

ON A NUMBER OF BRANCHES OF PERIODIC SOLUTIONS

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We present estimates for the number of unbounded branches of periodic solutions in a problem on bifurcation from infinity with degeneration of the linear part of order two and suggest sufficient conditions for the existence of many branches.

1. Problem statement

Consider the scalar equation

$$x'' + \lambda x = b(t) + f(x) \quad (1)$$

with a continuous function $b(t) \equiv b(t + 2\pi)$, a continuous bounded function f and the scalar parameter λ . We say that Eq. (1) has an *unbounded branch of periodic solutions with the limit $\xi = \xi(t)$ as $\lambda \rightarrow 1$* if for every sufficiently large $r > 0$ Eq. (1) with some $\lambda = \lambda_r$ has a solution $x_r(t) \equiv x_r(t + 2\pi)$ such that $\|x_r\|_{L^2} = r$ and $\lambda_r \rightarrow 1$, $\|x_r/r - \xi\|_{L^2} \rightarrow 0$ as $r \rightarrow \infty$, where $L^2 = L^2(0, 2\pi)$. If such a branch exists, then $\lambda = 1$ is a bifurcation point in the problem on bifurcation of 2π -periodic solutions of Eq. (1) from infinity. It is important to note that due to the boundedness of f the linearization at infinity brings Eq. (1) with $\lambda = 1$ to the equation $x'' + x = 0$, which has the two-dimensional set of 2π -periodic solutions $r \sin(t + \varphi)$, i.e. we have a bifurcation problem with degeneration of the main linear part of order two.

Let $f = f_0 + f_1$ and the continuous bounded functions f_0, f_1 satisfy

$$\lim_{x \rightarrow -\infty} f_0(x) = f_-, \quad \lim_{x \rightarrow +\infty} f_0(x) = f_+; \quad \lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x f_1(s) ds = 0 \quad (2)$$

with $f_- \neq f_+$. Set $\bar{f} = 2|f_- - f_+|$ and consider the Fourier expansion

$$b(t) = B_0 + \sum_{k=1}^{\infty} B_k \sin(kt + \beta_k), \quad B_k \geq 0, \quad k \in \mathbb{N}, \quad (3)$$

of the forcing b . If $B_1 > 0$, $B_1 \neq \bar{f}$, then Eq. (1) has 2 unbounded branches of periodic solutions as $\lambda \rightarrow 1$; this follows, e.g. from the results of [1].

Here we study the case $B_1 = 0$, which turns out to be “richer”.

2. Main result

Using the Fourier expansion (3), define the continuous 2π -periodic function

$$\chi(\varphi) = \sum_{k=3,5,7,\dots} (1 - k^2)^{-1} B_k \sin(k\varphi - \beta_k). \quad (4)$$

An isolated zero φ_0 of the function χ is called *regular* if in some neighborhood of φ_0 this function has different signs for $\varphi < \varphi_0$ and for $\varphi > \varphi_0$.

Theorem 2.1. *Let $B_1 = 0$. Let $f = f_0 + f_1$ and relations (2) hold with $f_- \neq f_+$. Let function (4) have $K > 0$ regular zeros $\varphi_1, \dots, \varphi_K$ on the segment $[0, 2\pi)$. Then Eq. (1) has at least K unbounded branches of periodic solutions with the limits $\xi_k(t) = \pi^{-1/2} \sin(t + \varphi_k)$, $k = 1, \dots, K$, as $\lambda \rightarrow 1$.*

From the Sturm – Hurwitz Theorem it follows that any real continuous 2π -periodic function (3) has at least $2n$ zeros on the segment $I = [0, 2\pi)$ if $B_0 = \dots = B_{n-1} = 0$. Therefore function (4) has at least 6 zeros on I .

If all zeros of function (4) are isolated, then it has at least 6 regular zeros on I and by Theorem 2.1 Eq. (1) has at least 6 unbounded branches of 2π -periodic solutions as $\lambda \rightarrow 1$. For $b(t) = \cos(2t) + \sin(nt)$ with any odd $n > 1$ formula (4) implies $\chi(\varphi) = (1 - n^2)^{-1} \sin(n\varphi)$; here there are $2n$ regular zeros on I and $2n$ unbounded branches of periodic solutions. The number K of regular zeros of function (4) on I is even for any function b .

In Theorem 2.1 all the branches are continuous in the sense of [3].

3. Remarks

a. Starting from the pioneering work [2], interest of many authors is attracted to the equation

$$x'' + x = b(t) + f(x), \quad (5)$$

i.e. Eq. (1) with the fixed $\lambda = 1$; this is the so-called *resonance* case. The relation $B_1 \neq \bar{f}$ implies an *a priori* estimate $\|x\|_{C^1} \leq \text{const}$ for the norms of all 2π -periodic solutions (if any) of Eq. (5). If $B_1 < \bar{f}$, then Eq. (5) has at least one such solution; the topological index at infinity of the completely continuous vector fields related to the periodic problem is non-zero. If $B_1 > \bar{f}$, then Eq. (5) may have no 2π -periodic solutions; the topological

index at infinity is zero. In particular, there are no 2π -periodic solutions if $B_1 > \bar{f}$ and $f_+ < f(x) < f_-$ for all $x \in \mathbb{R}$.

b. We do not use the variational structure of the periodic problem for Eq. (1). The conclusion of Theorem 2.1 is valid for the perturbed equation of the form, e.g. $x'' + x = b(t) + f(x) + (\lambda - 1)g(t, x, x', x(t - \tau), \lambda)$ with any continuous function $g(t, \dots) = g(t + 2\pi, \dots)$ satisfying the global estimate $|g(\dots)| \leq \delta$ for a sufficiently small $\delta > 0$. Generically, this perturbation is non-potential and destroys the variational structure for $\lambda \neq 1$.

Analogs of Theorem 2.1 are valid for other problems, e.g. periodic problems for vector equations, problems on subharmonics, two-point problems etc. For vector equations the problems are generically non-potential.

c. The proof of Theorem 2.1 is by topological methods. The main point is to reduce the problem to operator equations, which are topologically non-degenerate. The number of unbounded branches of periodic solutions of (1) is defined by a scalar branching equation. In case $B_1 > 0$, one obtains this equation just by multiplying Eq. (1) with x' , integrating over the period and passing to the limit as $\|x\|_C \rightarrow \infty$ in the resulting equation $\langle x', b \rangle = 0$; here $\langle \cdot, \cdot \rangle$ is the scalar product in $L^2(0, 2\pi)$. One arrives at $X_1 B_1 \sin(\varphi - \beta_1) = 0$, where $X_1 \sin(t + \varphi)$ and $B_1 \sin(t + \beta_1)$ are the first harmonics in the Fourier expansions of x and b . Here there are 2 solutions $\varphi = \beta_1$ and $\varphi = \beta_1 + \pi$, which define 2 unbounded branches of 2π -periodic solutions of (1).

If $B_1 = 0$ as in Theorem 2.1, then the limit of $\langle x', b \rangle = 0$ as $\|x\|_C \rightarrow \infty$ is $0 = 0$ and the above scheme does not work. The trick is to use in place of $\langle x', b \rangle = 0$ the equivalent equation $\langle u', f(x) + (1 - \lambda)x \rangle = 0$, where u is a unique 2π -periodic solution of $u'' + u = b(t)$ satisfying $\langle u, \sin t \rangle = \langle u, \cos t \rangle = 0$. Passing to the limit in $\langle u', f(x) + (1 - \lambda)x \rangle = 0$ as $\|x\|_C \rightarrow \infty$, $\lambda \rightarrow 1$, one obtains the branching equation $\chi(\varphi) = 0$.

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