and (2.4), the minimum number of these balls satisfies the estimate

$$\begin{aligned} N_{r(\mathscr{E}_j+B(0,\sigma\sqrt{n+1}))}(2r\sigma\sqrt{n+1}) &= N_{(\mathscr{E}_j+B(0,\sigma\sqrt{n+1}))}(2\sigma\sqrt{n+1}) \\ &\leq N_{\mathscr{E}_j}(\sigma\sqrt{n+1}) \leq 2^n\omega_i(t^*,u_j)\sigma^{-i} \\ &\leq 2^n\bar{\omega}_i(t^*)\sigma^{-i} \end{aligned} \tag{2.11}$$

for some i, $0 \leq i \leq n$.

We set $s = 2\sigma\sqrt{n+1}$. Then, in view of (2.9),(2.11), and (2.8),

$$\begin{aligned} N_X(sr) &\leq N_X(r) \max_{1\leq j\leq N_X(r)} N_{r(\mathscr{E}_j+B(0,\sigma\sqrt{n+1}))}(2r\sigma\sqrt{n+1}) \\ &= N_X(r) \max_{1\leq j\leq N_X(r)} N_{(\mathscr{E}_j+B(0,\sigma\sqrt{n+1}))}(2\sigma\sqrt{n+1}) \\ &\leq N_X(r) \max_{1\leq k\leq n} (\bar{\omega}_k(t^*)2^n\sigma^{-k}) \\ &= 2^{n+D}(n+1)^{D/2} \max_{1\leq d\leq n} (\bar{\omega}_d(t^*)\sigma^{D-k})s^{-D}N_X(r) \\ &\leq s^{-D}N_X(r). \end{aligned}$$

Since $s \leq \text{const} < 1$, it follows that $N_X(s^k r) \leq s^{-kD}N_X(r)$ for each integer k. For an arbitrary $\varepsilon > 0$ let k be such that $s^{k+1}r < \varepsilon \leq s^k r$. Then $N_X(\varepsilon) \leq (1/\varepsilon)^D((r/s)^D N_X(r))$. From Definition 1.2 we see that $\dim_F X \leq D$ and hence $\dim_F X \leq b/p$, which gives (2.6) by letting $p \to a$. The proof is complete. □

Corollary 2.1. *Suppose that $q(n) \geq 0$ and $q(n+1) < 0$ and the piecewise linear function $q(d)$ (see (2.2) and (2.3)) is concave (or, more generally, lies below the straight line joining the points $(n,q(n))$ and $(n+1,q(n+1))$). Then*

$$\dim_F X \leq d_0 = n + \frac{q(n)}{q(n)-q(n+1)}. \tag{2.13}$$

Proof. We denote by $f(d)$ the linear function whose graph is the straight line described in the corollary. Then, by concavity, $q(d) \leq f(d)$ and $q(d_0) = f(d_0) = 0$. Estimate (2.13) now follows from Theorem 2.1. □

The next result will be used in applications most often.

Corollary 2.2. *If* $q(m) \le f(m)$, *for* $m = 1, \ldots,$ *where* $f(d)$ *is a concave function of the continuous variable* d *and* $f(d_*) = 0$, *then*

$$\dim_F X \le d_*.$$

3. Applications to evolution equations

The examples below are given as an illustration of the main theorem. We did not try to improve the physical results published in the literature.

We first briefly recall the formal scheme from [8] for estimating the sums of the first m global Lyapunov exponents, that is, the numbers $q(m)$.

If the semigroup S_t generated by the equation

$$\partial_t u = F(u), \quad u(0) = u_0 \tag{3.1}$$

has an invariant set X, then

$$q(m) \le \limsup_{t\to\infty} \sup_{u_0\in X} \sup_{\substack{\xi_i\in H\\ i=1,\ldots,m}} \left(\frac{1}{t} \int_0^t \operatorname{Tr} F'_u(S_\tau u_0) \circ Q_m(\tau)\, \mathrm{d}\tau \right). \tag{3.2}$$

Here $Q_m(\tau)$ is the orthogonal projection in H onto $\operatorname{Span}(U_1(\tau), \ldots, U_m(\tau))$, where the U_i are the solutions of the first variation equation

$$\partial_t U_i = F'_u(S_t(u_0)) \cdot U_i(t), \quad U_i(0) = \xi_i. \tag{3.3}$$

3.1. *Reaction–diffusion system*

We consider the following system:

$$\partial_t u = \nu a \Delta u - f(u) + g, \quad u|_{\partial\Omega} = 0, \tag{3.4}$$

where $x \in \Omega \Subset \mathbb{R}^n$, $a = \{a_{ij}\}$ is a constant $N \times N$-matrix with positive symmetric part $(a + a^*)/2 \ge I$, $\nu > 0$, and u, f, g are N-dimensional vector functions. We set $H = (L_2(\Omega))^N$ and suppose that $g \in H$. We suppose that $f(v)$ and $f'_v(v)$ are of class $C(\mathbb{R}^N, \mathbb{R}^N)$ and the following conditions hold for all $v, z \in \mathbb{R}^N$:

$$-C \le f(v) \cdot v, \quad -Dz \cdot z \le f'_v(v) z \cdot z, \quad |f'_v(v)| \le C(|v|^{p-2} + 1) \tag{3.5}$$

for some positive C and D. The exponent p satisfies the inequality $p \le 2n/(n-2)$ for $n \ge 3$ and $p < \infty$ for $n \le 2$.

The system (3.4) with initial condition $u(0) = u_0 \in H_1 = (H^1_0(\Omega))^N$ has a unique solution and thereby generates the semigroup $S_t : H_1 \to H_1$. The semigroup S_t has a compact attractor $\mathscr{A} \Subset H$ which is bounded in $H_2 = (H^2(\Omega) \cap H^1_0(\Omega))^N$ (see [1], Theorem I.5.2).

If the following condition holds:

$$|f(v+z) - f(v) - f'_v(v)z| \le C(1 + |v|^{p_1} + |z|^{p_1})|z|^{1+\gamma},$$

where $p_1 < 4/(n-2)$, $(n > 2)$ and $\gamma > 0$ is sufficiently small, then the operators of the semigroup S_t are differentiable on $\mathscr{A}$ in H in the sense of (2.1).

Theorem 3.1. *The fractal dimension of* $\mathscr{A}$ *in* H *satisfies the estimate*

$$\dim_F \mathscr{A} \le N \left(\frac{n+2}{4\pi n}\right)^{n/2} \frac{1}{\Gamma(1+n/2)} |\Omega| \left(\frac{D}{v}\right)^{n/2}, \tag{3.6}$$

where $|\Omega|$ *denotes the n-dimensional measure of* Ω.

Proof. The first variation equation (3.3) corresponding to (3.4) has the form

$$\partial_t U = va\Delta U - f'_u(u(t))U.$$

Therefore, in this case

$$\operatorname{Tr} F'_u(u(t)) \circ Q_m(t) = \sum_{i=1}^m (va\Delta\varphi_i, \varphi_i) - \sum_{i=1}^m (f'_u(u(t))\varphi_i, \varphi_i)$$

$$\le -v \sum_{i=1}^m \|\nabla\varphi_i\|^2 + D \sum_{i=1}^m \|\varphi_i\|^2 \le -v \sum_{i=1}^m \lambda_i + Dm,$$

where the vector-functions $\varphi_i = \varphi_i(t) \in H \cap (H_0^1)^N$ are an orthonormal basis in $\mathrm{Span}(U_1(t), \ldots, U_m(t))$ (see (3.1)–(3.3)), and where we have taken into account (3.5), the property of a that $(a + a^*)/2 \ge I$ and the variational principle (see [8], Lemma VI.2.1): $\sum_{i=1}^m \|\nabla\varphi_i\|^2 \ge \sum_{i=1}^m \lambda_i$, the λ_i being the eigenvalues of the Dirichlet problem for the operator $-I\Delta$ in Ω. Clearly $\{\lambda_i\}_{i=1}^\infty = \{\Lambda_n, \ldots, \Lambda_n\}_{n=1}^\infty$, where the Λ_n are the eigenvalues of the scalar Dirichlet problem for $-\Delta$ in Ω and each Λ_n is repeated N times. This gives that

$$\sum_{i=1}^m \lambda_i = N \sum_{i=1}^k \Lambda_i + p\Lambda_{k+1} = p \sum_{i=1}^{k+1} \Lambda_i + (N-p) \sum_{i=1}^k \Lambda_i,$$

where $m = Nk + p$, $0 \le p < N$. Using the estimate

$$\sum_{i=1}^m \Lambda_i \ge c_0 |\Omega|^{-2/n} m^{1+2/n}, \quad c_0 = \frac{4\pi n}{n+2} (\Gamma(1+n/2))^{2/n}$$

from [6] we see that

$$\sum_{i=1}^m \lambda_i \ge c_0 |\Omega|^{-2/n} (p(k+1)^{1+2/n} + (N-p)k^{1+2/n})$$

$$= c_0 |\Omega|^{-2/n} N((p/N)(k+1)^{1+2/n} + ((N-p)/N)k^{1+2/n})$$

$$\ge c_0 |\Omega|^{-2/n} N(k + p/N)^{1+2/n} = c_0 |\Omega|^{-2/n} N^{-2/n} m^{1+2/n},$$

where we used the inequality $\theta(k+1)^{1+2/n} + (1-\theta)k^{1+2/n} \ge (k+\theta)^{1+2/n}$, $\theta = p/N$, which follows from the convexity of the function $k \to k^{1+2/n}$.

Therefore

$$q(m) \le f(m) = -vc_0 |\Omega|^{-2/n} N^{-2/n} m^{1+2/n} + Dm.$$

The function f is concave. The root of the equation $f(d) = 0$ is $d^* = N|\Omega|(D/c_0 v)^{n/2}$, and we see from Corollary 2.2 that (3.6) holds. □

Remark 3.1. The semigroup S_t is Lipschitz on $\mathscr{A}$ from H to H_1. Therefore, estimate (3.6) holds for the fractal dimension of $\mathscr{A}$ in H_1.

3.2. Navier–Stokes equations

We now consider the two-dimensional Navier–Stokes system

$$\begin{aligned} &\partial_t u + \sum_{i=1}^{2} u^i \partial_i u = \nu \Delta u - \nabla p + f, \\ &\operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0, \end{aligned} \tag{3.7}$$

where Ω is an arbitrary bounded domain $\Omega \Subset \mathbb{R}^2$.

We denote by P the orthogonal projection in $(L_2(\Omega))^2$ onto the Hilbert space H which is the completion in $(L_2(\Omega))^2$ of the space of smooth solenoidal vector functions with compact supports in Ω. Applying P we write (3.7) as an abstract evolution equation in H:

$$\partial_t u + \nu A u + B(u,u) = f, \quad u(0) = u_0, \tag{3.8}$$

where $A = -P\Delta$ and $B(u,v) = P(\sum_{i=1}^{2} u^i \partial_i v)$. We denote by λ_1 the first eigenvalue of A.

Eq. (3.8) generates the semigroup $S_t : H \to H$ which is differentiable in H and has a compact attractor $\mathscr{A} \Subset H$ (see [1,8] for the case of a smooth $\partial\Omega$ and [5] for the case of a non-smooth $\partial\Omega$).

Theorem 3.2. *The fractal dimension of $\mathscr{A}$ satisfies the following estimate in terms of the Grashof number $G = \|f\|/(\lambda_1 \nu^2)$:*

$$\dim_F \mathscr{A} \le \left(\frac{2}{\pi}\right)^{1/2} (\lambda_1 |\Omega|)^{1/2} \frac{\|f\|}{\lambda_1 \nu^2}, \tag{3.9}$$

where $|\Omega|$ is the area of Ω. If, in addition,

$$\frac{\|f\|}{\lambda_1 \nu^2} < \frac{1}{4} 3^{3/2} \pi^{1/2}, \tag{3.10}$$

then $\mathscr{A} = \bar{u}$, where $\bar{u}$ is the unique globally asymptotically stable stationary solution.

Proof. We have the following estimate for $q(m)$ (see [4], inequality (2.11)):

$$q(m) \le f(m) = -\frac{\nu \pi m^2}{2|\Omega|} + \frac{\|f\|^2}{\lambda_1 \nu^3}.$$

Now (3.9) follows from Corollary 2.2.

It was shown in [4] that $\mathscr{A} = \bar{u}$ if

$$\frac{\|f\|}{\lambda_1 \nu^2} < \frac{1}{c_0^2}, \tag{3.11}$$

where c_0 is the best constant in the inequality

$$\|\psi\|_{L_4(\Omega)} \le c_0 \|\psi\|^{1/2} \|\nabla \psi\|^{1/2}, \quad \psi \in H_0^1(\Omega).$$

Therefore to get (3.10) we merely substitute in (3.11) the best (to date) closed form estimate of c_0 from [7]: $c_0 < 2 \times 3^{-3/4}\pi^{-1/4}$, which is only about 2.5% greater than the sharp value of c_0 obtained numerically. The proof is complete.

Remark 3.2. The estimates of the Hausdorff and fractal dimension of the attractor

$$\dim_H \mathscr{A} \le c(\Omega)\frac{\|f\|}{\lambda_1 v^2}, \qquad \dim_F \mathscr{A} \le 2c(\Omega)\frac{\|f\|}{\lambda_1 v^2},$$

where the dimensionless constant $c(\Omega)$ depends only on the shape of Ω, were obtained by Temam [8]. It follows from Theorem 3.2 that

$$\dim_F \mathscr{A} \le c(\Omega)\frac{\|f\|}{\lambda_1 v^2}, \quad \text{where } c(\Omega) \le \left(\frac{2}{\pi}\right)^{1/2}(\lambda_1|\Omega|)^{1/2}.$$

By using the inequality

$$\lambda_1 \ge \frac{2\pi}{|\Omega|}$$

(see [4], inequality (1.6)), we get from (3.9) and (3.10) the following result.

Corollary 3.1. *The fractal dimension of* $\mathscr{A}$ *satisfies the following estimate*:

$$\dim_F \mathscr{A} \le c_1(\Omega)\frac{\|f\|\,|\Omega|}{v^2}, \tag{3.12}$$

where the constant $c_1(\Omega)$, *in general*, *depends on the shape of* Ω *and admits the following absolute upper bound*:

$$c_1(\Omega) \le \frac{1}{\pi}. \tag{3.13}$$

If, *in addition*,

$$\frac{\|f\|\,|\Omega|}{v^2} < \frac{1}{2}(3\pi)^{3/2}, \tag{3.14}$$

then $\mathscr{A} = \bar{u}$ *and* $\dim \mathscr{A} = 0$.

Remark 3.3. One can see, as in [4], that Theorem 3.2 and Corollary 3.1 hold if $\Omega \subset \mathbb{R}^2$ is an arbitrary domain with finite area.

Acknowledgements

This work was carried out with the financial support from the US Civillian Research & Development Foundation for the Independent States of the Former Soviet Union, Award NRM1-186 and the Russian Foundation for Fundamental Research, grant no. 97-01-00541.

References

[1] A.V. Babin, M.I. Vishik, Attractors of Evolution Equations, Nauka, Moscow, 1989 (English transl., North-Holland, Amsterdam, 1992).

[2] Z. Chen, A note on Kaplan–Yorke-type estimates on the fractal dimension of chaotic attractors, Chaos Solitons Fractals 3 (1993) 575–582.

[3] A. Douady, J. Oesterlé, Dimension de Hausdorff des attracteurs, C.R. Acad. Sci. Paris, Sér. A 290 (1980) 1135–1138.

[4] A.A. Ilyin, Attractors for Navier–Stokes equations in domains with finite measure, Nonlinear Anal. Theory Methods Appl. 27 (1996) 605–616.

[5] O.A. Ladyzhenskaya, First boundary value problem for Navier–Stokes equations in domain with non smooth boundaries, C.R. Acad. Sci. Paris 314 (1992) 253–258.

[6] P. Li, S.T. Yau, On the Schrödinger equation and the eigenvalue problem, Comm. Math. Phys. 8 (1983) 309–318.

[7] Sh.M. Nasibov, On optimal constants in some Sobolev inequalities and their applications to a nonlinear Schrödinger equation, Dokl. Akad. Nauk SSSR 307 (1989) 538–542 (English transl., Soviet Math. Dokl. 40 (1990) 110–115).

[8] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, 2nd Edition, Springer, New York, 1997.