POPOV ABSOLUTE STABILITY CRITERION FOR TIME-VARYING MULTIVARIABLE NONLINEAR SYSTEMS

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Abstract

This paper extends in a simple way the classical absolute stability Popov criterion to multivariable rational systems with time-varying memoryless nonlinearities subject to sector conditions. The proposed sufficient conditions are expressed in terms of easy-to-check Linear Matrix Inequalities, or under a frequency-domain form well-suited for robustness issues, and lead to simple graphical interpretations. Apart from the usual conditions, the results assume basically a sector condition on the derivative of the nonlinearities with respect to time. Results for local and global stability are given.

1 Introduction

We consider in this paper multivariable nonlinear control systems given by:

$$\dot{x} = Ax + Bu, \quad u = -\psi(t,y), \quad y = Cx \ . \qquad (1)$$

where $n, p \in \mathbb{N} \setminus \{0\}$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{p\times n}$. One denotes by H the corresponding matrix transfer function:

$$H(s) = C(sI - A)^{-1}B \ . \qquad (2)$$

One supposes throughout that the chosen representation is minimal [10]. Let the nonlinearity $\psi : \mathbb{R}^+ \times \mathbb{R}^p \to \mathbb{R}^p$ be time-dependent, *decentralized* [11] (that is: $\forall i \in \{1,\dots,p\}$, $\psi_i(t,y) = \psi_i(t,y_i)$), and let it fulfill the following *sector condition*:

$$\forall (t,y) \in \mathbb{R}^+ \times \mathbb{R}^p, \ \psi(t,y)^T(\psi(t,y) - Ky) \leq 0 \ , \qquad (3)$$

for a certain diagonal matrix $K = \operatorname{diag}\{K_i\} \geq 0$. Our goal is to find sufficient conditions for absolute stability (resp. for absolute stability with finite domain [11]), that is conditions under which the uniform global (resp. local) asymptotic stability of the null equilibrium of equations (1) is guaranteed, for any nonlinearity ψ fulfilling (3).

As is well known, asymptotic stability of all the *linear time-invariant* systems obtained when choosing $\psi(t,y) = \operatorname{diag}\{k_i\}y, \quad 0 \leq k_i \leq K_i$, is not even sufficient to deduce the stability of the *stationnary* systems fulfilling (3), see counterexamples in [16] or [2, p. 86-88]. Circle criterion [7] furnishes sufficient condition of absolute stability expressed in the frequency domain, namely

$$I + KH(s) \quad \text{is Strictly Positive Real (SPR)} \ . \qquad (4)$$

Popov criterion [17] constitutes a refinement of the latter, valid only for *stationnary* nonlinearities fulfilling (3). Instead of assumption (4), it supposes that there exists a diagonal matrix η such that

$$I + (I + \eta s)KH(s) \quad \text{is SPR} \ . \qquad (5)$$

Concerning absolute stability of systems with *time-varying nonlinearities*, one may consult [13] for a review of the period 1968-1977, surveying an important number of contributions, especially from Eastern Europe. Pyatnitskii has shown [18] that this property is *equivalent* to the asymptotic stability of all the *time-varying linear* systems of the considered class, that is for the maps $\psi(t,y) = \operatorname{diag}\{k_i(t)\}y, \quad 0 \leq k_i(t) \leq K_i$. This characterization gives rise to a class of sufficient conditions for absolute stability of rational systems, (see e.g. [14], also [12, 3] for some frequential conditions without restrictions on the rate of variation of the nonlinearity).

On the other hand, it is possible to obtain absolute stability for smaller classes of time-varying nonlinearities, by making restrictions on $\frac{\partial \psi}{\partial t}$, an a priori knowledge which may be available, according to the system under study. This is for instance the case when checking the stability of limit cycles, as estimates on the periodic orbit may be available.

The idea of restricting the variation of ψ has been applied in some papers, e.g. [15, 21]. Criteria have been proposed by Rekasius *et al.* [20] and Hul'chuk *et al.* [9], and Bertoni *et al.* [4] proposed a slightly weaker result. In these papers, the restriction on $\frac{\partial\psi}{\partial t}$ takes the form of a *generalized sector condition* (see formula (6) below), and the generalization of Popov criterion is made by addition in (5) of some terms related to this sector.

In the present paper, we propose some simple and tractable absolute stability criteria, valid for systems with time-varying nonlinearities whose variation wrt time is constrained in the same manner than in [20, 9]. Theorem 1 gives a criterion for absolute stability, expressed under the form of Linear Matrix Inequality (LMI), a standard class of problems for which sound numerical methods have been developed [6] Then we prove that it is stronger than the one provided in [20, 9], which is given in Theorem 2 under a frequency domain form as well as LMI form. The graphical criterion for scalar systems given in [4] is extended. At last, a still weaker criterion is provided in Theorem 3, which allows an interpretation in Popov plane. Proofs are presented in Section 3. Generalization to delay systems is made in [5].

We do not consider here problems of existence and uniqueness of the solutions, as these issues have been extensively studied. In the sequel is assumed the existence of *global solutions of* (1), i.e.: for all $\phi \in \mathbb{R}^n$, there exists a continuous function x defined on $[0, +\infty)$, absolutely continuous on $[0, +\infty)$, such that $x(0) = \phi$ and (1) is fulfilled almost everywhere on $[0, +\infty)$. The stability results given below concern the asymptotic behavior of these global solutions.

Notations In all the paper, $\|.\|$ denotes the euclidian norm or the associated matrix norm, the asterisk * denotes complex conjugation, I_r stands for the $r \times r$ identity matrix (simply I when no misunderstanding is possible) , The $\mathcal{H}_\infty$-norm of the transfer function H is denoted $\|H(s)\|_\infty$. For $z \in \mathbb{R}$, one denotes

$$|z|_+ \stackrel{\text{def}}{=} \sup\{z, 0\}, \quad |z|_- \stackrel{\text{def}}{=} \sup\{-z, 0\} .$$

At last, by convention, one extends the action of any map acting on scalar or scalar-valued functions to an operator acting on matrices or matrix-valued functions, obtained by componentwise application of the map. As an exemple, for any diagonal matrix η, one has:

$$|\eta|_\pm = \sup\{\pm\eta, 0\} = \operatorname{diag}\{\sup\{\pm\eta_i, 0\}\} = \operatorname{diag}\{|\eta_i|_\pm\} .$$

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2 Absolute stability criteria

Theorem 1 *Assume that there exists a convex open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ for which the following assumptions hold.*

(H0) *The function ψ is measurable and, for any $y \in \mathcal{O}$, $t \mapsto \psi(t, y)$ is locally Lipschitz (and hence t-a.e. differentiable), with a Lipschitz constant locally integrable wrt $y \in \mathcal{O}$.*

(H1) *The nonlinearity ψ is decentralized and there exists a diagonal matrix $K \geq 0$ such that*

$$\forall (t, y) \in \mathbb{R}^+ \times \mathcal{O}, \quad \psi(t, y)^T(\psi(t, y) - Ky) \leq 0 .$$

(H2) *There exists $\alpha > 0$ such that the poles of the transfer H have real part smaller than $-\alpha$.*

Assume that there exists diagonal matrices $D_j = \operatorname{diag}\{D_{j,i}\}$, $j \in \{1, 2, 3\}$, such that LMI (7) is feasible, in which P and η are the variables, and such that the following Hypothesis is fulfilled (with the same η and D_j)

(H3) *There exists $\gamma : \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{z \to 0} \gamma(z) = 0$ such that for almost any $t \in \mathbb{R}^+$, $\forall y \in \mathcal{O}$, $\forall i \in \{1, \ldots, p\}$,*

$$\eta_i \left(\int_0^{y_i} \frac{\partial\psi_i}{\partial t}(t, z)\, dz - D_{1,i} y_i^2 - D_{2,i} y_i \psi_i(t, y_i) - D_{3,i} \psi_i(t, y_i)^2 \right) \leq \|y\|^2 \gamma(\|y\|) . \quad (6)$$

Then, the origin of system (1) is uniformly locally asymptotically stable. Moreover, if $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, then the origin of system (1) is uniformly globally asymptotically stable.

LMI (7) is indeed piecewise linear wrt η. It gives rise to 2^p LMIs, according to the sign of the η_i.

Circle criterion is a particular case of Theorem 1 for $\eta = 0$, and Popov criterion for $D_j = 0$ (as $\frac{\partial\psi}{\partial t} = 0$).

An important case where (6) is fulfilled is the case where there exists a measurable diagonal Δ s.t.

for almost any $t \in \mathbb{R}^+$, $\forall y \in \mathcal{O}$,

$$y^T \eta \left(\frac{\partial\psi}{\partial t}(t, y) - \Delta(t) y \right) \leq \|y\|^2 \gamma(\|y\|) , \quad (11)$$

and the matrices D_j are then given by

$$D_1 = \frac{1}{2} \operatorname*{sup\,ess}_{t \geq 0} \{ \operatorname{sgn} \eta\ \Delta(t) \},\ D_2 = D_3 = 0 .$$

Matrix D_1 as below is nonnegative, otherwise condition (H1) would be violated. Condition (11) is a "local" sector condition (fulfilled e.g. if $\frac{\partial^2\psi_i}{\partial y_i \partial t}(t, 0)$ exists a.e. and is equal to $\Delta_i(t)$), "global" when $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$.

$$P > 0,\ \eta \stackrel{\text{def}}{=} \operatorname{diag}\{\eta_i\},\ R \stackrel{\text{def}}{=} R_P + \begin{pmatrix} 2C^T|\eta|\ \hat{D}_1KC & C^TK\hat{D}_2|\eta| \\ |\eta|\ \hat{D}_2KC & 2|\eta|\ \hat{D}_3K \end{pmatrix} < 0\ , \tag{7}$$

$$R_P \stackrel{\text{def}}{=} \begin{pmatrix} \hat{A}^TP + P\hat{A} & -PB \operatorname{sgn}\eta + C^TK + \hat{A}^TC^TK|\eta| \\ -\operatorname{sgn}\eta\ B^TP + KC + |\eta|KC\hat{A} + KC & -2I - \eta KCB - B^TC^TK\eta \end{pmatrix}\ , \tag{8}$$

$$\hat{A} \stackrel{\text{def}}{=} A + \frac{1}{2}B(\operatorname{sgn}\eta - I)KC \quad \hat{D}_1 \stackrel{\text{def}}{=} \operatorname{sgn}\eta\ D_1 - \frac{1}{2}(I - \operatorname{sgn}\eta)K(D_2 + KD_3), \tag{9}$$

$$\hat{D}_2 \stackrel{\text{def}}{=} D_2 + (I - \operatorname{sgn}\eta)KD_3, \quad \hat{D}_3 \stackrel{\text{def}}{=} \operatorname{sgn}\eta\ D_3\ . \tag{10}$$

It is not clear how to transform LMI (7) into a frequency condition. On the other hand, solvability of (7) is a consequence of the solvability of the LMI: $P > 0, \eta = \operatorname{diag}\{\eta_i\}$,

$$R_P + \begin{pmatrix} 2C^T||\eta|\hat{D}_1|_+KC & C^TK\hat{D}_2|\eta| \\ |\eta|\ \hat{D}_2KC & 2|\eta|\ \hat{D}_3K \end{pmatrix} < 0\ ,$$

as $K \geq 0$. Now, an easy change of variable shows that the latter problem is in turn solvable if the following LMI has solutions: $P > 0$, $\eta = \operatorname{diag}\{\eta_i\}$,

$$R_P + \begin{pmatrix} 0 & WB + C^TK\hat{D}_2|\eta| \\ B^TW + |\eta|\ \hat{D}_2KC & 2|\eta|\hat{D}_3K \end{pmatrix} < 0\ , \tag{12}$$

where R_P is given by (8), and $W = W^T \leq 0$ by

$$A^TW + WA = 2C^T||\eta|\ \hat{D}_1|_+KC\ . \tag{13}$$

The previous LMI may be transformed *equivalently* as a frequency condition, using Kalman-Yakubovich-Popov (KYP) Lemma as a central argument, as expressed by the following

Theorem 2 *Assume that there exists a convex open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ for which Hypotheses* (H0), (H1), (H2) *hold. Assume that there exists diagonal matrices η and D_j, $j \in \{1,2,3\}$, such that the transfer function matrix*

$$\begin{aligned} &I - \eta KD_3 + (I + \eta(sI + D_2))KH(s) \\ &-H^*(s)\sup\left\{\eta D_1; \frac{|\eta| - \eta}{2}K(D_2 + KD_3)\right\} KH(s) \text{ is SPR}\ , \end{aligned} \tag{14}$$

and such that Hypothesis (H3) *holds (with the same η and D_j). Then, the hypotheses of Theorem* 1 *are fulfilled, and hence its conclusions hold.*

For a scalar system, $p = 1$, and condition (14) is equivalent to $\exists \eta \in \mathbb{R}$, $\forall \omega \in \mathbb{R}$,

$$\begin{aligned} &\frac{1}{K} + \operatorname{Re} H(j\omega) - \eta(D_3 - D_2 \operatorname{Re} H(j\omega) + \omega \operatorname{Im} H(j\omega)) \\ &\quad + \sup\left\{\eta D_1; \frac{|\eta| - \eta}{2}K(D_2 + KD_3)\right\} |H(j\omega)|^2 \geq 0\ . \end{aligned}$$

The graphical interpretation is more general than in [4]:

If, apart from the regularity, sector and stability conditions (H0), (H1), (H2), (H3), *there exists a line of slope $1/\eta \in \mathbb{R} \cup \{\pm\infty\}$ passing through the point $(-\frac{1}{K}, 0)$ and lying to the left of the locus $(\operatorname{Re} H(j\omega), D_3 - D_2 \operatorname{Re} H(j\omega) + \omega \operatorname{Im} H(j\omega) + \operatorname{sgn}\eta \cdot \sup\{\operatorname{sgn}\eta \cdot D_1; \frac{1-\operatorname{sgn}\eta}{2}K(D_2 + KD_3)\}\ |H(j\omega)|^2)$ without intersecting it, then the uniform local stability property holds. If $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, then the uniform global stability property holds.*

An interesting problem is, given K, to determine the largest incertitude on $\frac{\partial\psi}{\partial t}$ under which absolute stability may be proved. In this case, the graphical criterion deduced from Theorem 2 can hardly be used, as the ordinate changes with the D_j. To overcome this drawback, condition (14) should rather be seen as a geometrical condition in the 3-dimensional space $(\operatorname{Re} H(j\omega), \operatorname{Im} H(j\omega), \omega \operatorname{Im} H(j\omega))$ obtained as the product of Nyquist and Popov planes, a condition not easy to interpret. We present in the sequel a weaker but simpler condition, located in the Popov plane.

Theorem 3 *Assume that there exists a convex open neighborhood $\mathcal{O}$ of 0 in $\mathbb{R}^p$ for which Hypotheses* (H0), (H1), (H2) *hold. Assume that there exists diagonal matrices $\eta \geq 0$ and D_j, $j \in \{1,2,3\}$, such that the transfer function matrix*

$$\begin{aligned} &I - \eta D_3 + (I + \eta(D_2 + sI))KH(s) \\ &-K\sup\left\{\eta D_1; \frac{|\eta| - \eta}{2}K(D_2 + KD_3); 0\right\}\ \|H(s)\|_\infty^2 \text{ is SPR}\ , \end{aligned} \tag{15}$$

and such that Hypothesis (H3) *holds. Then, the conclusions of Theorem* 2 *hold.*

The proof is straightforward, and left to the reader. One shows as above, that condition (15) is equivalent to the feasibility of a LMI.

As an example let us examine the case of a scalar system fulfilling (11), the general case (6) is similar. Formula (15) is equivalent to: $\exists \eta \in \mathbb{R},\ \forall \omega \in \mathbb{R}$,

$$\frac{1}{K} + \mathrm{Re}\, H(j\omega) - \eta \Bigg(\omega\, \mathrm{Im}\, H(j\omega) + \frac{1}{2}\, \mathrm{sgn}\eta\, \sup_{t\geq 0} \mathrm{ess}\, \{\, \mathrm{sgn}\eta\ \Delta(t)\}\ \|H(s)\|_\infty^2 \Bigg) \geq 0\ , \quad (16)$$

and this has a clear interpretation:

> *If, apart from the regularity, sector and stability conditions* (H0), (H1), (H2), (H3') *with $\eta \geq 0$ (resp. $\eta \leq 0$), a line of slope $1/\eta$ passing through the point $(-\frac{1}{K}, 0)$ lies above (resp. below) the Popov locus and may be translated vertically towards the locus by a distance $\frac{1}{2}\|H(s)\|_\infty^2 \sup_{t\geq 0} \mathrm{ess}\,\{\Delta(t)\}$ (resp. $-\frac{1}{2}\|H(s)\|_\infty^2 \inf_{t\geq 0} \mathrm{ess}\,\{\Delta(t)\}$) without intersecting it, then the uniform local stability property holds. If $\gamma \equiv 0$ and $\mathcal{O} = \mathbb{R}^p$, then the uniform global stability property holds.*

This is illustrated in Figure 1: in the left (resp. right) Popov diagram, (16) holds if $\sup_{t\geq 0} \mathrm{ess}\, \Delta(t) < \frac{2d}{\|H(s)\|_\infty^2}$ (resp. $\inf_{t\geq 0} \mathrm{ess}\, \Delta(t) > -\frac{2d}{\|H(s)\|_\infty^2}$. In the configurations shown in Figure 1, the quantity d involved is indeed *the least $z > 0$ such that one of the points $(-1/K, \pm z)$ belongs to the convex hull of the Popov locus $\mathfrak{P}$*, see [5].

Remark 4 • *The hypotheses of the criteria do not imply continuity wrt y neither of ψ (except in 0), nor of $\frac{\partial \psi}{\partial t}$.* • *When using Hypothesis* (H3)*, one may wish to consider matrices D_j, $j \in \{1, 2, 3\}$ fulfilling* (6) *only on $[t_0, +\infty)$ for a certain $t_0 > 0$, or even "at infinity", that is taking $t_0 \to +\infty$. Due to the strict inequalities involved, this is licit, provided, for the local stability results, that the flow is continuous in the neighborhood of* 0.

3 Extended sketch of the proofs

We give in the sequel extended sketch of the proofs of the local stability results. The global stability results are proved along the same lines.

3.1 Proof of Theorem 1

• Suppose first that $\eta \geq 0$. Denote $K = \mathrm{diag}\{K_i\}$, and consider the candidate Lyapunov function

$$V(t, x) \stackrel{\text{def}}{=} x^T P x + 2 \sum_{i=1}^{p} \eta_i K_i \int_0^{(Cx)_i} \psi_i(t, z)\ dz\ .$$

Using (H1) and denoting abusively $\dot V = \frac{d}{dt}[V(t, x(t))]$, one deduces that, as long as $y(t) \in \mathcal{O}$, $\dot V$ is t-a.e. smaller than

$$\begin{pmatrix} x \\ \psi(t,y) \end{pmatrix}^T R_P \begin{pmatrix} x \\ \psi(t,y) \end{pmatrix} + 2 \sum_{i=1}^{p} \eta_i K_i \int_0^{y_i(t)} \frac{\partial \psi_i}{\partial t}(t, z)\ dz\ .$$

When proving Popov criterion, only the first term appears in the bound on $\dot V$, and applying KYP Lemma leads to (5).

Use of Hypothesis (H3) leads, as long as $y(t) \in \mathcal{O}$, to

$$\dot V \leq \begin{pmatrix} x \\ \psi(t,y) \end{pmatrix}^T R \begin{pmatrix} x \\ \psi(t,y) \end{pmatrix} + o(\|y\|^2) \quad t\text{-a.e.},$$

On the other hand, for any $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^n$,

$$x^T P x \leq V(t, x) \leq x^T (P + C^T K \eta K C) x\ . \qquad (17)$$

One then deduces that there exists $\varepsilon > 0$ such that,

$$\dot V + \varepsilon V \leq 0\ ,$$

as long as $Cx(t) \in \mathcal{O}$. The hypotheses imply that $t \mapsto V(t, x(t))$ is absolutely continuous, because $\forall t, t' \in \mathbb{R}^+$ s.t. $y(t), y(t') \in \mathcal{O}$, $\forall i \in \{1, \dots, p\}$,

$$\left| \int_0^{y_i(t)} \psi_i(t, z)\ dz - \int_0^{y_i(t')} \psi_i(t', z)\ dz \right| \leq |t - t'| \int_0^{y_i(t)} L_i(z)\, dz + K_i \max\{|y_i(t)|, |y_i(t')|\}\ |y_i(t) - y_i(t')|\ ,$$

where L_i is the Lipschitz constant of ψ, defined by Hypothesis (H1). This implies

$$V(t, x(t))\ e^{\varepsilon t} - V(0, x(0)) = \int_0^t (\dot V + \varepsilon V)\ e^{\varepsilon \tau}\ d\tau \leq 0,$$

as long as $Cx(t) \in \mathcal{O}$. The hypotheses on the set $\mathcal{O}$, together with (17), permit to conclude that system (1) is uniformly locally exponentially stable.

• If now the inequality $\eta \geq 0$ is not verified, then consider instead of $\psi(t, y)$ the new input [19]

$$\hat\psi(t, y) \stackrel{\text{def}}{=} \mathrm{sgn}\,\eta\ \psi(t, y) + \frac{1}{2}(1 - \mathrm{sgn}\,\eta) K y\ .$$

One applies the previous results to the transformed system. Direct computations show that this amounts to apply the result already proved to the system obtained with $\hat\eta \stackrel{\text{def}}{=} |\eta|$ (resp. $\hat A$, $\hat B \stackrel{\text{def}}{=} B\ \mathrm{sgn}\,\eta$, $\hat D_1$, $\hat D_2$, $\hat D_3$) instead of η (resp. A, B, D_1, D_2, D_3). To verify this, check that

$$Ax - B\psi(t, Cx) = \hat A x - \hat B \hat\psi(t, Cx)\ ,$$

and that the following identities are valid

$$\psi(t, y)^T (\psi(t, y) - Ky) = \hat\psi(t, y)^T (\hat\psi(t, y) - Ky)\ ,$$

$$\eta_i \frac{\partial \psi_i}{\partial t}(t, y) = \hat\eta_i \frac{\partial \hat\psi_i}{\partial t}(t, y)\ ,$$

$$\eta_i (D_{1,i} y_i^2 + D_{2,i} y_i \psi_i(t, y_i) + D_{3,i} \psi_i(t, y_i)^2) = \hat\eta_i (\hat D_{1,i} y_i^2 + \hat D_{2,i} y_i \hat\psi_i(t, y_i) + \hat D_{3,i} \hat\psi_i(t, y_i)^2)\ .$$

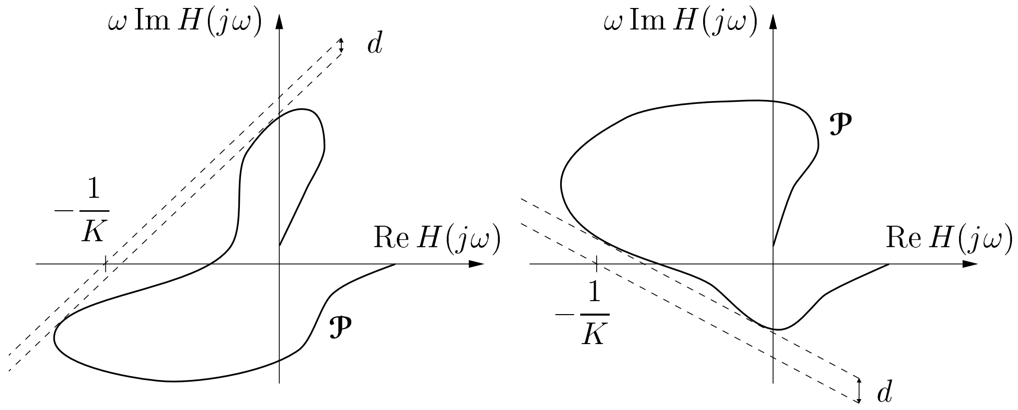

Figure 1: Graphical stability criterion in the Popov plane

3.2 Proof of Theorem 2

1st case: $\eta \geq 0$. The principle of the proof consists in showing that (14) is realized for a certain $\eta \geq 0$ *iff* LMI (12) is feasible.

Suppose first that one may apply KYP Lemma to this LMI. One gets that its solvability is equivalent to $(B^TW + (I + \eta D_2)KC + \eta KCA)(sI - A)^{-1}B + I + \frac{1}{2}\left(\eta K(CB - D_3) + (B^TC^T - D_3)K\eta\right)$ is SPR. Putting in display the fragment $G_P(s) \stackrel{\text{def}}{=} (KC + \eta KCA)(sI - A)^{-1}B + I + \frac{1}{2}(\eta KCB + B^TC^TK\eta)$ appearing in Popov criterion, this is equivalent to: $\exists\varepsilon > 0$, $\forall\omega \in \mathbb{R}$,

$$\begin{aligned}
\varepsilon I &< G_P(j\omega) + G_P^*(j\omega) - 2\eta D_3K + \eta D_2KH(j\omega) \\
&\quad +H^*(j\omega)KD_2\eta \\
&\quad -B^T(j\omega I + A^T)^{-1}WB + B^TW(j\omega I - A)^{-1}B \\
&= G_P(j\omega) + G_P^*(j\omega) - 2\eta D_3K + \eta D_2KH(j\omega) \\
&\quad +H^*(j\omega)KD_2\eta - B^T(j\omega I + A^T)^{-1}(W(j\omega I - A) \\
&\quad -(j\omega I + A^T)W)(j\omega I - A)^{-1}B \\
&= G(j\omega) + G^*(j\omega)\ ,
\end{aligned}$$

by use of (13), where G is the transfer matrix in (14).

It hence remains to show the applicability of KYP Lemma. The pair $\{A, B\}$ is controllable, due to the minimality of the representation (A, B, C). The observability condition necessitates two steps.

• Suppose first that the poles of H are simple. Due to the strict inequality involved in (12), the diagonal components of η may indeed be chosen in a certain *open set* $\mathcal{E}$ of the set of the nonnegative diagonal matrices. If the pair $\{B^TW+KC+\eta KCA, A\}$ is observable for a certain η in $\mathcal{E}$, we are done. Otherwise, due to Popov-Belevitch-Hautus eigenvector test [10], $\forall\eta \in \mathcal{E}$, $\exists(v,\lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}$, $Av = \lambda v$, $(B^TW + KC + \eta KCA)v = 0$. Now, as the poles of H are simple, the eigenvalues of A are also simple, and one may deduce that $\exists(v,\lambda) \in \mathbb{R}^n \setminus \{0\} \times \mathbb{R}$, $\forall m \in \mathbb{N}$, $\exists\eta_m \in \mathcal{E}$, $Av = \lambda v$, $(B^TW_m+KC+\eta_mKCA)v = 0$, where the symmetric nonpositive matrix W_m verifies $A^TW_m + W_mA = C^TK\eta_m|D_1|_+C$. One may hence differentiate wrt η_m the previous *piecewise affine*, which gives $KCv = 0$. This leads to $Cv = CAv = \cdots = CA^{n-1}v$, which contradicts the observability assumption of the pair $\{C, A\}$.

• The case where the poles of H are not simple, is reduced to the previous one by adequate pole-shifting. Roughly speaking, one writes (1) as $\dot{x} = (A+BJ)x - B(\psi(t, y(t)) + Jx)$. If J is "small", the stability and strict positiveness assumptions remain true, but one may hope to have obtained simple poles. Some technical complications come from the necessity to write $\psi(t, y(t)) + Jx$ as the output of a *decentralized* nonlinearity.

A way to circumvent this complication is the following. Decompose J in columns as $J = J_1 + \cdots + J_n$, where $(J_i)_{j,k} = \delta_i^k J_{j,k}$ (Kronecker symbol). Define the following matrices (the subscript a stands for *augmented*):

$$B_a \stackrel{\text{def}}{=} \begin{pmatrix} B & \sqrt{\|J\|}B & \ldots & \sqrt{\|J\|}B \end{pmatrix},\ C_a \stackrel{\text{def}}{=} \begin{pmatrix} C \\ \frac{1}{\sqrt{\|J\|}}J_1 \\ \vdots \\ \frac{1}{\sqrt{\|J\|}}J_n \end{pmatrix},$$

$$\psi_a(t, y_a) \stackrel{\text{def}}{=} \begin{pmatrix} \psi(t, \begin{pmatrix} I_p & 0_{p\times pn} \end{pmatrix} y_a) \\ \begin{pmatrix} 0_{pn\times p} & I_{pn} \end{pmatrix} y_a \end{pmatrix} \text{ for } y_a \in \mathbb{R}^{p(n+1)}\ .$$

Then, one may rewrite system (1) under the form

$$\dot{x} = (A + BJ)x - B_a\psi_a(t, y_a), \quad y_a = C_ax\ .$$

The new nonlinearity ψ_a is decentralized and verifies $\forall(t, y_a) \in \mathbb{R}^+ \times \mathcal{O} \times \mathbb{R}^{pn}$, $\psi_a(t, y)^T(\psi_a(t, y_a) - K_ay_a) \leq 0$, with $K_a \stackrel{\text{def}}{=} \text{diag}\{K, I_{pn}\}$. Also,

$$\frac{\partial\psi_a}{\partial t}(t, y_a) = \begin{pmatrix} \frac{\partial\psi}{\partial t}(t, \begin{pmatrix} I_p & 0_{p\times pn} \end{pmatrix} y_a) \\ 0_{pn\times 1} \end{pmatrix},$$

in such a way that a condition similar to (H3) may be written down for ψ_a, with

$$\eta_a \stackrel{\text{def}}{=} \text{diag}\{\eta, 0_{pn\times pn}\}, \quad D_{a,j} \stackrel{\text{def}}{=} \text{diag}\{D_j, 0_{pn\times pn}\}\ .$$

2nd case: $\eta \not\geq 0$. Let us consider the input $\hat{\psi}$ instead of ψ, as in Proof of Theorem 1 (same notations are used). We express frequency condition (14) in terms of the data rather than the hatted quantities, as in [2].

Lemma 5 *Let us denote*

$$\hat{H}(s) \stackrel{\text{def}}{=} C(sI - \hat{A})^{-1}\hat{B},\ \hat{I} \stackrel{\text{def}}{=} \text{sgn}\ \eta,\ \hat{J} \stackrel{\text{def}}{=} (I - \hat{I})/2\ .$$

Then,

$$\hat{H}(s) = (I + H(s)\hat{J}K)^{-1}H(s)\hat{I} = H(s)(I + \hat{J}KH(s))^{-1}\hat{I}\ .$$

Lemma 5 is a consequence of the feedback structure involved when changing the input, see [5] for a complete proof. This result permits to prove that $I - \hat{\eta}K\hat{D}_3 + (I + \hat{\eta}(sI + \hat{D}_2))K\hat{H}(s) - \hat{H}^*(s)|\hat{\eta}\hat{D}_1|_+K\hat{H}(s)$ is SPR *iff* (14) holds, see [5] for details. This achieves the proof.

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