Periodic solutions of equations with oscillating nonlinearities

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Abstract

In this paper we consider the 2π-periodic problem for the equation

$$x'' + n^2 x = f(x) - b(t)$$

where n is a positive integer, $b(t)$ is continuous and 2π-periodic, and $f(x)$ is bounded and continuous. We give a new formulation for the Lazer-Leach conditions for the existence of 2π-periodic solutions, and new sufficient conditions for the existence of unbounded sequences of such solutions.

1 Introduction

In this paper we consider the 2π-periodic problem for the equation

$$x'' + n^2 x = f(x) - b(t) \tag{1}$$

where n is a positive integer, $b(t)$ is continuous and 2π-periodic, and $f(x)$ is bounded and continuous. The corresponding pioneering work is due to Lazer and Leach [6], who proved the existence of at least one 2π-periodic solutions under one of the the conditions

$$|\overline{b}| < 2\left(\liminf_{x\to+\infty} f(x) - \limsup_{x\to-\infty} f(x)\right),$$

or

$$|\overline{b}| < 2\left(\liminf_{x\to-\infty} f(x) - \limsup_{x\to+\infty} f(x)\right),$$

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where

$$\overline{b} = \int_0^{2\pi} e^{int} b(t)\, dt.$$

Further sufficient conditions upon f and b for the existence of at least one 2π-periodic solution are described for example in [2, 7] and their references.

In the same paper [6], Lazer and Leach have also proved that if h is not constant and if

$$|\overline{b}| \geq 2(\sup_{\mathbb{R}} f - \inf_{\mathbb{R}} f),$$

then equation (1) has no 2π-periodic solution. In a recent work, Alonso and Ortega [1] have shown that, when local uniqueness of the Cauchy problem holds, this last condition implies that every solution of (1) satisfies

$$\lim_{|t|\to\infty} \left[x^2(t) + x'^2(t)\right] = +\infty,$$

and that the unboundedness of sufficiently large solutions follows from a weaker condition involving the asymptotic properties of f.

In this paper, we prove sufficient conditions for the existence of 2π-periodic solutions, which can be expressed using, instead of the asymptotic properties of f, those of the (odd) function Ψ defined by

$$\Psi(\xi) = \int_0^{2\pi} \sin nt\, f(\xi \sin nt)\, dt.$$

So, the Lazer-Leach conditions are written in the form

$$|\overline{b}| < \liminf_{\xi\to+\infty} \Psi(\xi) \quad \text{or} \quad |\overline{b}| < -\limsup_{\xi\to+\infty} \Psi(\xi),$$

and we show that the inequalities

$$\limsup_{\xi\to+\infty} \Psi(\xi) > |\overline{b}| > \liminf_{\xi\to+\infty} \Psi(\xi) > 0,$$

or

$$\limsup_{\xi\to+\infty} \Psi(\xi) > |\overline{b}| > 0 \geq \liminf_{\xi\to+\infty} \Psi(\xi),$$

or corresponding ones at $-\infty$, which occur from some oscillating nonlinearities f, imply the existence of an unbounded sequence of 2π-periodic solutions.

The proofs are based on topological degree arguments and careful asymptotic estimates.

2 Main result

For $f : \mathbb{R} \to \mathbb{R}$ continuous and bounded, set

$$\Psi(\xi) = \int_0^{2\pi} \sin nt\, f(\xi \sin nt)\, dt, \tag{2}$$

and

$$\psi_+ = \liminf_{\xi\to+\infty} \Psi(\xi), \qquad \psi^+ = \limsup_{\xi\to+\infty} \Psi(\xi). \tag{3}$$

Some techniques for the computation of those numbers and some examples are given in Section 4. Obviously,

$$\Psi(\xi) = \int_0^{2\pi} \sin nt\, f_*(\xi \sin nt)\, dt = 2\int_0^{\pi} \sin t\, f_*(\xi \sin t)\, dt$$
$$= \int_0^{2\pi} \sin(nt+\theta)\, f_*(\xi \sin(nt+\theta))\, dt,$$

for all $\theta \in \mathbb{R}$, where $f_*(x) = \frac{1}{2}[f(x) - f(-x)]$ is the odd part of $f(x)$. This means that the function $\Psi(\xi)$ is odd and

$$\psi_+ = -\limsup_{\xi\to-\infty} \Psi(\xi), \qquad \psi^+ = -\liminf_{\xi\to-\infty} \Psi(\xi).$$

For $b : \mathbb{R} \to \mathbb{R}$ continuous and 2π-periodic, and n a positive integer, set

$$\overline{b} = \int_0^{2\pi} e^{int} b(t)\, dt.$$

Theorem 1. *If one of the following four relations is valid:*

$$\psi^+ > |\overline{b}| > \psi_+ > 0, \qquad -\psi_+ > |\overline{b}| > -\psi^+ > 0,$$
$$\psi^+ > |\overline{b}| > 0 \geq \psi_+, \qquad -\psi_+ > |\overline{b}| > 0 \geq -\psi^+,$$

then equation (1) *has an unbounded sequence of 2π-periodic solutions. If one of the two conditions*

$$\psi_+ > |\overline{b}| \text{ or } -\psi^+ > |\overline{b}|$$

holds, then equation (1) *has at least one 2π-periodic solution, and the set of such solutions is bounded.*

If $|\overline{b}| > \max\{|\psi_+|, |\psi^+|\}$, then the set of 2π-periodic solutions of (1) is also bounded. It may be nonempty, as shown by the following example (given in [1] for another purpose)

$$x'' + x = \sin x - \sin(\sin t),$$

for which $\psi_+ = \psi^+ = 0$ (see Section 4), $|\overline{b}| = \int_0^{2\pi} \sin t \sin(\sin t)\, dt > 0$, and which admits the 2π-periodic solution $x(t) = \sin t$. It may be empty, as shown by the example

$$x'' + x = \frac{x}{1+|x|} - c \sin t,$$

with $c > \frac{4}{\pi}$ for which $\psi_+ = \psi^+ = 4$ (see Section 4), and

$$|\overline{b}| = \pi c > 4 = 2\left[\sup_{x\in\mathbb{R}} \frac{x}{1+|x|} - \inf_{x\in\mathbb{R}} \frac{x}{1+|x|}\right],$$

so that the non existence of a 2π-periodic solution follows from the Lazer-Leach result quoted in the Introduction.

If $|\overline{b}| = \pm\psi_+$ or $|\overline{b}| = \pm\psi^+$, then the knowledge of the numbers $\psi_\pm$ and $\overline{b}$ is not sufficient to answer the question of the boundedness or unboundedness of the set of possible 2π-periodic solutions.

The case where $\psi^+ > |\overline{b}| = 0 > \psi_+$, or $-\psi_+ > |\overline{b}| = 0 > -\psi^+$, will be discussed heuristically in Section 7.

The proof of Theorem 1 will be given in Section 6, after some preliminary notions and results have been introduced, and some generalizations will be discussed in Section 8.

3 An associated planar mapping and its equivalent formulation

It follows immediately from the Fredholm alternative for the associated linear problem that if x is a 2π-periodic solution of (1), then

$$\int_0^{2\pi}[f(x(t))-b(t)]\cos nt\,dt = 0 = \int_0^{2\pi}[f(x(t))-b(t)]\sin nt\,dt. \tag{4}$$

This suggests the introduction of the orthogonal projector in $L^2 = L^2((0,2\pi),\mathbb{R})$ defined by

$$Pu(t) = \frac{1}{\pi}\int_0^{2\pi}\cos n(t-s)\,u(s)\,ds.$$

which is such that the system of equations (4) is equivalent to the abstract equation

$$Pf(x) - Pb = 0. \tag{5}$$

If we denote by Π_n the two-dimensional space spanned by the function $\cos nt$ and $\sin nt$, then $P(L^2) = \Pi_n$, and it will be useful to study the restriction of equation (5) to Π_n. Of course, every element of Π_n can be written equivalently in the form $\xi\sin(nt+\varphi)$, with $\xi \geq 0$ and $\varphi \in \mathbb{R}$, and hence we are interested in obtaining an expression for $Pf(\xi\sin(nt+\varphi))$.

Lemma 1. *For each $\xi \geq 0$ and $\varphi \in \mathbb{R}$, we have*

$$Pf(\xi\sin(n(\cdot)+\varphi))(t) = \frac{1}{\pi}\sin(nt+\varphi)\Psi(\xi). \tag{6}$$

where Ψ is defined in (2).

Proof. Since

$$Pf(\xi\sin(n(\cdot)+\varphi))(t) = \frac{1}{\pi}\int_0^{2\pi}\cos n(t-s)f(\xi\sin(ns+\varphi))\,ds$$

$$= \frac{1}{\pi}\int_{\varphi/n}^{2\pi+\varphi/n}\cos(n(t-s)+\varphi)f(\xi\sin ns)\,ds = \frac{1}{\pi}\int_0^{2\pi}\cos(n(t-s)+\varphi)f(\xi\sin ns)\,ds$$

$$= \frac{1}{\pi}\int_0^{2\pi}\Big(\cos(nt+\varphi)\cos ns + \sin(nt+\varphi)\sin ns\Big)\,f(\xi\sin ns)\,ds,$$

and

$$\int_0^{2\pi}\cos ns f(\xi\sin ns)\,ds = 0,$$

we have

$$Pf(\xi\sin(n(\cdot)+\varphi))(t) = \frac{1}{\pi}\sin(nt+\varphi)\int_0^{2\pi}\sin ns\,f(\xi\sin ns)\,ds = \frac{1}{\pi}\sin(nt+\varphi)\Psi(\xi).$$

and the proof is complete.

4 The computation of ψ_+ and ψ^+

We give here some recipes how to calculate the asymptotic values ψ_+ and ψ^+ of Ψ defined in (3) when $f(x)$ is given.

Lemma 2. *Let*

$$f(x) = f_1(x) + f_2(x) + f_3(x) + f_4(x) + f_5(x),$$

where $f_1(x) = \frac{1}{2}[f(x) + f(-x)]$ is the even part of $f(x)$, the other $f_i(x)$ are odd,

$$f_2(x) \to 0 \text{ as } x \to \infty,$$

$f_3(x)$ satisfies

$$f_3(x) \to \pm\overline{f} \neq 0 \text{ as } x \to \pm\infty,$$

and $f_4(x)$ has a sublinear primitive, i.e.

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f_4(u)\, du = 0. \tag{7}$$

Then

$$\psi_+ = 4\overline{f} + \liminf_{\xi\to+\infty} \int_0^{2\pi} \sin nt\, f_5(\xi \sin nt)\, dt, \quad \psi^+ = 4\overline{f} + \limsup_{\xi\to+\infty} \int_0^{2\pi} \sin nt\, f_5(\xi \sin nt)\, dt.$$

Proof. This statement can be proved in the following way. First of all, if $f_1(x)$ is even, then

$$\int_0^{2\pi} \sin nt\, f_1(\xi \sin nt)\, dt = 0.$$

If $f_2(x) \to 0$ as $x \to \infty$, then

$$\lim_{\xi\to\infty} \int_0^{2\pi} \sin nt\, f_2(\xi \sin nt)\, dt = 0.$$

If $f_3(x) \to \pm\overline{f}$ as $x \to \pm\infty$, then

$$\lim_{\xi\to\pm\infty} \int_0^{2\pi} \sin nt\, f_3(\xi \sin nt)\, dt = \pm 4\overline{f}.$$

And if $f_4(x)$ has sublinear primitive $F_4(x)$, then

$$\lim_{\xi\to\infty} \int_0^{2\pi} \sin nt\, f_4(\xi \sin nt)\, dt = 0. \tag{8}$$

The last formula follows from the relations

$$\int_0^{2\pi} \sin nt\, f_4(\xi \sin nt)\, dt = \int_0^{2\pi} \sin t\, f_4(\xi \sin t)\, dt = 4 \int_0^{\pi/2} \sin t\, f_4(\xi \sin t)\, dt$$

$$= 4 \int_\varepsilon^{\pi/2-\varepsilon} \sin t\, f_4(\xi \sin t)\, dt + 4 \int_{[0,\varepsilon]\cup[\pi/2-\varepsilon,\pi/2]} \sin t\, f_4(\xi \sin t)\, dt$$

$$\leq 4\int_{\varepsilon}^{\pi/2-\varepsilon} \tan t \, d\frac{F_4(\xi \sin t)}{\xi} + 8\varepsilon \sup |f_4(x)|$$

$$= c\varepsilon + 4\left(\tan t \, \frac{F_4(\xi \sin t)}{\xi}\Bigg|_{t=\varepsilon}^{t=\pi/2-\varepsilon} \right) - 4\int_{\varepsilon}^{\pi/2-\varepsilon} \frac{F_4(\xi \sin t)}{\xi \cos^2 t}\, dt = c\varepsilon + o(1).$$

Since ε is arbitrary, (8) is proved.

For example $f_4(x)$ has a sublinear primitive if $f_4(x)$ is periodic or almost periodic with zero mean value. Other examples of odd functions with sublinear primitives are $\sin\sqrt{|x|}\,\mathrm{sign}\, x$ and $\sin(x^3)$, but the primitive of the function $\sin\ln(1+x)$ $(x > 0)$ is not sublinear. Nonlinearities with sublinear primitives were considered in [1] in the related context of the problem of existence of unbounded solutions.

5 An asymptotic lemma for trigonometric integrals

To relate $P(f(x) - Pb)$ and its restriction to Π_n, we need a result on the asymptotic behavior of some trigonometric integrals, which is a variation of the one used in [1].

Let $C^1_{2\pi}$ be the Banach space of 2π-periodic real functions of class C^1 with its usual norm $\|\cdot\|_{C^1}$.

Lemma 3. *For any $c > 0$ the following equality is valid:*

$$\lim_{\xi\to\infty} \sup_{\|h\|_{C^1}\leq c;\ \theta,\varphi\in\mathbb{R}} \left| \int_0^{2\pi} \sin(nt+\theta)\left(f(\xi\sin(nt+\varphi+h(t))) - f(\xi \sin(nt+\varphi))\right) dt \right| = 0. \tag{9}$$

Proof. First of all let us note that

$$\lim_{\xi\to\infty} \sup_{\|h\|_{C^1}\leq c;\ \theta,\varphi\in\mathbb{R}} \left| \int_0^{2\pi} \sin(nt+\theta)\left(f(\xi\sin(nt+\varphi+h(t))) - f(\xi \sin(nt+\varphi))\right) dt \right|$$

$$\leq \lim_{\xi\to\infty} \sup_{\|h\|_{C^1}\leq c} \left| \int_0^{2\pi} \sin nt \left(f(\xi \sin nt + h(t)) - f(\xi \sin nt)\right) dt \right|$$

$$+ \lim_{\xi\to\infty} \sup_{\|h\|_{C^1}\leq c} \left| \int_0^{2\pi} \cos nt \left(f(\xi \sin nt + h(t)) - f(\xi \sin nt)\right) dt \right|.$$

Therefore it is sufficient to prove the equality

$$\lim_{\xi\to\infty} \sup_{\|h\|_{C^1}\leq c} \left| \int_0^{2\pi} g(t) \left(f(\xi \sin nt + h(t)) - f(\xi \sin nt)\right) dt \right| = 0 \tag{10}$$

for $g(t) = \sin nt$ and for $g(t) = \cos nt$. Let us choose an $\varepsilon > 0$ and let us show that the supremum in (10) is less than ε for sufficiently large $|\xi|$:

$$\sup_{\|h(t)\|_{C^1}\leq c} \left| \int_0^{2\pi} g(t) \left(f(\xi \sin nt + h(t)) - f(t, \xi \sin nt)\right) dt \right| < \varepsilon. \tag{11}$$

For this, let us split the interval $[0, 2\pi]$ into a finite number of subintervals $[a_i, b_i]$ and $[b_i, a_{i+1}]$ in the following way. The intervals (b_i, a_{i+1}) contain the set $\{t \in [0, 2\pi] : \cos nt =$

$0\}$, the union of these intervals can have any arbitrary small measure, so they can be choosen such that

$$\sup |f(x)| \int_{\cup[b_i,a_{i+1}]} |g(t)|\, dt < \varepsilon/2. \tag{12}$$

Suppose that the points a_i and b_i are fixed up to the end of the proof of the lemma. For any $[a_i, b_i]$ the estimate

$$\inf_{t\in[a_i,b_i]} |n \cos nt| \geq \delta > 0$$

holds. This means that the function $\sin nt$ is strictly monotone on every $[a_i, b_i]$, and, for sufficiently large $|\xi|$ ($|\xi| > 2c\delta^{-1}$) the function $\xi \sin nt + h(t)$ is also strictly monotone, and $|\xi n \cos nt + h'(t)| > 1/2\ |\xi|\delta$. Consider the integrals

$$\mathcal{J}_i = \int_{a_i}^{b_i} g(t) f(\xi \sin nt + h(t))\, dt.$$

Fix any of them and doin this integral, the change of variables $t = t_\xi(\tau)$ defined by the formula $\xi \sin n\tau = \xi \sin nt + h(t)$:

$$\mathcal{J}_i = \int_{t_\xi^{-1}(a_i)}^{t_\xi^{-1}(b_i)} \frac{g(t_\xi(\tau))\, f(\xi \sin n\tau)}{t'_\xi(\tau)} d\tau.$$

The function $t_\xi(\tau)$ is one-to-one, $t_\xi(\tau) \to \tau$ and $t'_\xi(\tau) \to 1$ uniformly in τ as $|\xi| \to \infty$. Now

$$t_\xi^{-1}(a_i) \to a_i, \quad t_\xi^{-1}(b_i) \to b_i,$$

and $g(t_\xi(\tau)) \to g(\tau)$ due to the continuity of $g(\cdot)$. Consequently,

$$\mathcal{J}_i - \int_{a_i}^{b_i} g(\tau) f(\xi \sin n\tau)\, d\tau \to 0$$

for every i. This together with (12) proves (11) and the lemma.

6 Proof of Theorem 1

The proof of Theorem 1 is made in several steps. First of all we replace our periodic problem with an equivalent operator equation of Hammerstein type

$$x(t) = A[ax - f(x) + b]$$

with some linear operator A, and $a = n^2 + 1$.

At the next step, for the cases $\psi_+ > |\bar{b}|$ and $-\psi^+ > |\bar{b}|$ we calculate an index at infinity of the vector field $x - A[ax - f(x) + b]$, and show that this index is different from 0, which proves the corresponding conclusion of Theorem 1. The index is calculated with the use of usual homotopic methods: we prove that the possible solutions of some homotopy $\Theta(\lambda, x)$ are a priori bounded, with $\Theta(1, x) = x - A[ax - f(x) + b]$ and $\Theta(0, x)$ a vector field with a Landesman-Lazer type nonlinearity, whose index was already calculated (see [6, 7, 3]).

The cases $\psi^+ > |\bar{b}| > \psi_+ > 0$, $-\psi_+ > |\bar{b}| > -\psi^+ > 0$, $\psi^+ > |\bar{b}| > 0 \geq \psi_+$ and $-\psi_+ > |\bar{b}| > 0 \geq -\psi^+$ are more difficult. We prove that for any ξ_0, there exist $\xi_* > \xi_0$ and

$\xi^* > \xi_*$ such that problem (1) has a solution $x(t) = \xi \sin(nt+\varphi) + h(t)$, where $\xi \in (\xi_*, \xi^*)$. This implies the corresponding conclusion of Theorem 1.

Step 1: Equivalent equation.

Consider, in the space $L^2 = L^2((0, 2\pi), \mathbb{R})$, the linear operator $x = Au$ defined for each $u \in L^2$ by the solution x of the problem

$$-x'' + x = u(t), \qquad x(0) - x(2\pi) = 0 = x'(0) - x'(2\pi).$$

This operator acts in L^2, is completely continuous in L^2, self-adjoint and positively semidefined. This operator is an integral one which can be considered in various spaces: it acts continuously from L^2 to $C^1_{2\pi}$ and from C^0 to $C^2_{2\pi}$. Its spectrum consists of the eigenvalues $1,\ 2,\ 5,\ \ldots\ n^2+1,\ \ldots$. The eigenvalue 1 is simple, with the corresponding eigenfunction constant, and the other eigenvalues have multiplicity two, with the corresponding eigenfunctions of the form $\xi \sin(nt+\varphi)$ with arbitrary $\xi \geq 0$ and $\varphi \in \mathbb{R}$. Such eigenfunctions span a plane, already denoted as Π_n.

The operator equation equivalent to (1) has the form

$$x = A[(n^2+1)x - f(x) + b]. \tag{13}$$

Any solution $x \in L^2$ of this operator equation is a classical solution of problem (1), any classical solution of (1) is a solution $x \in L^2$ of this operator equation.

The part $x = aAx$ of equation (13) which is linear at infinity is degenerate: the linear operator $I - aA$ has the nontrivial 2-dimensional kernel Π_n introduced in Section 3. Below we use the already introduced orthogonal projector P on the plane Π_n, and the projector $Q = I - P$.

Now the two different cases of Theorem 1 will be considered separately.

Step 2 : Existence of a solution.

Let $\psi_+ > |\bar{b}|$. Consider the function

$$s(\xi) = \begin{cases} 1, & \text{if } \ \xi \geq 1, \\ \xi, & \text{if } \ |\xi| < 1, \\ -1, & \text{if } \ \xi \leq -1. \end{cases}$$

and the homotopy

$$\Theta(\lambda, x) = x - A\left[ax - \lambda f(x) - \frac{(1-\lambda)}{4}\psi_+ s(x) + b\right].$$

Now we have to do two things, namely to prove an a priori estimate for all possible zeros of the homotopy $\Theta(\lambda, x)$, and to study the vector field $\Theta(0, x)$.

a. *A priori estimate.* Suppose that $x(t) = \xi \sin(nt+\varphi) + h(t)$ where $h(t) = Qx(t)$ and $\Theta(\lambda, x) = 0$. Then $Q\Theta(\lambda, x) = 0$ and $P\Theta(\lambda, x) = 0$. The equality $Q\Theta(\lambda, x) = 0$ implies the estimate

$$\|h\|_{C^1} \leq c < \infty.$$

The equality $P\Theta(\lambda, x) = 0$ has the form

$$\lambda P f(x) - \frac{(1-\lambda)}{4}\psi_+ P s(x) + Pb = 0. \tag{14}$$

If $\xi \to +\infty$, then according to Lemma 3, $Pf(x) - Pf(\xi \sin(n(\cdot)+\varphi)) \to 0$; analogously $Ps(x) - Ps(\xi \sin(n(\cdot)+\varphi)) \to 0$. Due to (6)

$$Pf(\xi \sin(n(\cdot)+\varphi))(t) = \frac{1}{\pi}\Psi(\xi)\sin(nt+\varphi), \quad Ps(\xi \sin(n(\cdot)+\varphi)) = \frac{4}{\pi}\sin(nt+\varphi).$$

Therefore

$$\liminf_{\xi\to+\infty}\left\|\lambda Pf(x) - \frac{(1-\lambda)}{4}\psi_+ Ps(x)\right\|_{L^2} = \liminf_{\xi\to+\infty}\|\sin(n(\cdot)+\varphi)\|_{L^2}\frac{1}{\pi}(\lambda\Psi(\xi)+(1-\lambda)\psi_+) \geq$$

$$\geq \frac{1}{\sqrt{\pi}}(\lambda\psi_+ + (1-\lambda)\psi_+) = \frac{\psi_+}{\sqrt{\pi}}$$

and (since $\sqrt{\pi}\,\|Pb\|_{L^2} = |\overline{b}|$) this contradicts to (14). This proves the required a priori estimate.

b. *The vector field* $\Theta(0,x)$. The vector field $\Theta(0,x)$ has the form $x-[(ax-\psi_+ s(x)/4+b]$. The nonlinearity in this field has limits at infinity: $s(x) \to \pm 1$ as $x \to \pm\infty$. In previous papers, the index at infinity of such vector fields was calculated (see, e.g. [6, 7, 3]), and it was shown that if $|\overline{b}| < \psi_+$, then $|\operatorname{ind}_\infty \Theta(0,\cdot)| = 1$.

c. *The end of Step* 4. The a priori estimate guarantees that the value of $\operatorname{ind}_\infty \Theta(\lambda,\cdot)$ does not depend on λ. Now we see that

$$\operatorname{ind}_\infty \Theta(1,\cdot) = \operatorname{ind}_\infty \Big(I - A[(n^2+1)I - f(\cdot)+b]\Big) = \operatorname{ind}_\infty \Theta(0,\cdot) \neq 0,$$

and the general properties of the index at infinity imply the corresponding conclusion of Theorem 1.

Step 3 : Unbounded set of solutions.

Let either $\psi^+ > |\overline{b}| > \psi_+ \geq 0$ or $\psi^+ > |\overline{b}| > 0 \geq \psi_+$ or $-\psi_+ > |\overline{b}| > -\psi^+ \geq 0$ or $-\psi_+ > |\overline{b}| > 0 > -\psi^+$. We consider the cases $\psi^+ > |\overline{b}| > \psi_+ \geq 0$ and $\psi^+ > |\overline{b}| > 0 \geq \psi_+$, and omit the proof of the last two cases, which are completely analogous. Both these cases guarantee the existence of unbounded sequences ξ_k and ξ^k with $\xi_{k+1} > \xi^k > \xi_k$ such that

$$\Psi(\xi_k)+\varepsilon < |\overline{b}| < \Psi(\xi^k)-\varepsilon, \qquad \Psi(\xi_k) < \Psi(\xi) < \Psi(\xi^k), \quad \xi_k < \xi < \xi^k \tag{15}$$

for some fixed $\varepsilon > 0$. Without loss of generality, suppose that any ξ_k is sufficiently large so that the supremum in formula (9) is small enough for $\xi \geq \xi_k$.

Below we prove that the rotation γ of the vector field $x - A[(n^2+1)x - f(x)+b]$ on the boundary of the set $\Omega_k = \{\|Qx\| \leq R_1+1\} \times \{\xi\sin(n(\cdot)+\varphi) : \ \xi \in [\xi_k, \xi^k]\} \subset L^2$ is defined and that $|\gamma| = 1$. This equality proves the remaining part of the theorem: any Ω_k contains its own solution of problem (1), and the sets Ω_k are disjoint. The constant R_1 will be chosen below, and it does not depend on k.

Let us fix some k and let us calculate $|\gamma|$ for this number k. Consider the homotopy

$$\Xi(x,\lambda) = x - A(n^2+1)x + Af(x) - Ab + \lambda A[Pf(Px) - f(x)], \qquad \lambda \in [0,1].$$

For $\lambda = 0$ this homotopy is our vector field $x - A[(n^2+1)x - f(x)+b]$, for $\lambda = 1$ it is equal to $\Xi(x,1) = x - A[(n^2+1)x - f(Px)+b]$.

Let us prove that the homotopy is nonzero on $\partial\Omega_k$. If it is not the case, then $\Xi(x,\lambda)=0$ for some $\lambda\in[0,1]$ and $x(t)=\xi\sin(nt+\varphi)+h(t)$. Therefore, $Q\Xi(x,\lambda)=0$ and $P\Xi(x,\lambda)=0$. The first equality implies the estimates

$$\|h\|_{L^2}\le R_1,\qquad \|h\|_{C^1}\le c \tag{16}$$

where the constants c and R_1 are independent from λ and ξ. With this definition of the constant R_1, we see that $Q\Xi(x,\lambda)$ is nonzero if $\|Qx\|_{L^2}=R_1+1$.

Now consider the remaining part of the set $\partial\Omega_k$, which is made of the sets $\{\|Px\|_{L^2}=\xi_k,\ \|Qx\|_{L^2}\le R_1+1\}$ and $\{\|Px\|_{L^2}=\xi^k,\ \|Qx\|_{L^2}\le R_1+1\}$. The equality $P\Xi(x,\lambda)=0$ can be rewritten as

$$P[f(x)-b]+\lambda P[f(Px)-f(x)]=0$$

or as

$$P[f(Px)-b]=(1-\lambda)P[f(Px)-f(x)]$$

But the last equality is impossible for large k: the left-hand side was calculated directly,

$$P[f(Px)-b]=\frac{1}{\pi}\sin(n(\cdot)+\varphi)\Psi(\xi)-Pb,$$

and is uniformly nonzero due to (15), and the right-hand side is arbitrary small for large ξ.

Now consider the vector field $\Xi(x,1)=x-A[(n^2+1)x-Pf(Px)+b]$. The rotation $\gamma(\Xi(\cdot,1),\partial\Omega_k)$ can be calculated with the use of rotation product formula (see, e.g., [5] Theorem 22.4, p. 117). This formula for our case takes the form

$$\gamma(\Xi(\cdot,1),\partial\Omega_k)=(-1)^\alpha\gamma(Pf(P(\cdot))-Pb,\partial Z_k),$$

where α is some integer. Here $Z_k=\{\xi_k\le\|Px\|_{L^2}\le\xi^k\}\subset\Pi_n$ is an annulus in the plane Π_n, and the value of $\gamma(Pf(P(\cdot))-Pb,\partial Z_k)$ can be calculated directly. Due to formulae (6) from the step 3 of the proof and (15), the homotopy Γ defined for $\lambda\in[0,1]$ by

$$\Gamma(\xi,\varphi,\lambda)$$
$$=\frac{1}{\pi}\left\{(1-\lambda)\Psi(\xi)+\lambda\left[\Psi(\xi_k)+\frac{\xi-\xi_k}{\xi^k-\xi^k}(\Psi(\xi^k)-\Psi(\xi_k))\right]\right\}(\cos\varphi,\sin\varphi)-\frac{1}{\pi}(\Re\overline{b},\Im\overline{b}),$$

is such that

$$\Gamma(\xi_k,\varphi,\lambda)=\frac{1}{\pi}\Psi(\xi_k)(\cos\varphi,\sin\varphi)-\frac{1}{\pi}(\Re\overline{b},\Im\overline{b})\ne 0,$$

for all $\varphi\in[0,2\pi]$ and $\lambda\in[0,1]$, and

$$\Gamma(\xi^k,\varphi,\lambda)=\frac{1}{\pi}\Psi(\xi^k)(\cos\varphi,\sin\varphi)-\frac{1}{\pi}(\Re\overline{b},\Im\overline{b})\ne 0,$$

for all $\varphi\in[0,2\pi]$ and $\lambda\in[0,1]$. Hence,

$$\gamma(Pf(P(\cdot))-Pb,\partial Z_k)=\gamma(\Gamma(\cdot,\cdot,0),\partial Z_k)=\gamma(\Gamma(\cdot,\cdot,1),\partial Z_k)=\pm 1,$$

as the 2-dimensional mapping $\Gamma(\cdot,\cdot,1)$ is one-to-one on Z_k, maps the annulus Z_k onto another annulus Z_k' containing the origin. The proof is complete.

Remark. Theorem 1 can be proved as well by applying the generalized continuation Theorem IV.1 of [2] with $Lx=x''+n^2x$ and $Nx=f(x)-b$.

7 The case where $\psi^+ > \overline{b} = 0 > \psi_+$

In this section we discuss the case where $\psi^+ > |\overline{b}| = 0 > \psi_+$, which is not covered by the conditions of Theorem 1. Here we try to underline the difference between the cases $\overline{b} = 0$ and $\overline{b} \neq 0$ and to give some reasons why these cases are really different.

Consider 2-dimensional equation

$$\Psi(|z|)\frac{z}{|z|} + \varepsilon(z) = z_0, \tag{17}$$

where z and z_0 are complex numbers, and $|\varepsilon(z)| \to 0$ as $|z| \to \infty$. Let

$$\psi_+ = \liminf_{\xi \to +\infty} \Psi(\xi) < 0, \qquad \psi^+ = \limsup_{\xi \to +\infty} \Psi(\xi) > 0.$$

Let $z_0 \neq 0$. Suppose $\xi_* < \xi^*$, $\Psi(\xi_*) < 0 < |z_0| < \Psi(\xi^*)$ and ξ_* is large enough. Equation (17) has at least one solution z satisfying $\xi_* < |z| < \xi^*$. This follows from usual degree arguments: the degree of the field

$$\Psi(|z|)\frac{z}{|z|} + \varepsilon(z) - z_0$$

on the annulus $G = \{z : \ \xi_* < |z| < \xi^*\}$ is nonzero, as it can be shown by an argument similar to the one used at the end of the proof of Theorem 1.

If $z_0 = 0$, then we cannot construct such annulus. Moreover, for $\varepsilon = 0$ such annulus contains the circles $\{z : \ |z| = \xi_0\}$ $(\Psi(\xi_0) = 0)$, unique or not, which consist from solutions of our equation. And the local topological index of any such circle is zero ! If $z_0 = 0$, and $\varepsilon(z) = |z|^{-2} z\, e^{i\pi/2}$, then equation (17) does not have any nontrivial solutions.

This means that without any information about $\varepsilon(z)$ we cannot study our equation for the case $z_0 = 0$. The reason of this fact is the even dimension of the space: in odd-dimensional space the situation is different.

Our main periodic problem at infinity is rather close to such a 2-dimensional equation. An equivalent integral equation can be represented as the product of the considered 2-dimensional equation and some infinite-dimensional one, which is asymptotically linear and non-degenerate. Then the topological properties of the equivalent equation coincide with the properties of the 2-dimensional one.

8 Generalizations

8.1. We have formulated the main result for problem (1). It is possible to reformulate it for higher order ordinary differential equations or for more general type equations from control theory as considered in [4]. Consider for example the equation

$$L\left(\frac{d}{dt}\right) x = M\left(\frac{d}{dt}\right)(f(x) - b(t)). \tag{18}$$

Here $L(p)$ and $M(p)$ are real coprime polynomials with constant coefficients, $\deg L > \deg M$. Suppose that the polynomial $L(p)$ has a pair of roots $\pm ni$ for some positive integer

n, and has no other such roots. Define the function $\Psi(\xi)$ and the values $\overline{b}, \psi_+, \psi^+$ as in Section 2.

Theorem 2. *If either $\psi^+ > |\overline{b}| > \psi_+ > 0$ or $-\psi_+ > |\overline{b}| > -\psi^+ > 0$ or $\psi^+ > |\overline{b}| > 0 \geq \psi_+$ or $-\psi_+ > |\overline{b}| > 0 \geq -\psi^+$, then equation (18) has an unbounded sequence of 2π-periodic solutions. If either $\psi_+ > |\overline{b}|$ or $-\psi^+ > |\overline{b}|$, then equation (18) has at least one 2π-periodic solution.*

8.2. Other natural generalizations of Theorems 1 and 2 can be obtained for nonlinearities with delays, derivatives or even with hysteresis. Consider, for example, the delay-differential equation

$$x''(t) + n^2 x(t) = f(x(t), x(t-r)) - b(t). \tag{19}$$

Put

$$\Psi(\xi) = \int_0^{2\pi} \sin nt\, f(\xi \sin nt, \xi \sin n(t-r))\, dt$$

and define the values $\overline{b}, \psi_+, \psi^+$ as in Section 2 from this function $\Psi(\xi)$.

Theorem 3. *If either $\psi^+ > |\overline{b}| > \psi_+ > 0$ or $-\psi_+ > |\overline{b}| > -\psi^+ > 0$ or $\psi^+ > |\overline{b}| > 0 \geq \psi_+$ or $-\psi_+ > |\overline{b}| > 0 \geq -\psi^+$, then equation (19) has an unbounded sequence of 2π-periodic solutions. If either $\psi_+ > |\overline{b}|$ or $-\psi^+ > |\overline{b}|$, then equation (19) has at least one 2π-periodic solution.*

8.3. It would be interesting to obtain some analog of the first part of Theorem 1 for essentially time-depending nonlinearities $f(t,x)$. The principal difficulty of this case is following.

The two-dimensional mapping

$$(\xi, \theta) \mapsto Pf(\cdot, \xi \sin(n(\cdot) + \theta))$$

must have some proper topological properties, which hold in the case where $f(t,x) = f(x) - b(t)$. Explicitely, $Pf(\xi \sin(n(\cdot)+\theta))(t) = (\Psi(\xi)/\pi)\sin(nt+\theta)$, and this mapping can be factorized as a composition of a mapping depending only on the variable ξ, and another one depending only on θ. This allows us to use the theorem on the product of degrees. Without any additional assumptions (or at least ideas !), both components of this mapping depend on both variables, and we do not see how to calculate the index in the general case.

Of course, the case where $f(t,x) = f(x) + f_1(t,x) + b(t)$, with

$$\lim_{\xi\to\infty} \sup_{\|h\|_{C^1} \leq c;\ \theta,\varphi\in\mathbb{R}} \int_0^{2\pi} \sin(nt+\theta)\, f_1(t, \xi\sin(nt+\varphi) + h(t))\, dt = 0$$

can be considered without any additional difficulties. This happens if $f_1(t,x) \to 0$ as $x \to \pm\infty$, and other examples can be obtained from the results of Section 4.

8.4. It would be interesting to prove some analogs of Theorem 1 for unbounded functions $f(x)$, either for finite or for infinite values ψ_+ and ψ^+.

8.5. If $|\overline{b}| > \max\{|\psi_+|, |\psi^+|\}$, then the index at infinity is defined for the planar vector field related to problem (1), and this index is equal to 0.

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