Index at infinity for vector fields with oscillating nonlinearities

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1 Introduction

This paper concerns an index at infinity calculation for some asymptotically linear completely continuous vector fields $x - T(x)$. If the main linear part $x - Ax$ of the field is non-degenerate, then the index at infinity of $x - T(x)$ is well-defined by the spectrum of this linear part. If this main linear part is degenerate (1 is an eigenvalue of the linear operator A), the index calculation can be done only with the use of some properties of the sublinear part of $T(x)$.

If this sublinear part is asymptotically homogeneous (e.g. satisfies some Landesman – Lazer type conditions), then the calculation of the index can be reduced to the calculation of the degree of some finite dimensional vector field Q (see [1]). If this vector field Q is also degenerate, one can use higher order terms.

Here in this paper we consider the situation where the sublinear terms are not asymptotically homogeneous. We have here the following possibilities: the index is defined and its calculation can be reduced to the index calculation of a finite dimensional field; the index can not be defined (there are some sequence x_n of zeros and $\|x_n\| \to \infty$); known information is insufficient to get any answer and we need to use higher order terms. The last possibility has in some natural sense "zero measure".

In this paper we consider only the case of one-dimensional degeneration of the linear part: 1 is a simple eigenvalue of the linear operator A. For this case we formulate an abstract theorem and give some general examples for vector fields of Hammerstein type and for a two point boundary value problem. The paper is organized as follows: in Sections 2 and 3 we formulate the main abstract result, in Section 4 we discuss the main condition (1), applications are given in Section 5, the proofs in Section 6 – 8 and some remarks in Section 9.

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2 Index at infinity

Consider a vector field $\Phi x = x - T(x)$ in a Banach space E. We suppose that the operator $T(x)$ is completely continuous, in which case the vector field $\Phi(x)$ is also called completely continuous. Suppose that $x \neq T(x)$ for $\|x\| \geq \rho$. Then the rotation (see [2]) of this field on the boundary of every ball $B(r, 0) = \{x \in E, \|x\| \leq r\}$ is defined for every $r \geq \rho$. This rotation has a common value, called the index at infinity of the field $\Phi(x)$ and is denoted as $\operatorname{ind}_\infty \Phi$.

The index at infinity can be used for the study of the equation $x = T(x)$: if $\operatorname{ind}_\infty \Phi$ is defined ($x \neq T(x)$ for $\|x\| \geq \rho$ for some ρ) and $\operatorname{ind}_\infty \Phi \neq 0$, then there exists at least one solution of the equation $x = T(x)$.

The homotopic invariance of the index makes the index applicable for the study of unbounded branches of solutions or for the study of asymptotic bifurcation points for equations with a parameter ([2]).

Consider the equation $x = T(x; \lambda)$ with a real parameter $\lambda \in \Lambda = [a, b]$. Let the operator $T(x; \lambda)$ be completely continuous with respect to both variables. If the index at infinity of the field $\Phi_\lambda = x - T(x; \lambda)$ is defined for two distinct values λ_1 and λ_2 and $\operatorname{ind}_\infty \Phi_{\lambda_1} \neq \operatorname{ind}_\infty \Phi_{\lambda_2}$, then on the interval $[\lambda_1, \lambda_2] \in \Lambda$ there exists an unbounded branch of solutions: for any $R > 0$ there exist $\lambda_R \in (\lambda_1, \lambda_2)$ and x_R, $\|x_R\| \geq R$ such that $x_R = T(x_R; \lambda_R)$. This branch is continuous in a natural sense. In other words, if $\operatorname{ind}_\infty \Phi_{\lambda_1} \neq \operatorname{ind}_\infty \Phi_{\lambda_2}$, then there exists at least one asymptotic bifurcation point on $[\lambda_1, \lambda_2]$, which can be unique or not. In this paper we describe situations, where asymptotic bifurcation points fill some nontrivial interval $J \subset [\lambda_1, \lambda_2]$.

3 Abstract results

Consider a Banach space E and a completely continuous vector field $x - T(x)$. Suppose that $T(x) = Ax + F(x)$, where A is a linear operator and $F(x)$ is bounded nonlinearity: $\|F(x)\| \leq r_1, \ x \in E$.

Suppose 1 is a simple eigenvalue of the linear operator A, let $Ae = e, \ \|e\| = 1$. There exists a unique projector P on the 1-dimensional subspace $E_1 = \{\alpha e, \ \alpha \in \mathbb{R}\}$, which commutes with A. Denote $Q = I - P$, $E_2 = QE$, the subspace E_2 has co-dimension 1, it is invariant for A, and in E_2 the value 1 is regular for the operator A. Denote by $p(x) : \ E \to E_1$ the functional defined by the relation $Px = p(x)e$.

Suppose another Banach space $\tilde{E} \subset E$ is given with a stronger norm: $\|x\|_E \leq const \, \|x\|_{\tilde{E}}, \ x \in E$.

Now we can formulate the main assumption for the nonlinearity $F(x)$: for any $c > 0$

$$\lim_{|\xi| \to \infty} \sup_{\|h\|_{\tilde{E}} \leq c} |p(F(\xi e + h)) - p(F(\xi e))| = 0. \tag{1}$$

This property will be discussed below, and is valid for various classes of concrete nonlinear operators which appear in applications.

Consider the scalar function

$$\Psi(\xi) = p(F(\xi e)),$$

which is bounded and continuous. Consider the four numbers:

$$\psi_\pm = \liminf_{\xi\to\pm\infty} \Psi(\xi), \quad \psi^\pm = \limsup_{\xi\to\pm\infty} \Psi(\xi). \tag{2}$$

Theorem 1. *Let the operator A act continuously from E to $\tilde{E}$. Let the nonlinearity $F(x)$ also act from E to $\tilde{E}$ and let it be bounded in $\tilde{E}$: $\|F(x)\|_{\tilde{E}} \le r_2$, $x \in E$. Let (1) be valid for any $c > 0$. Then*

a) if either $\psi^+ < 0 < \psi_-$, or $\psi^- < 0 < \psi_+$, then the index at infinity of the field $\Phi(x)$ is defined and $|\operatorname{ind}_\infty \Phi| = 1$;

b) if either $\psi_- > 0$ and $\psi_+ > 0$, or $\psi^- < 0$ and $\psi^+ < 0$, then the index at infinity of the field $\Phi(x)$ is defined and $\operatorname{ind}_\infty \Phi = 0$;

c) if either $\psi_- < 0 < \psi^-$, or $\psi_+ < 0 < \psi^+$, or both, then the index at infinity of the field $\Phi(x)$ can not be defined: there exists a sequence $x_n \in E$ such that $x_n = T(x_n)$ and $\|x_n\|_E \to \infty$.

If this theorem is inapplicable, then at least one of the numbers (2) has to be zero. Even if $\psi_- \cdot \psi^- = 0$ ($\psi_+ \cdot \psi^+ = 0$) Theorem 1 can be applicable if $\psi_+ < 0 < \psi^+$ ($\psi_- < 0 < \psi^-$).

Under the assumptions of Theorem 1, if $\psi_+ < 0 < \psi^+$, then the sequence x_n has the form $x_n = \xi_n e + h_n$, where $h_n \in E_2$ are uniformly bounded and $\xi_n \to +\infty$. If $\psi_- < 0 < \psi^-$, then $\xi_n \to -\infty$; if $0 \in (\psi_-, \psi^-) \cap (\psi_+, \psi^+)$, then there are two sequences $x_n^\pm = \xi_n^\pm e + h_n^\pm$, where $\xi_n^\pm \to \pm\infty$.

If $\psi_- = \psi^-$ and $\psi_+ = \psi^+$, then the condition c) of Theorem 1 can not take place, this is the case, when the nonlinearity is asymptotically homogeneous in the sense of [1].

4 Hammerstein type nonlinearities

In this section we formulate some sufficient conditions for the validity of the assumption (1) and the corresponding results for the index at infinity calculation of Hammerstein type vector fields in functional spaces.

Let $\Omega \subset \mathbb{R}^m$ be a domain and let $E = L^2 = L^2(\Omega, \mathbb{R}^n)$.

Suppose that the nonlinearity $F(x)$ is a Hammerstein type operator generated by the linear operator A (let us recall that we study the vector field $\Phi(x) = x - T(x) = x - Ax - F(x)$ where A is the asymptotic derivative of $T(x)$) and some bounded continuous function $f(t,x) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$:

$$F(x) = Af(t,x).$$

Theorem 2. *Let $\tilde{E} = E$ and*

$$mes\{t \in \Omega : \ e(t) = 0\} = 0. \tag{3}$$

Let the function $f(t,x)$ satisfy the Lipschitz condition

$$\|f(t,x) - f(t,y)\|_{\mathbb{R}^n} \le L(r)\,\|x-y\|_{\mathbb{R}^n}, \qquad \|x\|_{\mathbb{R}^n}, \|y\|_{\mathbb{R}^n} > r \tag{4}$$

where

$$\lim_{r\to\infty} L(r) = 0. \tag{5}$$

Then condition (1) *is valid for any* $c > 0$, *and moreover*

$$\lim_{|\xi|\to\infty} \sup_{\|h\|_{L^2}\le c} \|f(\xi e(t) + h(t)) - f(\xi e(t))\|_{L^2} = 0. \tag{6}$$

For scalar functions $e(t)$, defined for $\Omega = [a,b] \subset \mathbb{R}$ it is possible to obtain a less restrictive result.

Suppose the linear projector Px is generated by the function $g(t) \in L^2$:

$$Px = e(t) \int_a^b g(s)x(s)\, ds.$$

Theorem 3. *Let* $E = L^2 = L^2([a,b],\mathbb{R})$ *and* $\tilde{E} = C^1 = C^1([a,b],\mathbb{R})$. *Let* $e(t) \in C^1$ *and*

$$mes\{t \in [a,b]:\ e'(t) = 0\} = 0. \tag{7}$$

Let the function $f(t,x)$ *be uniformly continuous in* t *with respect to* $x \in \mathbb{R}$. *Then condition* (1) *is valid for any* $c > 0$*:*

$$\lim_{|\xi|\to\infty} \sup_{\|h(t)\|_{C^1}\le c} \left| \int_a^b g(t)[f(t,\xi e(t) + h(t)) - f(t,\xi e(t))]\, dt \right| = 0. \tag{8}$$

Theorem 3 does not contain any Lipschitz conditions like (4) – (5), which, of course, are very restrictive. The bounded function $f(t,x)$ is uniformly continuous in t with respect to x for example if $f(t,x) = a(t) + b(t)f(x)$.

From the proof given below one can see that Theorem 3 can be generalized for the case $E = L^2(\Omega,\mathbb{R}^n)$, $\Omega \subset \mathbb{R}^m$ where $m \ge n$. It seems that for $m < n$ we have essential difficulties and need some additional hypotheses for $f(t,x)$.

Theorem 1 and Theorems 2 and 3 allow us to formulate statements for the index at infinity calculation of Hammerstein type vector fields in functional spaces.

Corollary 1. *Let* $E = L^2(\Omega,\mathbb{R}^n)$ *where* Ω *is a domain in* $\mathbb{R}^m$. *Let the linear completely continuous operator* A *act in* E, *let* 1 *be a simple eigenvalue for* A, *let* $Ae(t) = e(t)$ *and let condition* (3) *be valid. Let a bounded continuous function* $f(t,x):\ \Omega\times\mathbb{R}^n \to \mathbb{R}^n$ *satisfy* (4) *and* (5). *Then for the vector field* $\Phi(x) = x - A(x + f(t,x(t)))$ *all conclusions of Theorem* 1 *are valid.*

Corollary 2. *Let* $E = L^2([0,\pi],\mathbb{R})$. *Let the linear completely continuous operator* $A:\ E \to E$ *act continuously from* E *to the space* $C^1(0,\pi)$. *Let* 1 *be a simple eigenvalue for* A, *let* $Ae(t) = e(t)$ *and let condition* (7) *be valid. Let a bounded continuous function* $f(t,x):\ \Omega\times\mathbb{R} \to \mathbb{R}$ *be uniformly continuous in* t *with respect to* x. *Then for the vector field* $\Phi(x) = x - A(x + f(t,x(t)))$ *all conclusions of Theorem* 1 *are valid.*

Corollary 1 is a direct consequence of Theorems 1 and 2, corollary 2 is a direct consequence of Theorems 1 and 3.

5 Two point boundary value problem

In this section we give some results for two point BVP at resonance. Consider the problem

$$x'' + n^2 x + f(t,x) = 0, \quad x(0) = x(\pi) = 0 \tag{9}$$

for some positive integer n and for some bounded continuous function $f(t,x) : [0,\pi] \times \mathbb{R} \to \mathbb{R}$. Suppose that $f(t,x)$ is uniformly continuous in t with respect to x. Put

$$e(t) = \sqrt{\frac{2}{\pi}} \sin nt, \qquad \Psi(\xi) = \int_0^\pi e(t)\, f(t, \xi e(t))\, dt$$

and define numbers (2).

Theorem 4. *If either $\psi^+ < 0 < \psi_-$, or $\psi^- < 0 < \psi_+$, then problem* (9) *has at least one solution, the set K of all solutions of* (9) *is bounded. If either $\psi_- < 0 < \psi^-$, or $\psi_+ < 0 < \psi^+$, or both, then problem* (9) *has an infinite number of solutions, the set K is unbounded.*

This theorem follows directly from Corollary 2.

We illustrate this theorem with the description of the set of solutions of the problem

$$x'' + n^2 x + f(t,x) - \lambda e(t) = 0, \quad x(0) = x(\pi) = 0 \tag{10}$$

with a scalar parameter λ. Define the numbers ψ_-, ψ^-, ψ_+, ψ^+ as in Theorem 4 and let $\psi_- < \psi^- < 0 < \psi_+ < \psi^+$. For every $\xi \in \mathbb{R}$ we can find at least one λ and a nonempty set $H(\xi)$ of functions $h(t)$ such that $x(t) = \xi e(t) + h(t)$ is a solution of problem (10).

On the plane $\{\lambda, \xi\}$ the graph of the mapping $\xi \mapsto \lambda$ can be found at Fig.1.

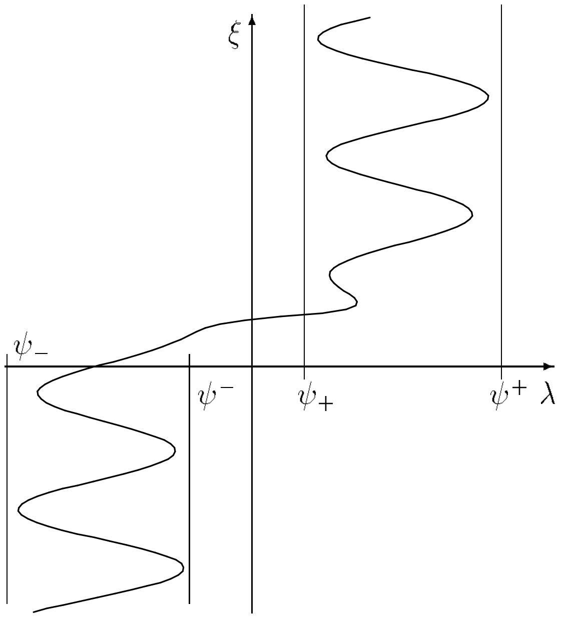

Fig.1

Here the set $[\psi_-, \psi^-] \cup [\psi_+, \psi^+]$ is the set of asymptotic bifurcation points for the problem (10). Of course, the behaviour of the graph for small values of ξ can be much more cumbersome, as we had no assumptions concerning the behaviour of the function $f(t,x)$ for small x. If $f'_x(t,x)$ is uniformly small enough (the constant depends only on n), then for any ξ there exist unique $h(t)$ and unique λ such that $x(t) = \xi e(t) + h(t)$ is a solution of problem (10).

The natural question arises: how to compute the numbers (2)? We want to mention, that if $f(t,x) = f_1(t,x) + f_2(t,x)$ and

$$\lim_{|\xi|\to\infty} \int_\Omega g(t)\, f_2(t, \xi e(t))\, dt = 0, \tag{11}$$

then numbers (2) computed for the functions $f(t,x)$ and $f_1(t,x)$ coincide. If the limits

$$\lim_{\xi\to\pm\infty} \int_\Omega g(t)\, f_2(t, \xi e(t))\, dt = \tilde{\psi}(\pm) \tag{12}$$

exist, then they are the difference between numbers (2) computed for the functions $f(t,x)$ and the same numbers, computed for $f_1(t,x)$.

This simple idea allows us to split the initial function $f(t,x)$ into a sum of functions such that it is possible either to see that (11) is valid or limits (12) exist and for one last function we need to do computations of different upper and lower limits.

Relation (11) is valid for various functions $f_2(t,x)$. Let us give some examples.

1. Let $f_2(t,x) \to 0$ as $|x| \to \infty$ uniformly with respect to t. Let $e(t)$ satisfy (3). Then (11) is valid.

2. Let $f_2(t,x)$ satisfy some Landesman–Lazer type conditions $f_2(t,x) \to \varphi^\pm(t)$ as $x \to \infty$. Let $e(t)$ satisfy (3). Then limits (12) exist and

$$\tilde{\psi}(\pm) = \int_{\{\pm e(t)\geq 0\}} g(t)\, \varphi^\pm(t)\, dt + \int_{\{\pm e(t)\leq 0\}} g(t)\, \varphi^\mp(t)\, dt.$$

3. Let $f_2(t,x) = f(x)$ be an even function, $t \in [0,\pi]$, $e(t) = \sqrt{2/\pi}\sin nt$ and n be even. Then for any ξ

$$\int_0^\pi e(t)\, f(\xi e(t))\, dt = 0.$$

Of course (11) is valid for this case.

4. Let $e(t) \in C^1$ and $e'(t)$ satisfy (7). Let again $f_2(t,x) = f(x)$ and let its primitive $\mathcal{F}(x)$ be sublinear:

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(u)\, du = 0. \tag{13}$$

Then (11) is valid.

Some analog of this statement is formulated in [3] in the context of unbounded solution existence. It can be proved in the following way. The set where $e'(t)$ is small has a small measure due to (7). This means that without loss of generality we can consider only the case, where the function $e(t)$ is monotone (for example, it increases), $e'(t) > \varepsilon$ and $g(t) \in C^1$. Then

$$\lim_{\xi\to\infty} \int_a^b g(t)\, f_2(t, \xi e(t))\, dt = \lim_{\xi\to\infty} \int_a^b g(t)\, f(\xi e(t))\, dt = \lim_{\xi\to\infty} \int_a^b \frac{g(t)}{\xi e'(t)}\, d\mathcal{F}(\xi e(t)) =$$

$$= \lim_{\xi\to\infty}\left(\frac{g(t)}{e'(t)}\,\frac{\mathcal{F}(\xi e(t))}{\xi}\Bigg|_{t=a}^{t=b}\right) - \lim_{\xi\to\infty}\int_a^b \left[\frac{d}{dt}\left(\frac{g(t)}{e'(t)}\right)\right]\frac{\mathcal{F}(\xi e(t))}{\xi}\,dt \to 0.$$

For example $f(x)$ has a sublinear primitive if $f(x)$ is periodic or almost periodic with zero average. Other examples: the functions $\sin\sqrt{|x|}$ and $\sin(x^2)$ also have sublinear primitives. The function $\sin(\ln(\ln(3+|x|)))$ has not sublinear primitive and the limit in (11) can be non-zero. If a primitive of a function $g(x)$ is bounded and

$$\lim_{x\to\infty}\frac{f''(x)}{(f'(x))^2} = 0,$$

then $g(f(x))$ has a sublinear primitive.

As a last example consider the problem

$$x'' + x + \sin(\ln(\ln(3+|x|))) + \frac{3}{\pi}\arctan x + 100\cos x = \lambda\sin t, \quad x(0) = x(\pi) = 0.$$

Here $p(x) = \int_0^\pi e(t)\,x(t)\,dt$. The function $100\cos x$ has a bounded primitive and satisfies (11), for the function $(3/\pi)\arctan x$ limits (12) exist and $\tilde{\psi}(\pm) = \pm 3\sqrt{2/\pi}$. And for the function $\sin(\ln(\ln(3+|x|)))$ the numbers (2) take the following values: $\psi_\pm = -2\sqrt{2/\pi}$, $\psi^\pm = 2\sqrt{2/\pi}$. This means that for the complete nonlinearity $\sin(\ln(\ln(3+|x|))) + 3/\pi\arctan x + 100\cos x$ our numbers take four different values $\psi_- = -5\sqrt{2/\pi}$, $\psi^- = -\sqrt{2/\pi}$, $\psi_+ = \sqrt{2/\pi}$ and $\psi^+ = 5\sqrt{2/\pi}$.

If $2 < |\lambda\pi| < 10$, then the set of solutions is unbounded; if $|\lambda\pi| < 2$, then the set of solutions is nonempty and bounded; if $|\lambda\pi| > 10$, then the set of solutions is bounded (may be empty).

6 Proof of Theorem 1

We prove the different conclusions of Theorem 1 separately.

a) Let $\psi^+ < 0 < \psi_-$ (the case $\psi^- < 0 < \psi_+$ can be considered in a similar way). Put

$$s(\xi) = \begin{cases} 1, & \text{if } \xi \ge 1, \\ \xi, & \text{if } |\xi| < 1, \\ -1, & \text{if } \xi \le -1. \end{cases} \tag{14}$$

Consider for $\lambda \in [0,1]$ the homotopy

$$\Xi_1(\lambda, x) = Qx - AQx - \lambda F(x) + (1-\lambda)s(p(x))e.$$

Obviously, $\Xi_1(1,x) = \Phi(x)$ and $\Xi_1(0,x) = Qx - AQx + s(p(x))e$. To prove the conclusion a) of Theorem 1 it is sufficient to see that $|\operatorname{ind}_\infty \Xi_1(0,\cdot)| = 1$ and to state an a priori estimate $\|x\| \le const < \infty$ for all the zeros x of the homotopy $\Xi_1(\lambda, x)$ for all λ's, $0 \le \lambda \le 1$.

The first step is rather simple: the equation $\Xi_1(0,x)$ has a unique solution $x = 0$, the index (in any sense) of this solution equals ± 1. This follows from the formula of the index product: the index of the origin in E_2 of the vector field $Qx - AQx$ is equal to $(-1)^\beta$, where β is a sum of multiplicities of all real eigenvalues of A which are strictly greater than 1 and the index of the origin in E_1 of $s(\xi)$ equals 1.

Now let us prove an a priori estimate. Let $x = \xi e + h$, $h \in E_2$ and $\Xi_1(\lambda, x) = 0$ for some $\lambda \in [0,1]$. This implies two equalities

$$Qx = AQx + \lambda QF(x) \tag{15}$$

and

$$p(\Xi_1(\lambda, x)) = -\lambda PF(x) + (1-\lambda)s(\xi)e = 0. \tag{16}$$

Equality (15), rewritten as $h = (I - AQ)^{-1}QF(x)$, implies the estimate

$$\|h\|_{\tilde{E}} \leq \|(I - AQ)^{-1}\|_{\mathcal{L}(\tilde{E}\cap E_2)} \sup_{x\in E} \|QF(x)\|_{\tilde{E}} \stackrel{\Delta}{=} c < \infty. \tag{17}$$

Consider equality (16). If $|\xi|$ is sufficiently large, then $s(\xi) = \operatorname{sign}\xi$. We consider only the case $\xi > 1$, the case $\xi < -1$ is analogous. Since

$$p(\Xi_1(\lambda, x)) = -\lambda p(F(x)) + (1-\lambda) =$$

$$= -\lambda(p(F(\xi e + h)) - p(F(\xi e))) - \lambda\Psi(\xi) + (1-\lambda) \geq o(1) - \lambda\psi^+ + 1 - \lambda$$

and $-\lambda\psi^+ + 1 - \lambda \geq \max\{1, -\psi^+\} > 0$ then for $\xi \to +\infty$ we have $p(\Xi_1(\lambda, x)) > 0$. This gives the a priori estimate for ξ, together with (17) it proves the conclusion a).

b) Let $\psi_- \geq \psi_+ > 0$. The case $\psi_+ > \psi_- > 0$ and the case of negative ψ^- and ψ^+ are analogous to the case considered. Consider for $\lambda \in [0,1]$ the homotopy

$$\Xi_2(\lambda, x) = Qx - AQx - \lambda F(x) - (1-\lambda)\psi_+ e.$$

Now $\Xi_2(1, x) = \Phi(x)$ and $\Xi_2(0, x) = Qx - AQx - \psi_+ e$, the equation $\Xi_2(0, x) = 0$ has no solutions and $\operatorname{ind}_\infty \Xi_2(0, \cdot) = 0$. To prove the conclusion b) of Theorem 1 it is sufficient to prove an a priori estimate $\|x\| \leq const < \infty$ for all the zeros x of the homotopy $\Xi_2(\lambda, x)$ for $0 \leq \lambda \leq 1$. This a priori estimate follows from the estimate (17) (it is valid here too) and the relations

$$p(\Xi_2(\lambda, x)) = -\lambda p(F(x)) - (1-\lambda)\psi_+ = -\lambda(p(F(\xi e + h)) - p(F(\xi e))) - \lambda\Psi(\xi) - (1-\lambda)\psi_+$$

$$\leq o(1) - \lambda\psi_+ - \psi_+ + \lambda\psi_+ = o(1) - \psi_+ < 0.$$

c) Let $\psi_+ < 0 < \psi^+$. Again, the case $\psi_- < 0 < \psi^-$ can be considered in a similar way. We prove that there exists a sequence of zeros $x_n = \xi_n e + h_n$ of the vector field $\Phi(x)$, satisfying $\|x_n\|_E \to \infty$.

Put

$$\varepsilon = \frac{1}{3}\min\{\psi^+, -\psi_+\}, \quad c = \left\|(I - A)^{-1}\right\|_{\mathcal{L}(\tilde{E}\cap E_2)} \sup_{x\in E} \|QF(x)\|_{\tilde{E}},$$

$$c_1 = \left\|(I - A)^{-1}\right\|_{\mathcal{L}(E_2)} \sup_{x\in E} \|QF(x)\|_E.$$

According to (1) choose a $\xi_0 > 0$ such that for $\xi \geq \xi_0$

$$|p(F(\xi e + h)) - \Psi(\xi)| < \varepsilon, \quad \|h\|_{\tilde{E}} \leq c. \tag{18}$$

According to the definition of the numbers ψ_+ and ψ^+ there exist numbers $\xi_* > \xi_0$ and $\xi^* > \xi_*$ such that

$$\Psi(\xi_*) < -2\varepsilon, \quad \Psi(\xi^*) > 2\varepsilon. \tag{19}$$

Consider the cylinder $\Omega = \{\|Qx\|_E \leq c_1 + 1, \ p(x) \in [\xi_*, \xi^*]\}$. Let us prove that the vector field $x - T(x)$ is non-zero on $\partial\Omega$. The boundary $\partial\Omega$ consists from 3 parts:

$$G_* = \{\|Qx\|_E \leq c_1 + 1, \ p(x) = \xi_*\}, \quad G^* = \{\|Qx\|_E \leq c_1 + 1, \ p(x) = \xi^*\},$$

$$G = \{\|Qx\|_E = c_1 + 1, \ p(x) \in [\xi_*, \xi^*]\}.$$

The vector field $\Phi(x)$ is non-zero on G because of the estimate $\|Qx\|_E \leq c_1$ of all solutions x of $Q\Phi(x) = 0$. The vector field $\Phi(x)$ is non-zero on G_* due to relations

$$p(\Phi(x)) = p(F(x)) - \Psi(p(x)) + \Psi(p(x)) < \varepsilon - 2\varepsilon = -\varepsilon < 0,$$

The vector field $\Phi(x)$ is non-zero on G^* due to relations

$$p(\Phi(x)) = p(F(x)) - \Psi(p(x)) + \Psi(p(x)) > -\varepsilon + 2\varepsilon = \varepsilon > 0.$$

In the last two formulas we used the following arguments: if $\Phi(x) = 0$, then $\|Qx\|_{\tilde{E}} \leq c$ and therefore

$$|p(F(x)) - \Psi(p(x))| < \varepsilon, \quad \Psi(p(x)) < -2\varepsilon, \qquad x \in G_*$$

and

$$|p(F(x)) - \Psi(p(x))| < \varepsilon, \quad \Psi(p(x)) > 2\varepsilon, \qquad x \in G^*.$$

We additionally proved that $p(\Phi(x)) < 0$ if $Q\Phi(x) = 0$ and $p(x) = \xi_*$ and $p(\Phi(x)) > 0$ if $Q\Phi(x) = 0$ and $p(x) = \xi^*$.

Now let us calculate the rotation $\gamma(\Phi, \partial\Omega)$ of non-degenerate vector field $\Phi(x)$ on $\partial\Omega$. This can be done with the use of Theorem 3 from [1]. Let us formulate this theorem in a form which is convenient for our very particular case.

Let X be a Banach space. Let some completely continuous operator $\mathcal{A} = \{A_1, A_2\}$ be defined on the set $\Omega = \Omega_1 \times \Omega_2$ where $\Omega_1 = B(0, r) \subset X$ and $\Omega_2 = [\xi_*, \xi^*]$. Let for any $\xi \in \Omega_2$ the vector field $x_1 - A_1(x_1, \xi)$ be non-degenerate on $\partial\Omega_1 = \{\|x\|_X = r\}$ and let the rotation $\gamma_1 = \gamma(x_1 - A_1(x_1, \xi), \partial\Omega_1)$ be non-zero. This γ_1 does not depend on ξ. Denote for any $\xi \in \Omega_2$ by $K(\xi)$ the non-empty set of solutions x_1 of the equation $x_1 = A_1(x_1, \xi)$.

Theorem [1]. *If $\xi_* - A_2(x_1, \xi_*) < 0$ for $x_1 \in K(\xi_*)$ and $\xi^* - A_2(x_1, \xi^*) > 0$ for $x_1 \in K(\xi^*)$, then $\gamma(\mathcal{A}, \partial\Omega) = \gamma_1$.*

This theorem implies for our situation the relations $\gamma(\Phi, \partial\Omega) = \gamma(Q\Phi, \{\|Qx\|_E = c_1 + 1\}) = \pm - 1 \neq 0$.

We proved that in the set Ω there is at least one zero of the vector field Φ. Denote it as x_1 and choose other numbers ξ_* and ξ^*, both greater than both old ones. Again, in the corresponding cylinder there exists at least one zero of Φ. Since we can choose arbitrary large numbers ξ_* and ξ^*, satisfying (18), there exists a sequence x_n of zeros with $\|x_n\|_E \to \infty$.

7 Proof of Theorem 2

Let us choose the function $\mu : [\xi_0, \infty) \to \mathbb{R}$ satisfying the following conditions

$$\mu(\xi) \to \infty,\ \xi \to \infty, \qquad \mu(\xi) \le \sqrt{\xi}, \qquad \mu(\xi) \le \frac{1}{\sqrt{L(\sqrt{\xi})}},$$

where $L(r)$ is the function from (4), satisfying (5). Since

$$\|f(t, \xi e(t) + h(t)) - f(t, \xi e(t))\|_{L^2}^2 =$$

$$= \int_\Omega \|f(t, \xi e(t) + h(t)) - f(t, \xi e(t))\|_{\mathbb{R}^n}^2 \, dt \le \int_{\Omega_1} \ldots + \int_{\Omega_2} \ldots + \int_{\Omega_3} \ldots,$$

where

$$\Omega_1 = \{t \in \Omega : \ \|e(t)\|_{\mathbb{R}^n} \le \frac{2}{\sqrt{|\xi|}}\}, \quad \Omega_2 = \{t \in \Omega : \ \|h(t)\|_{\mathbb{R}^n} \ge \mu(|\xi|)\},$$

$$\Omega_3 = \{t \in \Omega : \ \|e(t)\|_{\mathbb{R}^n} \ge \frac{2}{\sqrt{|\xi|}}, \ \|h(t)\|_{\mathbb{R}^n} \le \mu(|\xi|)\},$$

then it is necessary to estimate the last three integrals. According to (3)

$$\left|\int_{\Omega_1} \ldots\right| \le 4 \sup |f^2(t,x)| \cdot mes\, \Omega_1 \to 0,$$

according to Chebyshev inequality

$$\left|\int_{\Omega_2} \ldots\right| \le 4 \sup |f^2(t,x)| \cdot mes\, \Omega_2 \le 4 \sup |f^2(t,x)| \cdot \frac{\|h(t)\|_{L^2}^2}{\mu^2(|\xi|)} \to 0.$$

The estimate of the third integral follows from the inequality

$$\|\xi e(t) + h(t)\|_{\mathbb{R}^n} \ge |\xi| \, \|e(t)\|_{\mathbb{R}^n} - \|h(t)\|_{\mathbb{R}^n} \ge 2\sqrt{|\xi|} - \mu(|\xi|) \ge \sqrt{|\xi|},$$

which is valid for $t \in \Omega_3$. Now according to (4)

$$\left|\int_{\Omega_3} \ldots\right| \le \int_{\Omega_3} L^2(\sqrt{|\xi|}) \, \|h(t)\|_{\mathbb{R}^n}^2 \, dt \le mes\, \Omega \, L^2(\sqrt{|\xi|}) \, \mu^2(|\xi|) \le mes\, \Omega \, L(\sqrt{|\xi|}) \to 0.$$

Theorem 2 is proved.

8 Proof of Theorem 3

We have to prove that for any $\varepsilon > 0$ the supremum in (8) is less than ε for sufficiently large $|\xi|$. First let us choose a continuous $\tilde{g}(t)$ such that

$$2 \sup |f(t,x)| \int_a^b |g(t) - \tilde{g}(t)| dt < \varepsilon/3.$$

We need to prove that

$$\sup_{\|h(t)\|_{C^1}\le c}\left|\int_a^b \tilde{g}(t)[f(t,\xi e(t)+h(t))-f(t,\xi e(t))]\,dt\right|<\frac{2}{3}\varepsilon. \tag{20}$$

Now let us split the interval $[a,b]$ into a finite number of subintervals $[a_i,b_i]$ and $[b_i,a_{i+1}]$ in the following way. The intervals $[b_i,a_{i+1}]$ contain the set $\{t\in[a,b]: e'(t)=0\}$, according to (7) the union of these intervals can have arbitrary small measure. Let

$$2\sup|f(t,x)|\int_{\cup[b_i,a_{i+1}]}|\tilde{g}(t)|\,dt<\varepsilon/3. \tag{21}$$

Suppose that the points a_i and b_i are fixed up to the end of the proof. For any $[a_i,b_i]$ the estimate

$$\inf_{t\in[a_i,b_i]}|e'(t)|\ge\delta>0$$

holds. This means that the function $e(t)$ is strictly monotone on every $[a_i,b_i]$, for sufficiently large $|\xi|$ ($|\xi|>2c\delta^{-1}$) the function $\xi e(t)+h(t)$ is also strictly monotone, $|\xi e'(t)+h'(t)|>1/2\ |\xi|\delta$. Consider the integrals

$$\mathcal{J}_i=\int_{a_i}^{b_i}\tilde{g}(t)f(t,\xi e(t)+h(t))\,dt.$$

Fix any of them and do the change of variables $t=t_\xi(\tau)$ in this integral, defined by the formula $\xi e(\tau)=\xi e(t)+h(t)$:

$$\mathcal{J}_i=\int_{t_\xi^{-1}(a_i)}^{t_\xi^{-1}(b_i)}\frac{\tilde{g}(t_\xi(\tau))\,f(t_\xi(\tau),\xi e(\tau))}{t'_\xi(\tau)}d\tau.$$

The function $t_\xi(\tau)$ is one-to-one, $t_\xi(\tau)\to\tau$ and $t'_\xi(\tau)\to 1$ uniformly in τ as $|\xi|\to\infty$. Now

$$t_\xi^{-1}(a_i)\to a_i,\quad t_\xi^{-1}(b_i)\to b_i,$$

$\tilde{g}(t_\xi(\tau))\to\tilde{g}(\tau)$ due to the continuity of $\tilde{g}(\cdot)$, and

$$f(t_\xi(\tau),\xi e(\tau))\to f(\tau,\xi e(\tau))$$

due to the assumption of the uniform continuity of $f(t,x)$ in t. Consequently,

$$\mathcal{J}_i-\int_{a_i}^{b_i}\tilde{g}(\tau)f(\tau,\xi e(\tau)+h(\tau))\,d\tau\to 0$$

for every i. This together with (20) and (21) proves the theorem.

9 Remarks

1. The conclusions of Theorem 1 can be obtained for asymptotically linear degenerate vector fields with unbounded nonlinearities.

2. If we cannot calculate the exact values of the numbers (2) we can use some estimates of these numbers.

3. The main condition (1) is valid if

$$|p(F(x)) - p(F(y))| \leq L(r) \|x - y\|_E \,, \quad \|x\|_E \,, \|y\|_E \geq r \tag{22}$$

and (5) holds. Unfortunately we do not know any natural examples of nonlinearities, satisfying (22).

4. Analogs of Theorem 1 can be formulated for vector fields with non-compact operators (e.g. with condensing ones, see [2]).

5. The conditions for conclusion c) of Theorem 1 give us the information about the localization of solutions x_n .

References

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