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Vasiliev codes of length $n = 2^m$ and Steiner systems $S(n, 4, 3)$ of rank $n - m$ over $\mathbf{F}_2$[1]

Abstract

We study the extended binary perfect nonlinear Vasiliev codes of length $n = 2^m$ and Steiner systems $S(n, 4, 3)$ of rank $n - m$ over $\mathbb{F}_2$. The generalized concatenation (GC) construction of Vasiliev codes induces a variant of the doubling construction of Steiner systems $S(n, 4, 3)$ of rank $n - m$ over $\mathbb{F}_2$. We prove that any Steiner system $S(n = 2^m, 4, 3)$ of rank $n - m$ is obtained by such doubling construction and can be formed by the codewords of weight 4 of the corresponding Vasiliev codes. The length 16 is studied in details. We compute the full automorphism groups of all 12 non-equivalent Vasiliev codes of length 16. There are exactly 15 non-isomorphic systems $S(16, 4, 3)$ with rank 12 over $\mathbb{F}_2$. We compute the automorphisms groups for these Steiner systems.

§ 1. Introduction

An interesting open problems in algebraic coding theory is *the classification of non-linear binary perfect codes with Hamming parameters*. An interesting class of such codes is the Vasiliev codes [1]. According to Hergert [2] there are 19 non-equivalent Vasiliev's codes of length 15 (including the linear code), and according to Malugin [3] there are 13 non-equivalent extended Vasiliev's codes of length 16 (including the linear code). Another

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interesting question related to these codes is their group of automorphisms, which are not known even for the length $n = 16$. In [4] there are some upper and lower bounds for orders of the groups of symmetries of Vasiliev codes of any length $n = 2^m$. In [3] the orders of automorphism groups of all 18 non-equivalent Vasiliev codes of length 15 are found.

Another interesting open problem in combinatorial design theory is the classification of all non-isomorphic Steiner systems $S(16, 4, 3)$. In the previous papers [5, 6] we enumerated all systems with rank at most thirteen over the field $\mathbb{F}_2$. In particular, it has been proved that there are 15 non-isomorphic such systems with rank 12 and 4131 such non-isomorphic systems with rank 13.

The purpose of this paper is to find the full automorphism group for the 12 non-equivalent Vasiliev codes of length 16 and the corresponding 15 non-isomorphic Steiner systems $S(16, 4, 3)$, which are formed of codewords of weight four of Vasiliev codes of length 16. These systems are exactly the non-isomorphic systems of rank 12 over $\mathbb{F}_2$. For each Vasiliev code we give all Steiner systems $S(16, 4, 3)$ which belongs to this code. We also study the Steiner systems $S(n = 2^m, 4, 3)$ of rank $n - m$ over $\mathbb{F}_2$, connected with Vasiliev codes of length n. We describe a variant of the doubling construction of Steiner system $S(v, 4, 3)$ with fixed rank over $\mathbb{F}_2$. For the case $v = 2^m$ this construction provides all non-isomorphic Steiner systems $S(v, 4, 3)$ with rank $v - m$ over $\mathbb{F}_2$. Any such system belongs to some Vasiliev code of length v.

The paper is organized as follows. Preliminary results and terminology are given in § 2. In § 3 we describe the GC-construction of Vasiliev codes. The full automorphism groups of all Vasiliev codes of length 16 are given in § 4. In § 5 we give the doubling construction of Steiner systems $S(v, 4, 3)$ with the given rank over $\mathbb{F}_2$. In the case $v = 2^m$, this gives all non-isomorphic Steiner systems $S(v, 4, 3)$ of rank $v - m$. In § 6 we describe the automorphism groups of all 15 non-isomorphic Steiner systems $S(16, 4, 3)$ with rank 12 over $\mathbb{F}_2$. Vasiliev codes of length 16 and corresponding Steiner systems $S(16, 4, 3)$ obtained from codewords of weight four are considered in § 7.

§ 2. Preliminary results and terminology

We repeat briefly some results of [5] (see for details [5]). Let $J_n = \{1, 2, ..., n\}$ and let S_n be the full group of permutations of n elements. For any $i \in J_n$ and $\pi \in S_n$, define the image of i under the action of π by $\pi(i)$.

Let $E_a = \{0, 1, 2, 3\}$. Define the action of S_4 on E_a^4 as the permutations of coordinates of E_a^4. For any $\tau, \pi \in S_4$, we have $(\tau\pi)(\mathbf{a}) = \tau(\pi(\mathbf{a}))$.

Set $H_4 = S_4^4 = S_4 \times S_4 \times S_4 \times S_4$. Define the action of H_4 on E_a^4 component-wise, i.e. for any $\mathbf{a} = (a_1, a_2, a_3, a_4) \in E_a^4$ and $\mathbf{h} = (\pi_1, \pi_2, \pi_3, \pi_4) \in H_4$, we have

$$\mathbf{h}(\mathbf{a}) = (\pi_1(a_1), \pi_2(a_2), \pi_3(a_3), \pi_4(a_4)).$$

Let $G_4 = < S_4, H_4 >$ be the group generated by H_4 and S_4. Then $G_4 = S_4 \rtimes H_4 = H_4 \rtimes S_4$ is the semi-direct product of S_4 and H_4. Define the action of the group G_4 on E_a^4. For any $\mathbf{g} = \tau\mathbf{h} \in G_4$, and $\forall\, \mathbf{a} = (a_1, a_2, a_3, a_4) \in E_a^4$, we have

$$\begin{aligned} \mathbf{g}(\mathbf{a}) &= \tau(\mathbf{h}(\mathbf{a})) = \tau((\pi_1(a_1), \pi_2(a_2), \pi_3(a_3), \pi_4(a_4))) \\ &= (\pi_{\tau^{-1}(1)}(a_{\tau^{-1}(1)}), \pi_{\tau^{-1}(2)}(a_{\tau^{-1}(2)}), \pi_{\tau^{-1}(3)}(a_{\tau^{-1}(3)}), \pi_{\tau^{-1}(4)}(a_{\tau^{-1}(4)})). \end{aligned}$$

For any subset $X \subseteq E_a^4$ and element $\mathbf{g} \in G_4$, set $\mathbf{g}\,X = \{\mathbf{g}(\mathbf{x}) : \mathbf{x} \in X\}$.

Let $E = \{0, 1\}$. Define the action of S_n on E^n as the permutations of coordinates, i.e. for any $\mathbf{y} = (y_1, y_2, ..., y_n) \in E^n$ and $\pi \in S_n$, we have

$$\pi(\mathbf{y}) = (y_{\pi^{-1}(1)}, y_{\pi^{-1}(2)}, ..., y_{\pi^{-1}(n)}).$$

Observe that for any $\mathbf{y}_1, \mathbf{y}_2 \in E^n$ and $\pi \in S_n$

$$\pi(\mathbf{y}_1 + \mathbf{y}_2) = \pi(\mathbf{y}_1) + \pi(\mathbf{y}_2),$$

where + denotes the component-wise addition modulo 2.

Define the action of E^n on itself by shifts, i.e. for any $\mathbf{y} \in E^n$ and $\mathbf{h} \in E^n$, set $\mathbf{h}(\mathbf{y}) = \mathbf{h} + \mathbf{y}$. We denote this group of actions by H_n.

For $n = 16$ let $G = < S_{16}, H_{16} >$ be the group generated by H_{16} and S_{16}. Then $G = S_{16} \rtimes H_{16}$ is the semi-direct product of S_{16} and H_{16}. Arrange 16 coordinates into four *blocks* of four coordinates. Any $\mathbf{y} \in E^{16}$ can be written as $(\mathbf{y}_1|\mathbf{y}_2|\mathbf{y}_3|\mathbf{y}_4)$, where $\mathbf{y}_i = (y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}) \in$

E^4, $i = 1, 2, 3, 4$, and $\mathbf{y}_i$ is called the i-th block. Thus the first four coordinates belong to the first block, the second four coordinates belong to the second block, etc. We say that block is *even/odd* if its Hamming weight is even/odd. We say also that $\mathbf{y} \in E^{16}$ is even/odd, if its blocks $\mathbf{y}_i$, $i = 1, 2, 3, 4$, are even/odd.

Define the action of the group G_4 on E^{16}. For any $\mathbf{g} = \tau\mathbf{h} \in G_4$, and any $\mathbf{y} = (\mathbf{y}_1|\mathbf{y}_2|\mathbf{y}_3|\mathbf{y}_4) \in E^{16}$, set:

$$\begin{aligned}\mathbf{g}(\mathbf{y}) &= \tau\left(\mathbf{h}\left(\mathbf{y}\right)\right) = \tau\left(\pi_1(\mathbf{y}_1)|\pi_2(\mathbf{y}_2)|\pi_3(\mathbf{y}_3)|\pi_4(\mathbf{y}_4)\right) \\ &= (\pi_{\tau^{-1}(1)}(\mathbf{y}_{\tau^{-1}(1)})|\pi_{\tau^{-1}(2)}(\mathbf{y}_{\tau^{-1}(2)})|\pi_{\tau^{-1}(3)}(\mathbf{y}_{\tau^{-1}(3)})|\pi_{\tau^{-1}(4)}(\mathbf{y}_{\tau^{-1}(4)})),\end{aligned}$$

where (for $i = 1, 2, 3, 4$):

$$\pi_i\left(\mathbf{y}_i\right) = \pi_i\left(y_{i,1}, y_{i,2}, y_{i,3}, y_{i,4}\right) = (y_{i,\pi_i^{-1}(1)}, y_{i,\pi_i^{-1}(2)}, y_{i,\pi_i^{-1}(3)}, y_{i,\pi_i^{-1}(4)}).$$

Note that G_4 is a subgroup of S_{16}, and the index of G_4 in S_{16} is equal to $|S_{16}|/|G_4| = 16!/(4!)^5 = 2627625$. We will parameterize these cosets in the following way.

Proposition 1 [7]. *Any coset of G_4 in S_{16} has a representative $\pi \in S_{16}$, such that*

$$\begin{aligned}\pi(i+1) &< \pi(i+2) < \pi(i+3) < \pi(i+4), \qquad i = 0, 4, 8, 12, \\ 1 = \pi(1) &< \pi(5) < \pi(9) < \pi(13).\end{aligned}$$

We will also use another representation of the elements of S_{16}. Given $\pi \in S_{16}$, write it as the sequence $(\pi(1), \pi(2), ..., \pi(16))$. Then replace the indices of coordinates by the indices of their corresponding blocks. Recall that a coordinate whose index is 1, 2, 3, 4 belong to the 1-st block, i.e. in all positions j for which $\pi(j) \in \{1, 2, 3, 4\}$, we replace $\pi(j)$ by the element 1. The next four indices 5, 6, 7, 8 belong to the 2-nd block, and so on. Thus we write π as the sequence of block indices $(j_1, j_2, ..., j_{16})$, where

$$\{j_1, j_2, ..., j_{16}\} = \{1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4\}.$$

Definition 1 . *For binary vector* $\mathbf{y}$ *let* $\mathrm{wt}(\mathbf{y})$ *denote its Hamming weight. Let*

$$\mathbf{x} = (\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\mathbf{x}_4) \in E^{16}$$

where $\mathbf{x}_i = (x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4})$ *for* $i = 1, 2, 3, 4$. *We say that* $\mathbf{x}$ *satisfies* the parity rule, *if*

$$\text{wt}(\mathbf{x}_i) \equiv j \pmod 2, \quad i = 1, 2, 3, 4, \quad \textit{where} \ \ j \in \{0, 1\}. \tag{1}$$

Proposition 2 [6, 7]. *Suppose that* $C = \{(\mathbf{1000}), (\mathbf{0100}), (\mathbf{0010}), (\mathbf{0001})\}$, *where* $\mathbf{0} = 0000$, $\mathbf{1} = 1111$. *Suppose there exists a permutation* $\sigma \in S_{16}$ *such, that* σC *satisfies the parity rule (1). Then* σ *written as a sequence of block indices belongs to* G_4 *or a* G_4*-coset which has a representative of one of the following form:*

$$\begin{aligned}
\sigma_1 &= (1, 2, 3, 4 \,|\, 1, 2, 3, 4 \,|\, 1, 2, 3, 4 \,|\, 1, 2, 3, 4),\\
\sigma_2 &= (1, 2, 3, 4 \,|\, 1, 2, 3, 4 \,|\, 1, 1, 2, 2 \,|\, 3, 3, 4, 4),\\
\sigma_3 &= (1, 1, 2, 2 \,|\, 1, 1, 2, 2 \,|\, 3, 3, 4, 4 \,|\, 3, 3, 4, 4),\\
\sigma_4 &= (1, 1, 2, 2 \,|\, 1, 1, 3, 3 \,|\, 2, 2, 4, 4 \,|\, 3, 3, 4, 4),\\
\sigma_5 &= (1, 1, 2, 2 \,|\, 1, 1, 2, 2 \,|\, 3, 3, 3, 3 \,|\, 4, 4, 4, 4),\\
\sigma_6 &= (1, 1, 2, 2 \,|\, 1, 1, 3, 3 \,|\, 2, 2, 3, 3 \,|\, 4, 4, 4, 4).
\end{aligned}$$

Proposition 3 . *The sets of* σ_i*-type* G_4*-cosets of Proposition 2 are contained in the* (G_4-G_4)*-double cosets* $G_4\sigma_iG_4$, $i = 1, 2, \ldots, 6$ *where* $\sigma_i \in S_{16}$ *the fixed set of representatives*

$$\begin{aligned}
\sigma_1 &= (1, 5, 9, 13 \,|\, 2, 6, 10, 14 \,|\, 3, 7, 11, 15 \,|\, 4, 8, 12, 16),\\
\sigma_2 &= (1, 5, 9, 13 \,|\, 2, 6, 10, 14 \,|\, 3, 4, 7, 8 \,|\, 11, 12, 15, 16),\\
\sigma_3 &= (1, 2, 5, 6 \,|\, 3, 4, 7, 8 \,|\, 9, 10, 13, 14 \,|\, 11, 12, 15, 16),\\
\sigma_4 &= (1, 2, 5, 6 \,|\, 3, 4, 9, 10 \,|\, 7, 8, 13, 14 \,|\, 11, 12, 15, 16),\\
\sigma_5 &= (1, 2, 5, 6 \,|\, 3, 4, 7, 8 \,|\, 9, 10, 11, 12 \,|\, 13, 14, 15, 16),\\
\sigma_6 &= (1, 2, 5, 6 \,|\, 3, 4, 9, 10 \,|\, 7, 8, 11, 12 \,|\, 13, 14, 15, 16).
\end{aligned}$$

Proof. Follows from Proposition 1 and from the fact that permutations from G_4 do not change the parity, i.e. if X is a subset of E^{16} which satisfies the parity rule and for some $\sigma \in S_{16}$ the set σX also satisfies the parity rule, then the parity of $\mathbf{g}\sigma X$ is satisfied for any $\mathbf{g} \in G_4$. Moreover, the σ_i-type cosets of G_4 are contained in the double cosets $G_4\sigma_iG_4$, where σ_i, $i = 1, 2, \ldots, 6$ is the fixed set of σ_i-type permutations. $\triangle$

Definition 2 . *For any $\sigma \in S_{16}$, define its normalizer in G_4:*

$$\mathrm{N}(\sigma) \ = \ \{\mathbf{g} \in G_4 \ : \ \sigma^{-1}\mathbf{g}\sigma \in G_4\}.$$

Let E be a finite alphabet of size $q:\ E = \{0, 1, ..., q-1\}$. A q-ary code of length n is an arbitrary subset of E^n. Denote such q-ary code C of length n, with the minimal distance d and cardinality N as $(n, d, N)_q$-code. Denote by $\mathrm{wt}(\mathbf{x})$ the Hamming weight of the vector $\mathbf{x}$ over E. For a binary (i.e. $q = 2$) code C denote by $\langle C \rangle$ the linear envelope of words of C over $\mathbb{F}_2$. The dimension of space $\langle C \rangle$ is called the *rank* of C and is denoted $\mathrm{rank}(C)$. For a binary code C with zero vector call a *kernel* and denote $\mathrm{Ker}(C)$ the set of all vector $\mathbf{x}$ from C stabilizing this code: $C + \mathbf{x} = \{\mathbf{c} + \mathbf{x} :\ \mathbf{c} \in C\} = C$. It is clear, that $\mathrm{Ker}(C)$ is a linear space. For $q = 2$ let E^n_{ev} be the subset of E^n, containing all vectors of even weight. We need also constant weight codes. Denote a q-ary constant weight code W of length n, with weight of all codewords w, with minimal distance d and cardinality N by $(n, w, d, N)_q$-code. For $q = 2$ we denote such code W by (n, w, d, N)-code.

For vector $\mathbf{v} = (v_1, ..., v_n)$ over E denote by $\mathrm{supp}(\mathbf{v})$ its support, i.e. the set of indices with nonzero positions: $\mathrm{supp}(\mathbf{v}) = \{i :\ v_i \neq 0\}$.

§ 3. Vasiliev codes

We recall the GC-construction [8, 9] of binary perfect nonlinear codes, based on the following mapping from E^2 onto E^2:

$$\varphi(0, 0) = (00), \quad \varphi(0, 1) = (11), \quad \varphi(1, 0) = (10), \quad \varphi(1, 1) = (01).$$

Let we have the extended binary Hamming code H_t of length $t = 2^{m-1}$. Using the GC-construction [9], the code H_n, $n = 2t$ can be obtained as follows. Let $\mathbf{x} \in H_t$ and $\mathbf{e} \in E^t_{\mathrm{ev}}$ be two arbitrary vectors. Using $\mathbf{x} = (x_1, ..., x_t)$ and $\mathbf{e} = (e_1, ..., e_t)$ define the following vector $\mathbf{c} = \mathbf{c}(\mathbf{x}, \mathbf{e})$ of length n:

$$\mathbf{c} \ = \ (\varphi(x_1, e_1)|\varphi(x_2, e_2)|...|\varphi(x_t, e_t)). \tag{2}$$

It is easy to see [9] that the set

$$\{\mathbf{c}(\mathbf{x},\mathbf{e}):\ \mathbf{x}\in H_t,\ \mathbf{e}\in E^t_{\rm ev}\}$$

is the code H_n. It is easy now to define the class of all Vasiliev codes. The code H_n can be presented as the following partition:

$$H_n \ = \ \cup_{\mathbf{x}\in H_t} H_n(\mathbf{x}), \tag{3}$$

where $H_n(\mathbf{x})$ is the following set:

$$H_n(\mathbf{x}) \ = \ \cup_{\mathbf{e}\in E^t_{\rm ev}} \{\mathbf{c}(\mathbf{x},\mathbf{e})\}. \tag{4}$$

Definition 3 . *Let* $\mathbf{w}$ *be a binary vector of length* $n = 2\,t$, *divided into blocks of length* 2*:*

$$\mathbf{w} \ = \ (\mathbf{w}_1\,|\,\mathbf{w}_2\,|\,...\,|\,\mathbf{w}_t), \quad \mathbf{w}_i = (\mathbf{w}_{i,1},\mathbf{w}_{i,2}) \in E^2.$$

Say that $\mathbf{w}$ *is* 2-even *(respectively,* 2-odd*), if each block* $\mathbf{w}_i$*,* $i = 1,2,...,t$ *has an even (respectively, odd) weight.*

Now we construct the Vasiliev codes.

Proposition 4 [9]. *Let* $n = 2\,t = 2^m$ *and let* H_t *be the Hamming code of length* t*. Let the code* H_n *is partitioned into subcodes* $H_n(\mathbf{x})$ *according to (3) and (4). For any* $\mathbf{x}\in H_t$ *choose an arbitrary* 2*-even vector* $\mathbf{w}\,(\mathbf{x})$ *of length* n*. Then the set*

$$\cup_{\mathbf{x}\in H_t}(H_n(\mathbf{x}) \ + \ \mathbf{w}\,(\mathbf{x})), \tag{5}$$

is an extended binary perfect Vasiliev $(n,4,2^{n-m-1})$*-code* C*.*

As we know from [3], there are 12 non-equivalent extended Vasiliev codes of length 16, i.e. extended binary perfect $(16,4,2^{11})$-codes with rank exactly 12 over $\mathbb{F}_2$. In [7] we gave another GC-construction of all $(16,4,2^{11})$-codes of rank less or equal 13, based on mapping two quaternary MDS $(4,2,64)_4$-codes A and A' into binary [9, 10].

It is known (see [7], references) that there are 5 non-equivalent MDS codes $A_i : (4, 2, 64)_4$ over E_a. In [7] they are given in the so called canonical forms. These canonical MDS codes A_i define uniquely the canonical half-codes $C_i = C(A_i)$ with parameters $(16, 4, 1024)$. All the codes $C : (16, 4, 2048)$, constructed by this way, can be parameterized by three natural numbers i, j, k. The code (i, j, k), where $1 \le i \le j \le 5$ and $l = 1, 2, ..., m_2(i, j)$ is the code $C_{ij}^{(k)}$,

$$C_{ij}^{(k)} = (C_i + \mathbf{s},\, \mathbf{d}_{ij}^{(k)} C_j),$$

where: $\mathbf{s} = (1000|1000|1000|1000)$ is the fixed vector, $m_2(i, j)$ is the number of $(G_4 \rtimes H)$-orbits, and $\mathbf{d}_{ij}^{(k)}$ is the specially chosen element (double coset representative) of $G_4 = S_4 \rtimes (S_4)^4$ (see [7] for details). In [7] we give the numbers $m_2(i, j)$ of $(G_4 \rtimes H)$-orbits in the union of some double cosets of the group G_4 and the corresponding double cosets representatives $\mathbf{d}_{ij}^{(k)}$. This gives us an explicit construction of all codes $C_{ij}^{(k)}$ or codes (i, j, k) for $1 \le i \le j \le 5$ and $l \in \{1, 2, ..., m_2(i, j)\}$.

The next statement gives us 12 non-equivalent extended Vasiliev codes (i, j, k).

Proposition 5 [7] *The* 12 *non-equivalent extended Vasiliev codes of length* 16 *are the following* (i, j, k)*-codes* $(C_i + \mathbf{s},\ \mathbf{d}_{ij}^k\, C_j)$, $\quad \mathbf{d}_{ij}^k \in G_4$*:*

$$\begin{array}{cccccc} (1,1,2), & (1,1,4), & (1,2,1), & (1,2,2), & (1,2,5), & (1,5,1), \\ (2,2,1), & (2,2,2), & (2,2,5), & (2,3,40), & (3,3,3), & (3,3,9). \end{array}$$

§ 4. Automorphism groups of Vasiliev codes

Let C be any Vasiliev code and let $\mathrm{Ker}\,(C)$ be its kernel, i.e. $\mathrm{Ker}(C) = \{\mathbf{h} \in C :\ C + \mathbf{h} = C\}$. Then there exist a set of $[C : \mathrm{Ker}(C)]$ vectors $\{\mathbf{h}_i\}$ (denoted by $C/\mathrm{Ker}(C)$) in C so that C is the disjoint union of cosets of $\mathrm{Ker}(C)$

$$C \;=\; \bigcup_i\, (\mathbf{h}_i + \mathrm{Ker}(C)).$$

Definition 4 . *For any Vasiliev code* C *and* $\mathbf{h} \in H_{16}$ *let*

$$P(\mathbf{h}) \;=\; \mathrm{Stab}_{G_4}(C + \mathbf{h}).$$

In particular set $P = \mathrm{Stab}_{G_4}(C)$ *and* $P_i = P(\mathbf{h}_i) = \mathrm{Stab}_{G_4}(C + \mathbf{h}_i)$*.*

Note that if $\mathbf{h}' \in \mathrm{Ker}(C)$ then $P(\mathbf{h}+\mathbf{h}') = P(\mathbf{h})$.

Lemma 1 . *Let C be the Vasiliev code and*

$$(\tau, \mathbf{h}) \;=\; \tau\mathbf{h} \in \mathrm{Stab}_G(C) \subset G \;=\; S_{16} \rtimes H_{16}.$$

Let $\tau' \in P\tau P(\mathbf{h})$ and $\mathbf{h}' \in \mathrm{Ker}(C) + \mathbf{h}$. Then $(\tau', \mathbf{h}') \in \mathrm{Stab}_G(C)$.

Proof. Indeed suppose $\tau' = \mathbf{g}_1\tau\mathbf{g}_2$, where $\mathbf{g}_1 \in P$, $\mathbf{g}_2 \in P(\mathbf{h})$ and $\mathbf{h}' = \mathbf{h}_1 + \mathbf{h}$, where $\mathbf{h}_1 \in \mathrm{Ker}(C)$. We have that

$$(\mathbf{g}_1\tau\mathbf{g}_2, \mathbf{h}_1 + \mathbf{h})C = \mathbf{g}_1\tau\mathbf{g}_2(C + \mathbf{h}_1 + \mathbf{h}) = \mathbf{g}_1\tau(C + \mathbf{h}) = \mathbf{g}_1 C = C.$$

△

Lemma 2 . *Let C be the Vasiliev code and $\tau\mathbf{h} \in \mathrm{Stab}_G(C)$. Then $\tau = \mathbf{g} \in G_4$ or $\tau = \mathbf{x}^{-1}\sigma_3\mathbf{g} \in G_4\sigma_3G_4$, where $\mathbf{x} \in \mathrm{N}(\sigma_3)\backslash G_4/P$ is the $(\mathrm{N}(\sigma_3)$-P-double coset representative and $\mathbf{g}$ belongs to the $P(\mathbf{h})$-coset of G_4, uniquely determined by $\mathbf{x}$.*

Proof. Indeed, suppose $\tau\mathbf{h} \in \mathrm{Stab}_G(C)$, i.e. $\tau(C + \mathbf{h}) = C$. Since $\tau^{-1}(C)$ contains the zero codeword, vector $\mathbf{h}$ belongs to C. Therefore $C+\mathbf{h}$ satisfies the parity law, and $\tau(C+\mathbf{h})$ satisfies the parity law as well. By Proposition 2 the permutation τ belongs to G_4 or the σ_i-type coset of G_4, $i = 1, 2, ..., 6$ and by Proposition 3 we have that $\tau \in G_4\sigma_iG_4$ or $\tau \in G_4$. Direct calculations show that $\tau \in G_4$ or $\tau \in G_4\sigma_3G_4$. Thus, if $\tau = \mathbf{g} \in G_4$ then

$$C \;=\; \mathbf{g}(C + \mathbf{h}). \tag{6}$$

Next, suppose $\tau = \mathbf{x}^{-1}\sigma_3\mathbf{g}$, where $\mathbf{x}^{-1}, \mathbf{g} \in G_4$. Note that since G_4 is a group then $\mathbf{x} \in G_4$. Then equality $\mathbf{x}^{-1}\sigma_3\mathbf{g}(C + \mathbf{h}) \;=\; C$ is equivalent to

$$\sigma_3\mathbf{x}C \;=\; \mathbf{g}(C + \mathbf{h}). \tag{7}$$

Suppose that a pair $\mathbf{x}$ and $\mathbf{g}$ satisfies this equation. Let $\mathbf{x}' = \mathbf{x}_1\mathbf{x}\mathbf{g}_1$, where $\mathbf{x}_1 \in \mathrm{N}(\sigma_3)$ and $\mathbf{g}_1 \in \mathrm{Stab}_{G_4}(C)$. Let $\mathbf{x}_2 = \sigma_3\mathbf{x}_1\sigma_3$. Multiplying both sides of (7) by $\mathbf{x}_2$, we obtain

$$\mathbf{x}_2\sigma_3\mathbf{x}\mathbf{g}_1 C \;=\; \mathbf{x}_2\mathbf{g}(C + \mathbf{h}),$$

which is equivalent (since $\mathbf{x}_2\sigma_3 = \sigma_3\mathbf{x}_1$ and $\mathbf{g}_1 C = C$) to

$$\sigma_3\mathbf{x}'C \ = \ \mathbf{g}'(C+\mathbf{h}),$$

where $\mathbf{g}' = \mathbf{x}_2\mathbf{g}$. It follows that $\mathbf{x} \in \mathrm{N}(\sigma_3)\backslash G_4/P$ and $\mathbf{g}$ belongs to the $P(\mathbf{h})$-coset of G_4. To show that the $P(\mathbf{h})$-coset of $\mathbf{g}$ is uniquely determined by $\mathbf{x}$ suppose the opposite:

$$\begin{cases} \sigma_3\mathbf{x}C = \mathbf{g}(C+\mathbf{h}) \\ \sigma_3\mathbf{x}C = \mathbf{g}'(C+\mathbf{h}). \end{cases}$$

Thus $\mathbf{g}(C+\mathbf{h}) = \mathbf{g}'(C+\mathbf{h})$, which implies $\mathbf{g}^{-1}\mathbf{g}' \in P(\mathbf{h})$ i.e. $\mathbf{g}$, $\mathbf{g}'$ belong to the same $P(\mathbf{h})$-coset of G_4 which leads to a contradiction. $\triangle$

We need the following technical lemma:

Lemma 3 *For any $\mathbf{g}, \mathbf{x}_1, \mathbf{x}_2 \in G_4$ the equality*

$$\mathbf{x}_1\sigma_3 \ = \ \mathbf{x}_2\sigma_3\mathbf{g}$$

implies that $\mathbf{x}_2^{-1}\mathbf{x}_1 \in \mathrm{N}(\sigma_3)$.

Proof. Indeed, since $\sigma_3^2 = 1$, from $\mathbf{x}_1\sigma_3 = \mathbf{x}_2\sigma_3\mathbf{g}$ it follows that $\sigma_3\mathbf{x}_2^{-1}\mathbf{x}_1\sigma_3 = \mathbf{g}$. Thus $\sigma_3\mathbf{x}_2^{-1}\mathbf{x}_1\sigma_3 \in G_4$, i.e. $\mathbf{x}_2^{-1}\mathbf{x}_1 \in \mathrm{N}(\sigma_3)$. $\triangle$

Lemma 4 *Let C be a Vasiliev code and $\mathrm{Stab}_{G_4 \rtimes H_{16}}(C) = \mathrm{Stab}_{G_4 \rtimes H_{16}}(C)$ be its stabilizer group in $G_4 \rtimes H_{16}$. Let $P = \mathrm{Stab}_{G_4}(C)$ and $P_i = P(\mathbf{h}_i)$. Then*

$$\mathrm{Stab}_{G_4 \rtimes H_{16}}(C) \ = \ \bigcup_i P\mathbf{g}_i \rtimes (\mathbf{h}_i + \mathrm{Ker}(C)),$$

where $\{\mathbf{h}_i\}$ is the subset of the set $C/\mathrm{Ker}(C)$ representatives and $\mathbf{g}_i \in G_4/P_i$ is uniquely determined by $\mathbf{h}_i$. The disjoint union is taken over all i's so that $\mathbf{h}_i, \mathbf{g}_i$ satisfy equation (6).

Proof. Suppose $(\mathbf{g}, \mathbf{h}) = \mathbf{gh} \in G_4 \rtimes H_{16}$ is in the stabilizer group of C, i.e. $\mathbf{g}(C+\mathbf{h}) = C$ (it is clear that for some $\mathbf{h} \in C/\mathrm{Ker}(C)$ the equation $\mathbf{g}(C+\mathbf{h}) = C$ over $\mathbf{g}$ has no solution). Then for any $\mathbf{p} \in P$ and $\mathbf{h}' \in \mathrm{Ker}(C)$ the element $(\mathbf{pg}, \mathbf{h}+\mathbf{h}') \in G_4 \rtimes H_{16}$ is also in the stabilizer group of C (since $\mathbf{p}C = C$ and $\mathbf{h}' + C = C$). To show that the P-coset of $\mathbf{g}$ is uniquely determined by $\mathbf{h}$, suppose that $\mathbf{h} \in C/\mathrm{Ker}(C)$ and $\mathbf{g}_i$, $\mathbf{g}_j$ belong to the different P-cosets of G_4, i.e. $\mathbf{g}_j\mathbf{g}_i^{-1} \notin P$. Then

$$\begin{cases} C = \mathbf{g}_i(C+\mathbf{h}) \\ C = \mathbf{g}_j(C+\mathbf{h}) \end{cases}$$

Thus $\mathbf{g}_i^{-1}C = \mathbf{g}_j^{-1}C$, which implies $\mathbf{g}_j\mathbf{g}_i^{-1} \in P$ i.e. $\mathbf{g}_i$, $\mathbf{g}_j$ belong to the same P-coset of G_4 which leads to a contradiction. $\triangle$

Lemma 5 . *Let C be a Vasiliev code and $\mathrm{Stab}(C) = \mathrm{Stab}_G(C)$ ($G = S_{16} \rtimes H_{16}$) be its stabilizer group. Then*

$$\mathrm{Stab}(C) \;=\; \mathrm{Stab}_{G_4\rtimes H_{16}}(C) \cup \bigcup_i\bigcup_j P\mathbf{x}_{ij}^{-1}\sigma_3\mathbf{g}_{ij}P_i \rtimes (\mathbf{h}_i + \mathrm{Ker}(C)),$$

where $\mathbf{h}_i$'s is the subset of $C/\mathrm{Ker}(C)$ representatives; $\mathbf{x}_{ij} \in \mathrm{N}(\sigma_3)\backslash G_4/\mathrm{Stab}_{G_4}(C)$ and $\mathbf{g}_{ij} \in G_4/P_i$ is uniquely determined by $\mathbf{x}_{ij}$. The disjoint union is taken over all i's and j's so that $\mathbf{h}_i, \mathbf{x}_{ij}, \mathbf{g}_{ij}$ satisfy equation (7).

Proof. The double coset decomposition follows directly from Lemmas 1 and 2. To show that the union is disjoint, note that $\tau\mathbf{h} = \tau'\mathbf{h}'$ if and only if $\tau = \tau'$ and $\mathbf{h} = \mathbf{h}'$. Thus the union is disjoint for different $\mathbf{h}_i$'s. $\triangle$

Lemma 6 . *Let C be a Vasiliev code and $\mathrm{Stab}(C) = \mathrm{Stab}_G(C)$ be its stabilizer group and $|\mathrm{Stab}(C)|$ the number of elements. Following the notations of Lemma 5, we have*

$$|\mathrm{Stab}(C)| \;=\; |\mathrm{Stab}_{G_4\rtimes H_{16}}(C)| + |\mathrm{Ker}(C)|\cdot|P| \times \left(\sum_i\sum_j \frac{|P_i|}{|\mathbf{y}_{ij}^{-1}P\mathbf{y}_{ij}\cap P_i|}\right),$$

where $\mathbf{y}_{ij} = \mathbf{x}_{ij}^{-1}\sigma_3\mathbf{g}_{ij}$.

Proof. For any (P-P_i)-double coset we have $P\mathbf{y}_{ij}P_i = P\mathbf{y}_{ij}P_i\mathbf{y}_{ij}^{-1}\cdot\mathbf{y}_{ij}$ so that

$$P\mathbf{y}_{ij}P_i \;=\; \bigcup_k P\mathbf{y}_{ij}\mathbf{z}_{ijk}, \text{ where } \mathbf{z}_{ijk} \in \mathbf{y}_{ij}^{-1}P\mathbf{y}_{ij}\cap P_i\backslash P_i.$$

Thus the number of P-cosets in $P\mathbf{y}_{ij}P_i$ is equal to

$$\frac{|P_i|}{|\mathbf{y}_{ij}^{-1}P\mathbf{y}_{ij} \cap P_i|}.$$

Multiplying it by the number of elements of P-coset and by the number of elements of the group $\mathrm{Ker}(C)$, we obtain the formula. $\triangle$

Summarizing Lemmas 5 and 6, we have

Corollary 1 *Let C be a Vasiliev code. Then*

$$\mathrm{Stab}(C) \;=\; \mathrm{Stab}_{G_4 \rtimes H_{16}}(C) \cup \bigcup_i \bigcup_j \bigcup_k (P\mathbf{y}_{ij}\mathbf{z}_{ijk}, \mathbf{h}_i + \mathrm{Ker}(C)),$$

where $\mathbf{y}_{ij} = \mathbf{x}_{ij}^{-1}\sigma_3\mathbf{g}_{ij}$ and $\mathbf{z}_{ijk} \in \mathbf{y}_{ij}^{-1}P\mathbf{y}_{ij} \cap P_i \backslash P_i$.

Thus, we arrive to the following one of the main results of the paper.

Theorem 1 *The orders of the stabilizer groups $\mathrm{Stab}(i,j,k)$ of all extended Vasiliev (i,j,k)-codes C of length 16 are given in the following table (here $m(i,j,k)$ is the number of the $(P - \mathrm{Ker}\,(C))$-double cosets):*

code (i,j,k)	Stab_{G_4}	$\mathrm{Ker}(i,j,k)$	$m(i,j,k)$	$\lvert\mathrm{Stab}(i,j,k)\rvert$
$(1,1,2)$	768	512	4	3×2^{19}
$(1,1,4)$	1024	512	12	3×2^{21}
$(1,2,1)$	384	128	56	$7 \times 3 \times 2^{17}$
$(1,2,2)$	96	128	8	3×2^{15}
$(1,2,5)$	128	128	16	2^{18}
$(1,5,1)$	768	256	16	3×2^{20}
$(2,2,1)$	384	256	4	3×2^{17}
$(2,2,2)$	96	128	2	3×2^{13}
$(2,2,5)$	128	128	4	2^{16}
$(2,3,40)$	128	128	1	2^{14}
$(3,3,3)$	128	128	48	3×2^{18}
$(3,3,9)$	128	128	96	3×2^{19}

Now we are going to consider the structure of G_{16}-orbit of Vasiliev codes. In [7] we showed that all Vasiliev codes $(16, 4, 2048)$-codes C satisfy, the parity rule (see (1)). Following [7], denote by $\mathcal{C}$ the set of all such $(16, 4, 2048)$-codes, satisfying the parity rule, (with or without the zero word) and by $\mathcal{C}_0$ those with the zero word. Of course if a code $C \in \mathcal{C}$ does not have a zero word then $C + x \in \mathcal{C}_0$ $(x \in C)$ does.

Definition 5 *For any subgroup G of G_{16} and any $C \in \mathcal{C}$ the G-orbit of C in $\mathcal{C}$ is the subset of $\mathcal{C}$:*

$$\mathrm{Orb}_G(C) \;=\; \{\mathbf{g}\,C : \; \mathbf{g} \in G, \; \mathbf{g}\,C \in \mathcal{C}\}.$$

Since G is a group, if $C' \in \mathrm{Orb}_G(C)$ then

$$\mathrm{Orb}_G(C') \;=\; \mathrm{Orb}_G(C).$$

Since any code C without zero can be shifted to zero, it implies that any G_{16}-orbit of $\mathcal{C}$ has representatives from $\mathcal{C}_0$. Set

$$\mathrm{Orb}(C) \;=\; \mathrm{Orb}_{G_{16}}(C) \cap \mathcal{C}_0.$$

The following lemma examines the structure of orbits in details

Lemma 7 *Let C be a Vasiliev code with zero word, i.e. $C \in \mathcal{C}_0$ and let $P(\mathbf{h}) = \mathrm{Stab}_{G_4}(C + \mathbf{h})$. Then*

$$\mathrm{Orb}(C) \;=\; \bigcup_{\mathbf{h}} \bigcup_{\mathbf{x}} \bigcup_{\mathbf{g}} \mathbf{x}\sigma_3\mathbf{g}(C + \mathbf{h}),$$

where $\mathbf{h} \in C/\mathrm{Ker}(C)$, $\mathbf{x} \in G_4/\mathrm{N}(\sigma_3)$ and $\mathbf{g} \in G_4/P(\mathbf{h})$ is the disjoint union.

Proof. Indeed, suppose $C' = \tau(C + \mathbf{h}) \in \mathrm{Orb}(C)$, where $\tau \in S_{16}$ and $\mathbf{h} \in H_{16}$. Since C' has zero word, it follows that $\mathbf{h} \in C$, and in particular $\mathbf{h} \in C/\mathrm{Ker}(C)$. By Proposition 3 $\tau \in G_4\sigma_i G_4$, where $i = 1, \ldots, 6$. Direct calculations for Vasiliev codes show that only $i = 3$ occurs. Thus we can assume that $\tau = \mathbf{x}\sigma_3\mathbf{g}$, where $\mathbf{x}, \mathbf{g} \in G_4$. Furthermore

$$G_4\sigma_3 G_4 \;=\; \bigcup_{\mathbf{x} \in G_4/\mathrm{N}(\sigma_3)} \mathbf{x}\sigma_3 G_4.$$

Obviously $\mathbf{g}$ is defined up to the multiple of $P(\mathbf{h})$ on the right. The proof of Lemma 4 shows that different triples $(\mathbf{x}, \mathbf{g}, \mathbf{h})$ define different codes. $\triangle$

§ 5. Steiner systems $S(v, 4, 3)$ with given rank

A Steiner system $S(v, k, t)$ is a pair (X, B) where X is a v-set and B is a collection of k-subsets of X such that every t-subset of X is contained in exactly one member of B. A system $S(v, 3, 2)$ is called a Steiner triple system (briefly STS(v)) and a system $S(v, 4, 3)$ is called a Steiner quadruple system (briefly SQS(v)). The necessary condition for existence of an SQS(v) is that $v \equiv 2$ or 4 (mod 6).

A binary incidence matrix of a Steiner system $S(v, 4, 3)$ is the binary constant weight code, denoted by $C(v, 4, 4, v(v-1)(v-2)/24)$ which is strongly optimal [15]. In our notation the connection between the system (X, B) and the code C looks as follows:

$$B \;=\; \{\mathrm{supp}(\mathbf{v}) \subset X : \; \mathbf{v} \in C\}.$$

More generally, the following result is valid [15]: *the existence of a Steiner system* $S(v, k, t)$ *is equivalent to the existence of a constant weight code* $C(v, k, 2\,(k-t+1), N)$ *where* $N \;=\; \frac{\binom{v}{t}}{\binom{k}{t}}$. In this paper we will mainly use the presentation of $S(v, k, t)$ as the binary constant weight code and denote by C).

Let $S = S(v, 4, 3)$ be a Steiner system and let $C(v, 4, 4, v(v - 1(v - 2)/24)$, be the corresponding constant weight code (the incidence matrix of S). Denote by $\mathrm{rank}(S) = \mathrm{rank}(C)$ the dimension of the linear envelope of words of C over $\mathbb{F}_2$.

In the previous paper [6] we classified all Steiner systems $S(16, 4, 3)$ of rank less or equal 13 over F_2. In particular, we found 15 non-isomorphic such systems with rank exactly 12 (in [6] we give the explicit construction of all these systems). Any Steiner system $S(16, 4, 3)$ with rank less or equal to 13 can be constructed by GC-construction, based on mapping of two quaternary codes into binary ones. One of the codes is an MDS $(4, 2, 64)_4$-code A and the other code is a constant weight $(4, 2, 2, 18)$-code W. There are five non-equivalent MDS codes A_i, $i = 1, 2, 3, 4, 5$ [7], and eleven non-equivalent constant weight codes W_j, $j = 1, 2, ..., 11$ [6]. The code A_i defines uniquely the odd $(16, 4, 4, 64)$-code $C_i = C(A_i)$ and the code W_j defines uniquely the even $(16, 4, 4, 76)$-code $V_j = V(W_j)$. The resulting constant weight $(16, 4, 4, 140)$-code C is the following union:

$$C \;=\; C_i \;\cup\; C_i,\; \mathbf{d}_{ij}^{(k)}\, V_j, \quad k = 1, 2, ..., m_3(i, j),$$

where $\mathbf{d}_{ij}^{(k)}$ is the fixed representative in the $(P_i - Q_j)$-double coset decomposition of G_4, $P_i = \mathrm{Stab}_{G_4}(C_i)$ and $Q_j = \mathrm{Stab}_{G_4}(W_j)$ and $m_3(i,j)$ is the number of $(P_i - Q_j)$-double cosets of G_4. We refer to such Steiner system as (i,j,k).

The next statement gives us all 15 non-isomorphic Steiner systems (i,j,k) with rank 12 (see [6] for construction).

Proposition 6 [6] *The* 15 *non-isomorphic Steiner systems* $S(16,4,3)$ *with rank* 12 *over* $\mathbb{F}_2$ *are the following systems* (i,j,k)*:*

$$\begin{array}{ccccc} (1,1,2), & (1,1,4), & (1,2,1), & (1,2,2), & (1,2,5), \\ (1,2,14), & (1,2,15), & (1,6,1), & (1,6,2), & (1,6,5), \\ (2,1,5), & (2,2,2), & (2,2,15), & (3,2,158), & (3,2,162). \end{array}$$

It is well known that an SQS(2^m) system S, formed by the points and planes of the affine geometry AG$(m,2)$ (of dimension m over $\mathbb{F}_2$) has the minimal possible rank: $\mathrm{rank}(S) = 2^m - 1 - m$. Denote by L_m the system $S(2^m,4,3)$ formed by the points and planes of AG$(m,2)$. We describe now the variant of the classical doubling construction A (SQS$(v) \longrightarrow$ SQS$(2v)$ which gives, in particular, all SQS(2^m) with rank equal to $2^m - m$. This construction comes from GC-construction of extended (binary perfect nonlinear) Vasiliev $(n,4,2^{n-m-1})$-codes, which we described in § 3. It is known [11] that any code C with such parameters with rank(C) over F_2 equal $\mathrm{rank}(C) = 2^m - m$ is a Vasiliev code. From the other hand, it is well known that for any such (extended binary) perfect code C with zero codeword codewords of weight four form a Steiner system $S(n,4,3)$. Thus the construction which gives all non-equivalent Vasiliev codes with such rank induces the construction of Steiner systems SQS(2^m) with rank $2^m - m$. Furthermore it occurs that by this construction we obtain all non-isomorphic Steiner systems SQS(2^m) with rank $2^m - m$.

It is convenient to explain this construction in terms of corresponding constant weight codes (i.e. in terms of incidence matrices). Let we have the SQS(v) system S_v given by the pair (X,B),

$$X = \{1,2,...,v\}, \quad B = \{b_1,...,b_M\}, \quad M = v\,(v-1)(v-2)/24),$$

which we present by constant weight $(v, 4, 4, M)$-code denoted by C_v,

$$C_v \;=\; \{\mathbf{c}_1, ..., \mathbf{c}_M\}, \quad \mathbf{c}_i = (c_{i,1}, ..., c_{i,v})$$

where $\mathrm{supp}\,(\mathbf{c}_i) = b_i$ for $i = 1, ..., M$. We double each position of C_v by doubling the set X: to each element $i \in X$ we associate the pair of elements (i_1, i_2). To each codeword $\mathbf{c}_i \in C$ we associate the eight following vectors $V(\mathbf{c}_i)$ of length $2\,v$. For example, for the case when $\mathbf{c}_i = (1, 1, 1, 1, 0, 0, ..., 0)$ the set of eight vectors from $V(\mathbf{c}_i)$ look as follows:

$$\begin{aligned} V(\mathbf{c}_i) \;&=\; \{(1,0,1,0,1,0,1,0,0,0,...,0), \quad (0,1,0,1,0,1,0,1,0,0,...,0),\\ &\quad (1,0,1,0,0,1,0,1,0,0,...,0), \quad (0,1,0,1,1,0,1,0,0,0,...,0),\\ &\quad (1,0,0,1,1,0,0,1,0,0,...,0), \quad (0,1,0,1,1,0,1,0,0,0,...,0),\\ &\quad (1,0,1,0,0,1,0,1,0,0,...,0), \quad (0,1,0,1,1,0,1,0,0,0,...,0)\}. \end{aligned} \tag{8}$$

Thus, nonzero positions of $V(\mathbf{c}_i)$ are exactly four first pairs of positions of a new code of length $2\,v$. Now choose for any $\mathbf{c}_i$ the arbitrary 2-even vector $\mathbf{h}(\mathbf{c}_i)$ such that $\mathrm{supp}\,(\mathbf{h}(\mathbf{c}_i)) \subseteq \mathrm{supp}\,(V(\mathbf{c}_i)$. For the case of $\mathbf{c}_i$, which we consider, we can take, for example, any one from 16 vectors, having either $(0,0)$ or $(1,1)$ on the first four pair positions and $(0,0)$ at the all remaining $v-4$ pairs of positions. Finally, we define the set V of 2-even vectors, which consists of all 2-even vectors of weight four and length $2\,v$. This gives $\binom{v}{2} = v\,(v-1)/2$ vectors. Finally define the resulting constant weight code $C_{2\,v}$:

$$C_{2\,v} \;=\; V \cup \{\cup_{i=1}^{v}\{V(\mathbf{c}_i) + \mathbf{h}(\mathbf{c}_i)\}\};$$

here $V(\mathbf{c}_i) + h(\mathbf{c}_i) \,=\, \{\mathbf{c} + \mathbf{h}(\mathbf{c}_i)\,:\;\; \mathbf{c} \in V(\mathbf{c}_i)\}$. By construction $C_{2\,v}$ is a constant weight code where all codewords $\mathbf{c}$ have the weight four: $\mathrm{wt}(\mathbf{c}) = 4$ (indeed, adding of vector $\mathbf{h}(\mathbf{c}_i)$ does not change the weight: $(1,0) + (1,1) = (0,1)$ and $(0,1) + (1,1) = (1,0)$. The number of codewords of $C_{2\,v}$ is equal to

$$|C_{2\,v}| \;=\; 8 \times \frac{v\,(v-1)(v-2)}{24} \;+\; \binom{v}{2} \;=\; \frac{2\,v\,(2\,v-1)(2\,v-2)}{24},$$

i.e. how it should be for a Steiner system $S(2\,v, 4, 3)$. Now we have to check that the resulting constant weight code $C_{2\,v}$ has the minimal distance $d(C_{2\,v}) \,=\, 4$. Consider two arbitrary codewords of $C_{2\,v}$, say, $\mathbf{c}$ and $\mathbf{c}'$. Assume, first, that $\mathbf{c} \in V(\mathbf{c}_i)$ and $\mathbf{c}' \in V(\mathbf{c}_j)$. For the case $i \neq j$ it follows from the fact, that $d(\mathbf{c}_i, \mathbf{c}_j) \geq 4$ (indeed, C_v has the minimal distance $d(C_v) = 4$). For the case $i = j$ it follows from the construction (all 8 words of $V(\mathbf{c}_i)$ have the minimal distance 4). Now consider the case $\mathbf{c} \in V(\mathbf{c}_i)$ and $\mathbf{c}' \in V$. Since $\mathbf{c}'$ is 2-even of weight 4 and $\mathbf{c}$ contains exactly four 2-odd blocks, in the worst case supports of $\mathbf{c}$ and $\mathbf{c}'$ have two elements in common: $|\mathrm{supp}\,(\mathbf{c}) \cap \mathrm{supp}\,(\mathbf{c}')| = 2$, which implies that $d(\mathbf{c}, \mathbf{c}') = 4$. Finally, for the case, $\mathbf{c}, \mathbf{c}' \in V$ it follows from definition of 2-even vectors: two distinct 2-even vectors of same weight have the distance $d \geq 4$. Thus the resulting code is a constant weight $(2\,v, 4, 4, 2\,v(2\,v-1)(2\,v-2)/24)$-code $C_{2\,v}$, which correspond to a Steiner system $S(2\,v, 4, 3)$.

What is a rank of this system? If the original system $S_v = S(v, 4, 3)$ for the case $v = 2^m$ is the system L_v (points and planes of the affine geometry $\mathrm{AG}(m, 2)$), then the $\mathrm{rank}(L_v) = v - 1 - m$. In this case we have for the resulting system $S_{2\,v} = S(2\,v, 4, 3)$ that $\mathrm{rank}(S_{2\,v}) \leq 2\,v - m - 1$ for any choice of vectors $\mathbf{h}(\mathbf{c}) :\ \mathbf{c} \in C_v$. More exactly, if all vectors $h(c)$ are the zero vector, then the resulting system $S_{2\,v}$ is the system $L_{2\,v}$ and, hence has $\mathrm{rank}(L_{2\,v}) = 2\,v - m - 2$. In all other cases, i.e. when there are nonzero vectors $\mathbf{h}(\mathbf{c})$, the rank is equal to $\mathrm{rank}(S') = 2\,v - m - 1$. This follows from two following known results.

1) By construction the set of codewords of the code $C_{2\,v}$ is a subset of the extended Vasiliev $(2\,v, 4, 2^{2\,v-m-2})$-code, obtained by GC-construction [9], described in § 3.

2) The extended binary perfect nonlinear $(2\,v, 4, 2^{2\,v-m-2})$-code has the rank $2\,v - m - 1)$ if and only if it is a nonlinear extended Vasiliev code of length $2\,v = 2^{m+1}$ [11].

Now assume that v is arbitrary and that the original system $S(v, 4, 3)$ has a rank r. We want to show that under construction above the resulting system $S' = S(2\,v, 4, 3)$ has the rank $\mathrm{rank}(S') \leq r + v - 1$. To see it, we first note that the set of vectors V has the rank $v - 1$ over F_2. This is clear, since all vectors of weight 2 generate the space of all even vectors E^v_{ev} which has the rank $v - 1$. Now consider the contribution of all vectors from the set $V^* = \cup_{\mathbf{c} \in C_v} V(\mathbf{c})$. By construction of sets $V(\mathbf{c})$, it is clear that

$$\mathrm{rank}\,(V^*) \geq \mathrm{rank}\,(C_v). \tag{9}$$

Without loss of generality we can assume that $\mathbf{c}_1, ..., \mathbf{c}_r$ are linearly independent over F_2 vectors from C_v. Now consider the rank of the union $V \cup V(\mathbf{c}_i)$ for some $i \in \{1, ..., r\}$. We claim that $\mathrm{rank}\,(V \cup V(\mathbf{c}_i)) = v$ for any $i \in \{1, ..., r\}$. The fact that $\mathrm{rank}\,(V \cup V(\mathbf{c}_i)) \geq v$ is quite evident. Indeed, V consists of only 2-even vectors, and each vector from $V(\mathbf{c}_i)$ has exactly four 2-odd blocks. Now consider eight vectors from $V(\mathbf{c}_i)$. As we can see from the example of $V\,(\mathbf{c}_i)$ given in (8) above, for any two vectors $\mathbf{c}$ and $\mathbf{c}'$ from $V(\mathbf{c}_i)$ we have that $\mathbf{c} + \mathbf{c}'$ is a 2-even vector. Adding of any 2-even vector $\mathbf{h}\,(\mathbf{c}_i)$ to all vectors from $V\,(\mathbf{c}_i)$ does not change this property, since any 2-even vector is a linear combination of vectors from V. We conclude, therefore, that $\mathrm{rank}\,(V \cup V(\mathbf{c}_i)) \leq v$. This implies that $\mathrm{rank}\,(V \cup V(\mathbf{c}_i)) = v$.

Now it is clear that r linearly independent vectors $\mathbf{c}_1, ..., \mathbf{c}_r$ from C_v induce r linearly independent vectors, say $\mathbf{c}'_1 \in V\,(\mathbf{c}_1)$, ..., $\mathbf{c}'_r \in V\,(\mathbf{c}_r)$. As we have proved above, from any set $V\,(\mathbf{c}_i)$, $i = 1, ..., r$ only one vector contribute to the rank of $C_{2\,v}$. We conclude, therefore, that $\mathrm{rank}\,(C_{2\,v}) = v - 1 + r$.

Thus, we have proved the following result.

Theorem 2 . *For any Steiner system $S_v = S(v, 4, 3)$ of rank r over F_2, the construction, described above provides a Steiner system $S_{2\,v} = S(2\,v, 4, 3)$ of rank* $\mathrm{rank}\,(S_{2\,v}) \leq r + v$*. Furthermore, if all vectors $h(c)$ are zero vectors, then* $\mathrm{rank}\,(S_{2\,v}) = r + v - 1$*, otherwise* $\mathrm{rank}\,(S_{2\,v}) = r + v$*.*

The natural question is *under what conditions the Steiner system $S(v, 4, 3)$ with rank r is obtained by the construction, described above?* The next statement answers this question for the case $v = 2^m$ and $r = 2^m - m$.

Theorem 3 . *Let S be a Steiner system $S(2^m, 4, 3)$ with* $\mathrm{rank}\,(S) = 2^m - m$ *over $\mathbb{F}_2$. Then this system S is obtained from L_{m-1} by the construction, described above.*

Proof. Let S be a Steiner system $S(2^m, 4, 3)$ with $\mathrm{rank}\,(S) = 2^m - m$ over $\mathbb{F}_2$. Let C be the corresponding constant weight $(v, 4, 4, M)$-code where $v = 2^m$ and $M = v\,(v-1)(v-2)/24$. Let $v = 2\,u$, i.e. $u = 2^{m-1}$. Since $\mathrm{rank}\,(C) = 2^m - m$, the dual code $C^{\perp}$ is a linear code, say $[v, m, d^{\perp}]$-code. Taking into account the result of [12] we conclude immediately that

$d^{\perp} = v/2 = u$ and that for any codeword $\mathbf{x} \in C^{\perp}$ we have that $\mathrm{wt}(\mathbf{x}) = v/2$ or v. Without loss of generality we can choose as a basis of $C^{\perp}$ the following codewords:

$$\begin{aligned}
\mathbf{x}_1 &= (1111|1111|...|1111|1111|1111|1111|...|1111|1111),\\
\mathbf{x}_2 &= (1111|1111|...|1111|1111|0000|0000|...|0000|0000),\\
& \quad .... \qquad\qquad ... \qquad\qquad ...\\
\mathbf{x}_{m-2} &= (1111|1111|...|0000|0000|1111|1111|...|0000|0000),\\
\mathbf{x}_{m-1} &= (1111|0000|...|1111|0000|1111|0000|...|1111|0000),\\
\mathbf{x}_m &= (1100|1100|...|1100|1100|1100|1100|...|1100|1100).
\end{aligned}$$

Partition the coordinate set $J = \{1, 2, ..., v\}$ of the code C into $v/2 = u$ subsets $J_1, J_2, ..., J_u$, where $J_i = \{2\,i - 1, 2\,i\}$. Partition each codeword $\mathbf{c} \in C$ into u subsets:

$$\mathbf{c} \;=\; (\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, ... \,|\, \mathbf{c}_u), \quad \mathrm{supp}(\mathbf{c}_i) \;=\; J_i.$$

The code C partition into two subcodes C_{even} and C_{odd}:

$$\begin{aligned}
C_{\mathrm{even}} &= \{\mathbf{c} \in C : \; \mathbf{c} \text{ is a 2-even vector}\},\\
C_{\mathrm{odd}} &= \{\mathbf{c} \in C : \; \mathbf{c} \text{ otherwise}\}.
\end{aligned}$$

We claim that

$$|C_{\mathrm{even}}| \;=\; \binom{u}{2}, \quad |C_{\mathrm{odd}}| \;=\; \frac{u\,(u-1)(u-2)}{24} \times 8. \tag{10}$$

Indeed, recall that

$$|C| \;=\; \frac{v\,(v-1)(v-2)}{24} \;=\; \frac{u\,(u-1)(u-2)}{24} \times 8 \;+\; \binom{u}{2}, \quad v = 2\,u. \tag{11}$$

But the number of 2-even vectors of weight 4 and length $v = 2\,u$ is not more than $\binom{u}{2}$. From the other side, C is an incidence matrix of of $S(v, 4, 3)$. Hence for any vector $\mathbf{y}$ of weight 3, there is exactly one vector $\mathbf{c} \in C$ such that $\mathrm{supp}\,(\mathbf{y}) \in \mathrm{supp}\,(\mathbf{c})$. Choosing all possible such vectors $\mathbf{y}$ of weight 3 with exactly one 2-odd block, we obtain immediately that C_{even} should contain all possible 2-even vectors of length $v = 2\,u$ and weight 4. We conclude that

$$|C_{\mathrm{even}}| \;=\; \binom{u}{2}. \tag{12}$$

From (11) and (12) we deduce that

$$|C_{\rm odd}| \;=\; \frac{u\,(u-1)(u-2)}{24} \times 8. \qquad (13)$$

Consider $C_{\rm odd}$. First we note the following simple fact.

Lemma 8 *Let $W_{\rm odd}$ be a constant weight $(8,4,4,N_{\rm odd})$-code with the following property: all its codewords are 2-odd vectors. Then $N_{\rm odd} \leq 8$.*

Proof. Denote by $W_{\rm even}$ the following constant weight $(8,4,4,6)$-code, containing only 2-even vectors:

$$\begin{array}{ccc} (11\,|\,11\,|\,00\,|\,00), & (11\,|\,00\,|\,11\,|\,00), & (11\,|\,00\,|\,00\,|\,11), \\ (00\,|\,00\,|\,11\,|\,11), & (00\,|\,11\,|\,00\,|\,11), & (00\,|\,11\,|\,11\,|\,00). \end{array}$$

It is easy to see that for any code $W_{\rm odd}$ with $N_{\rm odd}$ codewords the union $W_{\rm odd} \cup W_{\rm even}$ is a constant weight $(8,4,4,N)$-code where $N = N_{\rm odd} + N_{\rm even}$). It is well known that $N \leq 14$; the optimal $(8,4,4,14)$-code corresponds to the Steiner system $S(8,4,3)$ [15]. Since $N_{\rm even} = 6$, it follows that $N_{\rm odd} \leq 8$ for any code $W_{\rm odd}$. $\triangle$

The set $C_{\rm odd}$ we divide into K subsets C_{i_1,i_2,i_3} where the integers $i_1, i_2, i_3 \in \{1, 2, ..., u\}$:

$$C_{i_1,i_2,i_3} \;=\; \{\mathbf{c} = (\mathbf{c}_1\,|\,...\,|\,c_u) \in C_{\rm odd}:\; {\rm supp}\,(\mathbf{c}_{i_s}) \in J_{i_s},\;\; s = 1,2,3\}.$$

We claim that for any set C_{i_1,i_2,i_3} there is $i_4 \in \{1, 2, ..., u\}$ such that for any $\mathbf{c} \in C_{i_1,i_2,i_3}$

$${\rm supp}\,(\mathbf{c}) \in \cup_{s=1}^{4} J_{i_s}.$$

By definition of a Steiner system, for any $\mathbf{c} \in C_{i_1,i_2,i_3}$ there is a number $i \in J$, $i \neq i_s$, $s = 1,2,3$ such that $c_i = 1$. By the definition of $C_{\rm odd}$ we have that $i \notin J_{i_s}$, $s = 1,2,3$. This implies that there is some $i_4 \in J$ such that $i \in J_{i_4}$. But any vector from $C_{\rm odd}$ is orthogonal over $\mathbb{F}_2$ to any vector from $C^{\perp}$. Using vectors $\mathbf{x}_k$, $k = 2, 3, ...m-1$, given above partition the set J into subsets of length 4:

$$J \;=\; J^{(1)} \cup J^{(2)} \cup ... \cup J^{(v/4)}.$$

Partition any codeword $\mathbf{c} = (c_1, c_2, ..., c_v) \in C$ into blocks of length 4 according to this partition:

$$\mathbf{c} \;=\; (\mathbf{c}_1 \,|\, \mathbf{c}_2 \,|\, ... \,|\mathbf{c}_{u/2}), \quad \mathrm{supp}\,(\mathbf{c}_i) \in J^{(i)}.$$

Now consider any $\mathbf{c} \in C_{i_1,i_2,i_3}$ and let $i \in J_{i_4}$ be such that $c_i = 1$. We claim that there are two numbers j_1 and j_2 such that

$$J^{(j_1)} \;=\; J_{\ell_1} \cup J_{\ell_2}, \;\; \mathrm{and} J^{(j_2)} \;=\; J_{\ell_3} \cup J_{\ell_4}$$

where $\{i_1, i_2, i_3, i_4\} = \{\ell_1, \ell_2, \ell_3, \ell_4\}$. If not, we immediately obtain contradiction with orthogonality of that $\mathbf{c}$ and any linear combination of vectors $\mathbf{x}_2, \mathbf{x}_3, ..., \mathbf{x}_{m-1}$. Furthermore, the number i_4 is the same for any $\mathbf{c}$ from C_{i_1,i_2,i_3}. Hence for any triple (i_1, i_2, i_3) from the set $J = \{1, 2, ..., u\}$ there is an unique 4-tuple from J containing it. Thus, any set C_{i_1,i_2,i_3} induces on the eight positions of J_{i_1}, J_{i_2}, J_{i_3}, and J_{i_4} a constant weight $(8, 4, 4, N_{i_1,i_2,i_3})$-code, which consists only of 2-odd vectors. By Lemma 8 any such code has the cardinality $N_{i_1,i_2,i_3} \leq 8$. We conclude, therefore, that

$$|C_{\mathrm{odd}}| \;\leq\; \frac{u\,(u-1)(u-2)}{24} \times 8. \tag{14}$$

From (13) and (14) we deduce that

$$|C_{\mathrm{odd}}| \;=\; \frac{u\,(u-1)(u-2)}{24} \times 8. \tag{15}$$

But there are $\frac{u\,(u-1)(u-2)}{6}$ distinct sets C_{i_1,i_2,i_3} and each set has not more than 8 codewords. We conclude from (15), that each such set C_{i_1,i_2,i_3} contains exactly 8 codewords. Since there are exactly $\frac{u\,(u-1)(u-2)}{24}$ distinct 4-tuples and any two distinct such 4-tuples have intersection not more that in two elements, we deduce that the set of 4-tuples (i_1, i_2, i_3, i_4) from J form a Steiner system $S(u, 4, 3)$. Thus, the Steiner system $S(2\,u, 4, 3)$ is obtained from $S(u, 4, 3)$ by construction, described above. $\triangle$

As a direct corollary of Theorems 2 and 3 and Proposition 4, we obtain the following result.

Theorem 4 . *Let S be a Steiner system $S(v, 4, 3)$, $v = 2^m$ with* $\mathrm{rank}\,(S) = v - m$ *over $\mathbb{F}_2$. Let C be the corresponding constant weight $(v, 4, 4, v\,(v-1)(v-2)/24)$-code. Then this code*

C is a subset of the Vasiliev $(v, 4, 2^{v-m})$-code obtained by the GC-construction, described by Proposition 4.

§ 6. Automorphism groups of Steiner systems $S(16, 4, 3)$ of rank 12

Definition 6 *For any Steiner system S let*

$$Q = \mathrm{Stab}_{G_4}(S).$$

For any Steiner system S write $\mathrm{Stab}(S)$ for $\mathrm{Stab}_{S_{16}}(S)$.

Lemma 9 *Let S be a Steiner system and $\tau \in \mathrm{Stab}(S) \subset G = S_{16}$. Let $\tau' \in Q\tau Q$. Then $\tau' \in \mathrm{Stab}(S)$.*

Proof. Similar to the proof of Lemma 1. △

Lemma 10 *Let S be a Steiner system of rank 12 and $\tau \in \mathrm{Stab}(S)$. Then $\tau = \mathbf{g} \in G_4$ or $\tau = \mathbf{x}\sigma_3\mathbf{g} \in G_4\sigma_3G_4$, where $\mathbf{x} \in Q\backslash G_4/\mathrm{N}(\sigma_3)$ is the $(G_4 - \mathrm{N}(\sigma_3))$-double coset representative and $\mathbf{g}$ belongs to the Q-coset of G_4, uniquely determined by $\mathbf{x}$.*

Proof. Indeed, suppose $\tau \in \mathrm{Stab}(S)$. By Proposition Lemma 5 of [9], the permutation τ belongs to G_4 or $G_4\sigma_3G_4$. Suppose $\tau = \mathbf{x}\sigma_3\mathbf{g}$, where $\mathbf{x}, \mathbf{g} \in G_4$. Then $\tau S = S$ is equivalent to

$$\mathbf{x}\sigma_3\mathbf{g}S = S. \tag{16}$$

We will show that if $\mathbf{x}$ satisfies this equation then any $\mathbf{x}' \in Q\mathbf{x}\mathrm{N}(\sigma_3)$ will also satisfy it. Indeed, let $\mathbf{x}' = \mathbf{p}\mathbf{x}\mathbf{x}_1$, where $\mathbf{x}_1 \in \mathrm{N}(\sigma_3)$ and $\mathbf{p} \in Q$. Let $\mathbf{x}_2 = \sigma_3\mathbf{x}_1\sigma_3 \in G_4$. Multiplying both sides of (16) by $\mathbf{p}$ we obtain

$$\mathbf{p}\mathbf{x}\sigma_3\mathbf{g}S = \mathbf{p}\mathbf{x}\mathbf{x}_1\mathbf{x}_1^{-1}\sigma_3\mathbf{g}S = \mathbf{x}'\sigma_3\mathbf{g}'S = S,$$

where $\mathbf{g}' = \mathbf{x}_2^{-1}\mathbf{g} \in G_4$. Next, we show that if $(\mathbf{x}, \mathbf{g})$ is a solution to (16), then $\mathbf{g}$ belongs to the unique Q-coset. Suppose for a given $\mathbf{x}$ there exist $\mathbf{g}$ and $\mathbf{g}'$ which satisfy (16). Thus

$$\mathbf{x}\sigma_3\mathbf{g}S = \mathbf{x}\sigma_3\mathbf{g}'S,$$

which is equivalent to $\mathbf{g}S = \mathbf{g}'S$, i.e. $\mathbf{g}^{-1}\mathbf{g}' \in Q$ or $\mathbf{g}' \in \mathbf{g}Q$. Thus, for a given $\mathbf{x}$, the Q-coset of $\mathbf{g}$ is uniquely determined by $\mathbf{x}$. △

Lemma 11 *Let S be a Steiner system and* $\mathrm{Stab}(S) = \mathrm{Stab}(S)$ *be its Automorphism group. Let* $Q = \mathrm{Stab}_{G_4}(S)$*. Then*

$$\mathrm{Stab}(S) \ = \ Q \cup \bigcup_i Q\mathbf{x}_i\sigma_3\mathbf{g}_iQ,$$

where $\mathbf{x}_i \in Q\backslash G_4/\mathrm{N}(\sigma_3)$ *and* $\mathbf{g}_i \in G_4/Q$ *is uniquely determined by* $\mathbf{x}_i$*. The elements* $\mathbf{x}_i, \mathbf{g}_i$ *run over all pairs that satisfy (16).*

Proof. The double coset decomposition follows directly from Lemmas 9 and 10. △

Lemma 12 *Let S be a Steiner system,* $\mathrm{Stab}(S) = \mathrm{Stab}(S)$ *its Automorphism group and* $|\mathrm{Stab}(S)|$ *the number of elements. Following the notations of Lemma 11, we have*

$$|\mathrm{Stab}(S)| \ = \ |Q| \times \left(1 \ + \ \sum_i \frac{|Q|}{|\mathbf{y}_i^{-1}Q\mathbf{y}_i \cap Q|}\right),$$

where $\mathbf{y}_i = \mathbf{x}_i\sigma_3\mathbf{g}_i$*.*

Proof. For any (Q-Q)-double coset we have $Q\mathbf{y}_iQ = Q\mathbf{y}_iQ\mathbf{y}_i^{-1} \cdot \mathbf{y}_i$ so that

$$Q\mathbf{y}_iQ \ = \ \bigcup_j Q\mathbf{y}_i\mathbf{z}_{ij}, \text{ where } \mathbf{z}_{ij} \in \mathbf{y}_i^{-1}Q\mathbf{y}_i \cap Q\backslash Q.$$

Thus the number of Q-cosets in $Q\mathbf{y}_iQ$ is equal to

$$\frac{|Q|}{|\mathbf{y}_i^{-1}Q\mathbf{y}_i \cap Q|}.$$

Multiplying it by the number of elements of Q-coset we obtain the formula. △

Summarizing Lemmas 11 and 12, we have

Corollary 2 *Let S be a Steiner system. Then*

$$\mathrm{Stab}(S) \ = \ Q \cup \bigcup_i \bigcup_j Q\mathbf{y}_i\mathbf{z}_{ij},$$

where $\mathbf{y}_i = \mathbf{x}_i\sigma_3\mathbf{g}_i$ *and* $\mathbf{z}_{ij} \in \mathbf{y}_i^{-1}Q\mathbf{y}_i \cap Q\backslash Q$*.*

Thus, we arrive to the following one of main results of the paper.

Theorem 5 *The orders of the automorphisms groups* $\mathrm{Stab}(i,j,k)$ *of all extended Steiner systems* (i,j,k) *of length* 16 *with rank* 12 *over* F_2 *are given in the following table:*

Steiner (i,j,k)	Stab_{G_4}	$\lvert\mathrm{Stab}(i,j,k)\rvert$
$(1,1,2)$	768	768
$(1,1,4)$	1024	3×1024
$(1,2,1)$	512	3×512
$(1,2,2)$	256	256
$(1,2,5)$	512	3×512
$(1,2,14)$	256	256
$(1,2,15)$	256	256
$(1,6,1)$	768	768
$(1,6,2)$	3072	7×3072
$(1,6,5)$	1024	3×1024
$(2,1,5)$	96	96
$(2,2,2)$	32	3×32
$(2,2,15)$	32	4×32
$(3,2,158)$	128	6×128
$(3,2,162)$	128	6×128

As it should be expected, the homogeneous system $(1,6,2)$ has the largest stabilizer group (see [6, proposition 15]), which has all derivative Steiner Triples Systems $S(15,3,2)$ with number 1 (see [6]). The other homogeneous system with the same derivative systems is the system $(1,1,1)$ with rank 11 over $\mathbb{F}_2$ (the points and planes of affine geometry $\mathrm{AG}(4,2)$).

Note that all Steiner systems $S(16,4,3)$ with rank 12 have the derivative Triple systems with numbers 1 and 2 only.

§ 7. Vasiliev codes of length 16 and Steiner systems $S(16,4,3)$ of rank 12

Now we give the Steiner systems $S(16,4,3)$, which can be formed by the codewords of weight 4 from all Vasiliev codes above.

Definition 7 *Let C be a code from $\mathcal{C}_0$. We say that a Steiner system $S = S(16,4,3)$ belongs to C if there exist $\mathbf{g} \in G_{16}$ such that $S \subset \mathbf{g}C$.*

Proposition 7 *If a Steiner system S belongs to C then there exist $C' \in \mathrm{Orb}(C)$ such that $S \subset C'$.*

Proof. There exist $\mathbf{g} = (\tau, \mathbf{h}) \in G_{16} = S_{16} \rtimes H_{16}$ such that

$$S \subset C', \text{ where } C' = \tau(C + \mathbf{h}).$$

This implies that C' has zero word, i.e. $\mathbf{h} \in C$. Moreover since S satisfies the parity rule it implies that C' satisfies the parity rule as well. Thus $C' \in \mathcal{C}_0$. $\triangle$

The following statement gives the interrelationship between Vasiliev codes of length 16 and Steiner systems $S(16,4,3)$.

Theorem 6 *Extended Vasiliev codes (i,j,k) of length 16 and rank 12 over $\mathbb{F}_2$ contain the following $S(16,4,3)$ systems (i',j',k') of rank 12 or less over $\mathbf{F}_2$. The first column of the table below gives the Vasiliev codes and the corresponding Steiner systems are given at the other columns of the same row):*

$(1,1,2)$	$(1,1,2)$				
$(1,1,4)$	$(1,1,4)$				
$(1,2,1)$	$(1,1,1)$	$(1,2,1)$	$(1,6,2)$		
$(1,2,2)$	$(1,1,2)$	$(1,2,2)$	$(1,2,14)$	$(1,6,1)$	$(2,1,2)$
$(1,2,5)$	$(1,1,4)$	$(1,2,5)$	$(1,2,15)$	$(1,6,5)$	
$(1,5,1)$	$(1,6,2)$	$(1,6,5)$			
$(2,2,1)$	$(1,2,1)$	$(1,2,5)$	$(1,2,14)$		
$(2,2,2)$	$(2,1,2)$	$(2,2,2)$			
$(2,2,5)$	$(1,2,2)$	$(1,2,15)$	$(2,2,15)$		
$(2,3,40)$	$(2,2,2)$	$(2,2,15)$	$(3,2,158)$	$(3,2,162)$	
$(3,3,3)$	$(1,6,1)$	$(3,2,162)$			
$(3,3,9)$	$(3,2,158)$				

As we see from the table above the Vasiliev $(1,2,2)$-code contains 5 non-isomorphic Steiner systems $S(16,4,3)$ (namely, $(1,1,2)$, $(1,2,2)$, $(1,2,14)$, $(1,6,1)$, and $(2,1,2)$) all of them with rank 12 over F_2. The Vasiliev $(1,2,1)$-code contains 3 non-isomorphic Steiner systems $S(16,4,3)$, one of rank 11 (the system $(1,1,1)$) and two of rank 12 (the systems $(1,2,1)$ and $(1,6,2)$).

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